Existence of the best n -term approximants for structured dictionaries

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Abstract. This paper investigates the existence of elements of the best n -term approximation in infinite dimensional Hilbert spaces. The notion of *uniform linear independence* (ULI) for a dictionary is introduced. It is shown that if the dictionary used for approximation satisfies the Bessel inequality and has the ULI property, then for every element of the Hilbert space there exists an element of the best n -term approximation. It is also shown that if a dictionary does not satisfy the ULI property, then there exists an arbitrarily small compact perturbation of this dictionary for which the elements of the best n -term approximation need not exist. The obtained results are applied to frames.

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1. Introduction. Nonlinear n -term approximation is a relevant topic as far as recent developments in approximation theory are concerned. A popular method of obtaining n -term approximants is referred to as *greedy approximation*. See the survey [10] for a comprehensive overview of this approach to nonlinear n -term approximation. Even though various greedy algorithms can be used to obtain n -term approximants with various properties, one very general question concerning nonlinear n -term approximation so far has not been answered: *When do the best n -term approximants exist?* This kind of question is important and interesting from a theoretical point of view for any approximation scheme. While the existence of the best approximants in the linear setting is a problem with an immediate solution (as such approximants belong

to a bounded subset of a finite dimensional subspace), this is not the case with nonlinear approximation. Firstly, one can easily construct dictionaries such that the best n -term approximants do not exist, which means that additional assumptions about the dictionary are necessary. Secondly, the sets of nonlinear n -term approximants have a more complicated structure than finite dimensional linear subspaces.

This paper also concerns classes of dictionaries called *frames*, *Bessel systems*, and *Riesz bases*. Such dictionaries play a significant role in approximation theory as well as numerous applications, especially signal processing. The development of the theory of frames goes back to the now famous paper of Duffin and Schaeffer [7]. A detailed treatment of Bessel systems, frames, and Riesz bases can be found in the textbook [3].

The organization of this paper is as follows: Section 1 recalls the required definitions and facts concerning n -term approximation and various types of dictionaries, and introduces the *uniform linear independence* property of a dictionary. In Section 2 the two main results are formulated and a number of corollaries is shown. Section 3 is devoted to the proofs of the main theorems.

2. Preliminaries. Throughout this paper we assume that H denotes an infinite dimensional separable Hilbert space with the scalar product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. For a set A the notation $\#A$ denotes the cardinality of A .

Definition 2.1. A countable subset $\Phi = \{\phi_j\}_{j \in \mathbb{N}} \subset H$ is a *dictionary* if it has the following properties

- (i) $\overline{\text{span } \Phi} = H$,
- (ii) $0 < \inf\{\|\phi_j\| : j \in \mathbb{N}\} < \sup\{\|\phi_j\| : j \in \mathbb{N}\} < +\infty$.

Definition 2.2. For a subset $A \subset \mathbb{N}$ we use the notation $\Phi_A = \{\phi_j : j \in A\}$. P_A will denote the operator of orthogonal projection from H onto $\overline{\text{span}(\Phi_A)}$.

Definition 2.3. For a dictionary $\Phi = \{\phi_j\}_{j \in \mathbb{N}}$ we define the *sets for nonlinear n -term approximation*:

$$\Sigma_n(\Phi) = \left\{ \sum_{k=1}^n a_k \phi_{j_k} : a_k \in \mathbb{C}, j_k \in \mathbb{N} \text{ for } k = 1, 2, \dots, n \right\}$$

The definition of a dictionary is very general. We will also be considering the following types of countable subsets of H :

Definition 2.4. A countable subset $\Phi = \{\phi_j\}_{j \in \mathbb{N}} \subset H$ is a *Bessel system* if there exists a constant $B < +\infty$ such that for all $x \in H$

$$\sum_{j \in \mathbb{N}} |\langle x, \phi_j \rangle|^2 \leq B \|x\|^2.$$

A dictionary which is a Bessel system will be referred to as a *Bessel dictionary*.

Definition 2.5. A dictionary $\Phi = \{\phi_j\}_{j \in \mathbb{N}} \subset H$ is a *frame* if there exist constants $A > 0$ and $B < +\infty$ such that for all $x \in H$

$$A \|x\|^2 \leq \sum_{\phi \in \Phi} |\langle x, \phi \rangle|^2 \leq B \|x\|^2.$$

The constants A and B are called *lower* and *upper frame bounds* for Φ .

We now define a condition for dictionaries connected with the notion of linear independence:

Definition 2.6. A dictionary $\Phi = \{\phi_j\}_{j \in \mathbb{N}} \subset H$ has the *uniform linear independence property for m* ($\text{ULI}(m)$) if there exists a constant c_m such that for every linearly independent set Φ_A , where $A \subset \mathbb{N}$ is of cardinality m and every set of scalars $\{a_j\}_{j \in A}$

$$\left\| \sum_{j \in A} a_j \phi_j \right\| \geq c_m \left(\sum_{j \in A} |a_j|^2 \right)^{\frac{1}{2}}.$$

A dictionary has the *uniform linear independence property (ULI)* if it has $\text{ULI}(m)$ for all $m \in \mathbb{N}$.

Because the inverse inequality always holds with the constant $C_n = \sqrt{n} \cdot \sup\{\|\phi\| : \phi \in \Phi\}$, $\text{ULI}(m)$ means that for all linearly independent subsets $\Phi_A \subset \Phi$ of cardinality m the Hilbert space norms on the spaces $\text{span}\{\phi_j : j \in A\}$ are *uniformly* equivalent to the ℓ_2 coefficient norms in the basis $\{\phi_j : j \in A\}$.

We briefly discuss some examples of dictionaries which have ULI.

Definition 2.7. A dictionary $\Phi = \{\phi_j\}_{j \in \mathbb{J}}$ is a *Riesz basis* if there exist constants $A, B > 0$ such that for any finite sequence of scalars $\{c_k\}$ one has

$$A \sum_k |c_k|^2 \leq \left\| \sum_k c_k \phi_k \right\|^2 \leq B \sum_k |c_k|^2. \quad (2.1)$$

Proposition 2.8. Every Riesz basis Φ is a Bessel dictionary which has ULI.

Proof. ULI follows directly from the definition. That a Riesz basis is also a Bessel system is shown in [3, Theorem 3.6.3]. $\square$

Another class of dictionaries with ULI is given by the following definition (originally introduced in [1]):

Definition 2.9. A frame $\Phi = \{\phi_j\}_{j \in \mathbb{N}}$ is a *Riesz frame* if there exists a constant $A > 0$ such that for any $C \subset \mathbb{N}$ the system $\{\phi_j\}_{j \in C}$ is a frame in $\text{span}\{\phi_j\}_{j \in C}$ with lower frame bound A .

Proposition 2.10. Every Riesz frame Φ is a dictionary which has ULI.

Proof. From [3, Lemma 5.5.4] it follows that every finite linearly independent subset of Φ is a Riesz basis in its linear span with the lower bound constant in (2.1) equal to the lower Riesz frame bound A . $\square$

We also present a simple example of an overcomplete dictionary which has ULI:

Example 2.11. Let $\{x_n\}_{n \in \mathbb{N}} \subset H$ be an orthonormal basis. The dictionary $\Phi = \{\phi_j\}_{j \in \mathbb{N}}$ is defined as follows:

$$\phi_j = \begin{cases} x_{2n} & \text{if } j = 4n, \\ x_{2n+1} & \text{if } j = 4n + 1, \\ \frac{1}{\sqrt{2}}(x_{2n} + x_{2n+1}) & \text{if } j = 4n + 2, \\ \frac{1}{\sqrt{2}}(x_{2n} - x_{2n+1}) & \text{if } j = 4n + 3. \end{cases}$$

Overcompleteness of Φ is obvious. From a simple calculation it follows that for a finite subset $A \subset \mathbb{N}$ and scalars $a_j, j \in A$, the following estimate holds:

$$\left\| \sum_{j \in A} a_j \phi_j \right\| \geq \left(1 - \frac{1}{\sqrt{2}}\right)^{\frac{1}{2}} \left(\sum_{j \in A} |a_j|^2 \right)^{\frac{1}{2}}.$$

Hence this dictionary has ULI.

Finally, we recall the notion of *coherence* of a dictionary:

Definition 2.12. For a dictionary $\Phi = \{\phi_j\}_{j \in \mathbb{N}}$ such that $\|\phi_j\| = 1$ for all $j \in \mathbb{N}$, its *coherence* $\eta(\Phi)$ is defined as the number

$$\eta(\Phi) = \sup\{|\langle \phi_i, \phi_j \rangle| : i \neq j\}.$$

The notion of coherence has appeared in numerous results concerning n -term approximation and sparse representations, see for instance [8, 5, 6]. The following fact is shown in [6, Lemma 2.1]:

Proposition 2.13. Assume that the dictionary $\Phi = \{\phi_j\}_{j \in \mathbb{N}}$ with $\|\phi_j\| = 1$ for all n has coherence $\eta = \eta(\Phi)$. Then Φ has $ULI(\mu)$ for all $\mu \leq \eta^{-1} + 1$ and

$$c_\mu \geq 1 - \eta(\mu - 1).$$

Moreover, for any $A \subset \mathbb{N}$ such that $\#A \leq \eta^{-1} + 1$ the set Φ_A is linearly independent.

To close this section, we note that it would be interesting to know the extent to which dictionaries like Gabor and wavelet frames have ULI. From [2, Theorem 4.7] it follows that every redundant regular Gabor frame is not a Riesz frame. However, ULI is a weaker condition as compared to the Riesz frame condition, so the question whether Gabor frames have ULI is open. A partial answer may be obtained by applying Proposition 2.13 to *intrinsically localized frames* introduced in [9].

3. Formulation of results. The main purpose of this paper is to prove the two theorems below. The first one shows that the elements of the best n -term approximation exist for Bessel dictionaries which have ULI.

Theorem 3.1. Assume that $\Phi = \{\phi_j\}_{j \in \mathbb{N}} \subset H$ is a Bessel dictionary which has $ULI(m)$. Then for any $x \in H$ there exists $\tilde{x}_m \in \Sigma_m(\Phi)$ such that

$$\|x - \tilde{x}_m\| = \text{dist}(x, \Sigma_m(\Phi)).$$

The second theorem shows that if a Bessel dictionary Φ does not have ULI, then there exists an arbitrarily small perturbation of Φ which results in a dictionary for which the best approximants do not exist.

Theorem 3.2. *Assume that H is infinite dimensional and for the dictionary $\Phi = \{\phi_j\}_{j \in \mathbb{N}} \subset H$ the $ULI(m)$ property fails. Then there exists $\mu \leq m$ such that for any $\varepsilon > 0$ there exists a dictionary $\Psi = (\psi_j)_{j \in \mathbb{N}}$ such that*

$$\sum_{j \in \mathbb{N}} \|\phi_j - \psi_j\|^2 < \varepsilon$$

and the set $\Sigma_\mu(\Psi)$ is not closed in the norm topology of H .

From Theorem 3.1 we immediately get

Corollary 3.3. *If Φ is a frame in H which has $ULI(m)$ for some $m \in \mathbb{N}$, then for every $x \in H$ there exists $\tilde{x}_m \in \Sigma_m(\Phi)$ such that*

$$\|x - \tilde{x}_m\| = \text{dist}(x, \Sigma_m(\Phi)).$$

Because every Riesz basis and Riesz frame has ULI, we also have

Corollary 3.4. *If Φ is a Riesz basis or Riesz frame in H , then for every $x \in H$ and every $m \in \mathbb{N}$ there exists $\tilde{x}_m \in \Sigma_m(\Phi)$ such that*

$$\|x - \tilde{x}_m\| = \text{dist}(x, \Sigma_m(\Phi)).$$

Regarding the non-existence of the elements of the best m -term approximation for frames, we have

Corollary 3.5. *If $\Phi = \{\phi_j\}_{j \in \mathbb{N}}$ is a frame in H for which the $ULI(m)$ property fails, then there exists $\mu \leq m$ such that for any $\varepsilon > 0$ there exists a frame $\Psi = \{\psi_j : j \in \mathbb{N}\}$ in H such that*

$$\sum_{j \in \mathbb{N}} \|\phi_j - \psi_j\|^2 < \varepsilon^2$$

and the set $\Sigma_\mu(\Psi)$ is not closed.

Proof. We apply Theorem 3.2 to obtain the system Ψ . Observe that the operator $K : \ell^2(\mathbb{N}) \rightarrow H$ defined as

$$K((c_j)_{j \in \mathbb{N}}) = \sum_{j \in \mathbb{N}} c_j(\phi_j - \psi_j)$$

is compact. From [4, Theorem 4.2] it now follows that Ψ is a frame in its linear span. Because Ψ was constructed to be complete in H , it therefore is a frame in H . $\square$

4. Proofs of theorems. We begin with three lemmas:

Lemma 4.1. *Assume that $(A_n)_{n \in \mathbb{N}}$ is a sequence of subsets of $\mathbb{N}$, each of cardinality $m < +\infty$. Then there exists a sequence $(n_k)_{k \in \mathbb{N}}$ of natural numbers, and subsets $B, C_{n_k} \subset \mathbb{N}, k \in \mathbb{N}$, such that for each k*

$$A_{n_k} = B \cup C_{n_k}, \quad \text{and} \quad B \cap C_{n_k} = \emptyset$$

and $C_{n_k} \cap C_{n_{k'}} = \emptyset$ for $k \neq k'$.

Proof. The lemma is easily proved by induction with respect to m . For $m = 1$ denote $A_n = \{a_n\}$. The sequence (a_n) either contains a constant subsequence, or a monotonically increasing one and the claim follows.

Assume now that the lemma is true for m . We show it is true for $m + 1$. Let $a = \liminf_{n \rightarrow \infty} \min A_n$. If $a < \infty$ then we can choose a sequence $(\bar{n}_k)_{k \in \mathbb{N}}$ such that $a \in A_{\bar{n}_k}$ for all k . The sets $\bar{A}_{\bar{n}_k} = A_{\bar{n}_k} \setminus \{a\}$ are of cardinality m . Hence, there exists a subsequence (n_k) of $(\bar{n}_k)$ and pairwise disjoint sets $\bar{B}, \bar{C}_k$ such that $\bar{A}_{n_k} = \bar{B} \cap \bar{C}_{n_k}$. For $B = \bar{B} \cup \{a\}$ we have $A_{n_k} = B \cap C_{n_k}$.

If $a = +\infty$, then we can choose a sequence (n_k) such that the sets A_{n_k} are pairwise disjoint. $\square$

Lemma 4.2. *Let $\Phi = \{\phi_j\}_{j \in \mathbb{N}}$ be a Bessel dictionary. Assume that $u \in H, 0 < r < \|u\|, C_n \subset \mathbb{N}, n = 1, 2, \dots$ are pairwise disjoint and of the same cardinality $\mu < \infty$,*

$$u_n = \sum_{j \in C_n} a_{n,j} \phi_j, \quad \text{for } n \in \mathbb{N},$$

where $a_{n,j}$ are scalars such that $\sup\{|a_{n,j}| : n \in \mathbb{N}, j \in C_n\} < \infty$. Then there are only finitely many n such that $\|u - u_n\| < r$.

Proof. Let $M = \sup_{n,j} |a_{n,j}|$. We estimate

$$\begin{aligned} r \|x\| &\geq |\langle u, u - u_n \rangle| = \left| \|u\|^2 - \sum_{j \in C_n} a_{n,j} \langle u, \phi_j \rangle \right| \\ &\geq \|u\|^2 - \left(\sum_{j \in C_n} |a_{n,j}|^2 \right)^{\frac{1}{2}} \left(\sum_{j \in C_n} |\langle u, \phi_j \rangle|^2 \right)^{\frac{1}{2}} \\ &\geq \|u\|^2 - \sqrt{\mu} M \left(\sum_{j \in C_n} |\langle u, \phi_j \rangle|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Hence

$$\sum_{j \in C_n} |\langle u, \phi_j \rangle|^2 \geq \frac{1}{\mu M^2} \|u\|^2 (\|u\| - r)^2.$$

As the subsets C_n are pairwise disjoint, this estimate contradicts the assumption that Φ is a Bessel system. $\square$

Lemma 4.3. *Assume that the dictionary $\Phi = \{\phi_j\} \subset H$ has $\text{ULI}(m)$ for some $m \in \mathbb{N}, x \in H, A_n \subset \mathbb{N} (n \in \mathbb{N})$ are of cardinality m and the sets Φ_{A_n} are linearly independent for all n . Let the scalars $a_{n,j}, n \in \mathbb{N}, j \in A_n$, be such that*

$$P_{A_n} x = \sum_{j \in A_n} a_{n,j} \phi_j.$$

Then $\sup\{|a_{n,j}| : n \in \mathbb{N}, j \in C_n\} < \infty$.

Proof. We estimate

$$\|x\| \geq \|P_{A_n}x\| = \left\| \sum_{j \in A_n} a_{n,j} \phi_j \right\| \geq c_m \left(\sum_{j \in A_n} |a_{n,j}|^2 \right)^{\frac{1}{2}} \geq c_m \max_{j \in A_n} |a_{n,j}|.$$

□

4.1. Proof of Theorem 3.1. We may assume that $x \neq 0$. For a given $m \in \mathbb{N}$ denote $d_m = \text{dist}(x, \Sigma_m(\Phi))$. If $d_m \geq \|x\|$, then we may put $\tilde{x}_m = 0$. Hence, we assume further on that $d_m < \|x\|$.

There exist subsets $A_n \subset \mathbb{N}, n \in \mathbb{N}$, such that $d \leq \|x - P_{A_n}x\| \rightarrow d_m$ with $n \rightarrow \infty$. Let us write

$$P_{A_n}x = \sum_{j \in A_n} a_{k,j} \phi_j.$$

We prove the theorem by induction with respect to m . For $m = 1$ we have $A_n = \{j_n\}$. If the sequence $(j_n)_n$ contains a constant subsequence, we are done, otherwise we apply Lemma 4.2 for $u = x, C_n = A_n$ and $u_n = P_{A_n}x$. That the coefficients a_{n,j_n} are bounded follows from Lemma 4.3.

Assume now that $m > 1$ and the Theorem holds for $\mu \in \{1, \dots, m-1\}$. If for infinitely many n the sets Φ_{A_n} are linearly dependent, then $d_m = d_\mu$ for some $\mu < m$ and we may select $\tilde{x}_m = \tilde{x}_\mu$. The existence of such $\tilde{x}_\mu$ is guaranteed by the inductive assumption.

Hence, we assume that Φ_{A_n} are linearly independent for all n . By Lemma 4.1 we may assume that $A_n = B \cup C_n$ with $\#C_n = \mu \leq m, B \cap C_n = \emptyset$ for all n , and $C_n \cap C_{n'} = \emptyset$ for $n \neq n'$.

If $\mu = 0$, then we may put $\tilde{x}_m = P_{A_n}x = P_Bx$.

For $\mu > 0$ we write $x_n = P_{A_n}x = y_n + z_n$, where

$$y_n = \sum_{j \in B} a_{n,j} \phi_j, \quad z_n = \sum_{j \in C_n} a_{n,j} \phi_j.$$

We have $y_n = 0$ if $\mu = m$. By Lemma 4.3 the sequence (y_n) is bounded, and hence we may assume that $y_n \rightarrow y$ with $n \rightarrow \infty$ for some $y \in \Sigma_{m-\mu}(\Phi)$.

If $\|x - y\| \leq d_m$, we may put $\tilde{x}_m = y$. If $\|x - y\| > d_m$, we proceed in the following manner: for any $\varepsilon > 0$ there exists n_0 such that for $n > n_0$ we have $\|y - y_n\| < \varepsilon/2$ and

$$\|x - y - z_n\| \leq \|x - y_n - z_n\| + \|y_n - y\| < d_m + \varepsilon. \quad (4.1)$$

From the the fact that Φ has ULI(m) we get for any n and $i \in C_n$

$$|a_{n,i}| \leq \left(\sum_{j \in C_n} |a_{n,j}|^2 \right)^{\frac{1}{2}} \leq \left(\sum_{j \in A_n} |a_{n,j}|^2 \right)^{\frac{1}{2}} \leq \frac{1}{c_m} \|P_{A_n}x\| \leq \frac{1}{c_m} \|x\|. \quad (4.2)$$

We now apply Lemma 4.2 for any $r = d_m + \varepsilon/2$, where $d_m < \varepsilon < \|x - y\|$, $u = x - y$, and $u_n = z_n$. That the assumptions of the Lemma are satisfied follows from the estimates (4.1) and (4.2). We obtain that there are only finitely many natural numbers $n > n_0$ such that $\|x - y - z_n\| < r$. Because for $n > n_0$ we also have $\|P_{A_n}x - (y + z_n)\| = \|y - y_n\| < \varepsilon/2$, there are also only finitely

many $n > n_0$ for which $\|x - P_{A_n}x\| < d_m + \varepsilon$. This, however, contradicts the claim that $A_n = B \cup C_n$ with $\#C_n = \mu > 0$. Therefore, the case when $\mu > 0$ is not possible. $\square$

4.2. Proof of Theorem 3.2. Let μ be the smallest m such that the SLI property of the system Φ fails for m . There exists a sequence of subsets $A_n \subset \mathbb{N}$ of cardinality μ and vectors $x_n \in \text{span}\{\phi_j : j \in A_n\}$ with $\|x_n\| = 1$, $x_n = \sum_{j \in A_n} a_{n,j} \phi_j$ and

$$\sum_{j \in A_n} |a_{n,j}|^2 \rightarrow +\infty \quad \text{with } n \rightarrow +\infty. \quad (4.3)$$

Step 1. We will show that there exists a sequence of different indices $j_n \in \mathbb{N}$ such that $|a_{n,j_n}| \rightarrow +\infty$. By Lemma 4.1 we may assume that

$$A_n = B \cup C_n \quad \text{with } B \cap C_n = \emptyset \quad (4.4)$$

and the sets C_n are pairwise disjoint. If $B \neq \emptyset$, we set

$$y_n = \sum_{j \in B} a_{n,j} \phi_j, \quad z_n = \sum_{j \in C_n} a_{n,j} \phi_j.$$

Because μ is minimal, there exists a constant $c < +\infty$ such that

$$\left(\sum_{j \in A'} |a_{n,j}|^2 \right)^{\frac{1}{2}} \leq c \|y_n\|, \quad \left(\sum_{j \in C_n} |a_{n,j}|^2 \right)^{\frac{1}{2}} \leq c \|z_n\|.$$

Together with (4.3) and the fact that $1 = \|x_n\| = \|y_n + z_n\|$ this implies that the sequences $(y_n)_n, (z_n)_n$ are unbounded. Without loss of generality we may assume that $\|z_n\| \rightarrow +\infty$. Because

$$\|z_n\| \leq C \left(\sum_{j \in C_n} |a_{n,j}|^2 \right)^{\frac{1}{2}} \quad \text{with } 0 < C \leq \sqrt{\mu} \cdot \sup_{\phi \in \Phi} \|\phi\|,$$

this implies that for some $j_n \in C_n$ we have $|a_{n,j_n}| \rightarrow +\infty$.

On the other hand, if $B = \emptyset$, then it follows from (4.3) that there exist $j_n \in A_n = C_n$ such that $|a_{n,j_n}| \rightarrow +\infty$. Because the sets C_n are disjoint, the indexes j_n are different for all $n \in \mathbb{N}$.

Step 2. We now construct a dictionary $\Psi = \{\psi_j : j \in \mathbb{J}\} \subset H$ such that $\sum_{j \in \mathbb{J}} \|\phi_j - \psi_j\|^2 \leq \varepsilon^2$ and the set $\Sigma_\mu(\Psi)$ is not closed. From a simple Baire Category argument it follows that for a countable dictionary Φ we have $\text{span } \Phi \neq H$, hence we may select $u \in H \setminus \bigcup_{n \in \mathbb{N}} \Sigma_n(\Phi)$ such that $\|u\| = 1$. There exists a sequence of vectors $u_n \in \Sigma_n(\Phi)$ such that $\|u - u_n\| \rightarrow 0$ with $n \rightarrow +\infty$ and $\|u_n\| = 1$.

For $j = j_n$ define $\nu(j) = n$. This means that $a_{n,j_n} = a_{\nu(j),j}$. From the sequence $(j_n)_n$ we select two disjoint subsequence $(k_n)_{n \in \mathbb{N}}$ and $(l_n)_{n \in \mathbb{N}}$ such that

$$\sum_{n \in \mathbb{N}} \frac{4}{|a_{\nu(k_n),k_n}|^2} \leq \varepsilon^2 \quad \text{and} \quad \sum_{n \in \mathbb{N}} \frac{4}{|a_{\nu(l_n),l_n}|^2} \leq \varepsilon^2. \quad (4.5)$$

Let $\{\phi_{m_n}\}_{n \in \mathbb{N}}$ be a subset of Φ such that $H = \overline{\text{span}\{\phi_{m_n}\}_{n \in \mathbb{N}}}$. We define a new dictionary $\Psi = \{\psi_j\}_{j \in \mathbb{N}}$:

$$\psi_j = \begin{cases} \phi_j & \text{for } j \notin \{k_n\}_{n \in \mathbb{N}} \cup \{l_n\}_{n \in \mathbb{N}}, \\ \phi_j + a_{\nu(j),j}^{-1}(u_n - x_{\nu(j)}) & \text{for } j = k_n, \\ \phi_j + a_{\nu(j),j}^{-1}(\|\phi_{m_n}\|^{-1} \phi_{m_n} - x_{\nu(j)}) & \text{for } j = l_n. \end{cases}$$

Then $\|\phi_j - \psi_j\|^2 = 0$ for $j \notin \{k_n\}_{n \in \mathbb{N}} \cup \{l_n\}_{n \in \mathbb{N}}$ and $\|\phi_j - \psi_j\|^2 \leq 4|a_{\nu(j),j}|^{-2}$ for $j \in \{k_n\}_{n \in \mathbb{N}} \cup \{l_n\}_{n \in \mathbb{N}}$. Therefore, by (4.5)

$$\sum_{j \in \mathbb{N}} \|\phi_j - \psi_j\|^2 \leq \varepsilon^2, \quad (4.6)$$

To prove that Ψ is a dictionary in H we need to show that Ψ is complete in H . Let us take any $n \in \mathbb{N}$. Then

$$x_{\nu(l_n)} = \sum_{j \in A_{\nu(l_n)}} a_{\nu(l_n),j} \phi_j$$

and for all $j \in A_{\nu(l_n)} \setminus \{l_n\}$ we have $\psi_j = \phi_j$, due to (4.4) and the construction of Ψ . We now have

$$\sum_{j \in A_{\nu(l_n)}} a_{\nu(l_n),j} \psi_j = \sum_{j \in A_{\nu(l_n)}} a_{\nu(l_n),j} \phi_j - x_{\nu(j)} + \|\phi_{m_n}\|^{-1} \phi_{m_n} = \|\phi_{m_n}\|^{-1} \phi_{m_n},$$

from which it follows that Ψ is complete in H . From (4.6) it follows that $\inf_{j \in \mathbb{N}} \|\psi_j\| \geq \inf_{j \in \mathbb{N}} \|\phi_j\| - \varepsilon$ and $\sup_{j \in \mathbb{N}} \|\psi_j\| \leq \sup_{j \in \mathbb{N}} \|\phi_j\| + \varepsilon$. Hence, Ψ is a dictionary in H .

Step 3. We now show that u is a limit of elements from $\Sigma_\mu(\Psi)$, but $u \in H \setminus \Sigma_\mu(\Psi)$. For the first claim observe that

$$\sum_{j \in A_{\nu(k_n)}} a_{\nu(k_n),j} \psi_j = u_n \rightarrow u \quad \text{as } n \rightarrow +\infty.$$

As regards the second claim, assume to the contrary, that $u = \sum_{j \in A} b_j \psi_j$ for some $A \subset \mathbb{N}$ of cardinality μ . For every $j \in \mathbb{N}$ we have $\psi_j \in \Sigma_{N_j}(\Phi)$ for some $N_j \in \mathbb{N}$. Hence $u \in \Sigma_N(\Phi)$ for some $N \in \mathbb{N}$, which contradicts how we selected u . $\square$

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