Contact Moishezon threefolds with second Betti number one

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Abstract. We prove that the only contact Moishezon threefold having second Betti number equal to one is the projective space.

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1. Introduction. A compact complex manifold X of dimension $2n + 1$ is said to be a *contact manifold* if there is a vector bundle sequence

$$0 \rightarrow F \rightarrow T_X \rightarrow L \rightarrow 0$$

where T_X is the tangent bundle of X and F a sub-bundle of rank $2n$ such that the induced map

$$\bigwedge^2 F \rightarrow L = T_X/F, \quad v \wedge w \mapsto [v, w]/F$$

is everywhere non-degenerate. Properly speaking (X, F) is a contact manifold; the line bundle L is called the *contact line bundle*. It is easy to see that $-K_X = (n + 1)L$, that is in case X is a threefold:

$$-K_X = 2L. \quad (1.1)$$

We refer e.g. to [12] and [2] for details. There are basically two methods to construct compact contact manifolds.

- A simple Lie group gives rise to a Fano contact manifold X by taking the unique closed orbit for the adjoint action of the Lie group on the projectivised Lie algebra; we refer to [1]. Unless the group is of type A , we have $b_2(X) = 1$. Specifically, this construction includes $\mathbb{P}^{2n+1}$, $\mathbb{P}(T_{\mathbb{P}^{n+1}})$, Grassmannians of lines on quadrics, and some exceptional homogeneous spaces.

- Given any compact complex manifold M , the projectivised tangent bundle $\mathbb{P}(T_M)$ is a contact manifold.

A famous conjecture of LeBrun and Salamon [13] claims that there are no other *projective* contact manifolds. If $b_2(X) \geq 2$, this is settled by [11] and [5]. For results in case $b_2(X) = 1$, we refer to [1–3, 7, 8, 12, 15]. Since there is no known example of a compact contact manifold not in the above list, one might wonder whether the projectivity assumption in the conjecture of LeBrun and Salamon is really necessary. Dropping the projectivity assumption, it seems reasonable to assume first that X is not too wild, i.e. X is in class $\mathcal{C}$, which is to say that X is bimeromorphic to a compact Kähler manifold.

In [6] it has been shown that a contact threefold in class $\mathcal{C}$ which is not rationally connected must be of the form $X = \mathbb{P}(T_M)$ with a Kähler surface M . Thus it remains in dimension 3 to treat rationally connected varieties in class $\mathcal{C}$. Notice that these are automatically Moishezon spaces, i.e., they carry three algebraically independent meromorphic functions, see Proposition 2.2. In fact, the rational connectedness of X implies that $H^2(X, \mathcal{O}_X) = 0$. In this short note we treat the case that $b_2(X) = 1$.

Theorem 1.2. *Let X be a smooth threefold in class $\mathcal{C}$ with $b_2(X) = 1$, which is contact for some choice of $F \subset TX$. Then $X \simeq \mathbb{P}^3$.*

In the projective case, this theorem has first been shown by Ye [16].

2. Preliminaries. We will make heavily use of the following theorem of Kollár [9] and Nakamura.

Proposition 2.1. *Let X be a smooth Moishezon threefold with $\text{Pic}(X) \simeq \mathbb{Z}$ and let $\mathcal{O}_X(1)$ be the big (= effective) generator of $\text{Pic}(X)$. Write $K_X = \mathcal{O}_X(m)$ with some integer m and assume $m < 0$. Then*

- (1) $m \geq -4$ and $m = -4$ if and only if $X = \mathbb{P}^3$.
- (2) $m = -3$ if and only if X is the 3-dimensional quadric.
- (3) If $m = -2$, then $h^0(X, \mathcal{O}_X(1)) \leq 7$.
- (4) $H^2(X, \mathcal{O}_X(1)) = H^3(X, \mathcal{O}_X(1)) = 0$.

For the proof, see [9, Theorem (5.3.4)], [9, Theorem (5.3.12)], and [9, Corollary (5.3.9)], respectively.

Next we collect some basic properties of rationally connected manifolds. Recall that a compact manifold in class $\mathcal{C}$ is *rationally connected* if two general points in X can be joined by a chain of rational curves. For the benefit of the reader, we list the following well-known properties and include indications on the proof.

Proposition 2.2. *Let X be a rationally connected manifold in class $\mathcal{C}$. Then the following holds.*

- (1) X is simply connected;
- (2) $H^q(X, \mathcal{O}_X) = 0$ for all $q \geq 1$; in particular X is Moishezon.
- (3) $\text{Pic}(X)$ does not have torsion; so if $b_2(X) = 1$, then $\text{Pic}(X) = \mathbb{Z}$.

Proof. (1) We refer to [4, Corollary 5.7]. Notice that in [4], the manifold is supposed to be Kähler. Since however X is bimeromorphically equivalent to a Kähler manifold, we may choose a birational holomorphic map $\hat{X} \rightarrow X$ with $\hat{X}$ Kähler, given by a sequence of blow-ups with smooth centers. Then we apply Campana's theorem on $\hat{X}$ and use the basic fact $\pi_1(\hat{X}) = \pi_1(X)$ (it suffices to check that for a single blow-up along a submanifold).

(2) Since X is rationally connected, there exists a rational curve $C \subset X$ such that the tangent bundle $T_X|_C$ is ample, see [10, IV.3.7] (the proof works for manifolds in class $\mathcal{C}$ as well). From this fact it follows easily

$$H^0(X, \Omega_X^q) = 0,$$

hence by Hodge duality $H^q(X, \mathcal{O}_X) = 0$ for $q > 0$. We refer to [11, IV.3] for details.

In order to show that X is Moishezon, observe that $H^q(\hat{X}, \mathcal{O}_{\hat{X}}) = 0$ for positive q , in particular $H^2(\hat{X}, \mathcal{O}_{\hat{X}}) = 0$. Thus by Kodaira's classical theorem $\hat{X}$ is projective and therefore X is Moishezon.

(3) Suppose $\text{Pic}(X)$ contains a torsion element. Thus there is a non-trivial line bundle M such that $M^{\otimes m} \simeq \mathcal{O}_X$ for some positive number m . As a consequence, there is a finite étale cover $f: \tilde{X} \rightarrow X$ such that $f^*(M) \simeq \mathcal{O}_{\tilde{X}}$. This contradicts the simply connectedness of X . $\square$

3. Proof of the theorem. To start the proof of the main theorem, we first observe that X is uniruled, see [6, Theorem 2.2]. Furthermore, X is rationally connected, for otherwise by the main Theorem in [6] X is isomorphic to $\mathbb{P}(T_M)$ with a Kähler surface M and $b_2(X) \geq 2$, a contradiction to our assumption. In particular by Proposition 2.2, X is Moishezon, simply connected and $\text{Pic} X \simeq \mathbb{Z}$. Let $\mathcal{O}_X(1)$ be the effective generator of $\text{Pic}(X)$. Since the canonical line bundle K_X is divisible by 2 by (1.1), we have

$$K_X = \mathcal{O}_X(m)$$

with an even integer m . Since X is uniruled, m must be negative, see [9, Theorems (5.3.2) and (5.3.3)]. Applying Proposition 2.1 (1), we simply have to exclude $m = -2$. So suppose $m = -2$, in other words the contact line bundle $L = \mathcal{O}_X(1)$. We will arrive at a contradiction with (3) of Proposition 2.1 by calculating the number of sections $h^0(X, L)$.

Since $c_3(X)$ is the Euler characteristic of X , we have $c_3(X) = b_0 - b_1 + b_2 - b_3 + b_4 - b_5 + b_6$ with $b_0 = b_6 = 1, b_1 = b_5 = 0$ (X being simply connected), and $b_2 = b_4 = 1$ by our assumption). Hence

$$c_3(X) = 4 - b_3 \leq 4. \quad (3.1)$$

Since the contact form gives an isomorphism $\bigwedge^2 F = L$, we have $c_1(F) = L$. From the short exact sequence $0 \rightarrow F \rightarrow TX \rightarrow L \rightarrow 0$ we obtain

$$(1 + c_1(X) + c_2(X) + c_3(X)) = (1 + L + c_2(F))(1 + L)$$

In degrees 3 and 2 we obtain, respectively:

$$c_3(X) = c_2(F) \cdot L \quad \text{and} \quad (3.2)$$

$$c_2(X) = c_2(F) + L^2. \quad (3.3)$$

The Riemann–Roch–Hirzebruch [14, Section XIX.4] formula for $\mathcal{O}_X$ gives:

$$\underbrace{\chi(\mathcal{O}_X)}_{=1 \text{ by Prop. 2.2}} = \frac{1}{24} \underbrace{c_1(X)}_{=2L \text{ by Prop. (1.1)}} \cdot c_2(X)$$

and thus:

$$L \cdot c_2(X) = 12, \text{ and} \quad (3.4)$$

$$L^3 \stackrel{\text{by (3.3)}}{=} \underbrace{L \cdot c_2(X)}_{=12 \text{ by (3.4)}} - \underbrace{L \cdot c_2(F)}_{=c_3(X) \text{ by (3.2)}} = 12 - c_3(X) \stackrel{\text{by (3.1)}}{\geq} 8. \quad (3.5)$$

Now Riemann–Roch–Hirzebruch for L reads:

$$\begin{aligned} \chi(L) &= \frac{1}{3!}L^3 + \frac{1}{2}L^2 \cdot \underbrace{\frac{-K_X}{2}}_{=L \text{ by (1.1)}} + L \cdot \frac{(-K_X)^2 + c_2(X)}{12} + \underbrace{\chi(\mathcal{O}_X)}_{=1 \text{ by Prop. 2.2}} \\ &= \frac{1}{6}L^3 + \frac{1}{2}L^3 + \frac{1}{3}L \cdot \underbrace{\left(\frac{-K_X}{2}\right)^2}_{=L^2 \text{ by (1.1)}} + \underbrace{\frac{L \cdot c_2(X)}{12}}_{=1 \text{ by (3.4)}} + 1 \\ &= \frac{1}{6}L^3 + \frac{1}{2}L^3 + \frac{1}{3}L^3 + 1 + 1 \\ &= L^3 + 2 \geq 10 \text{ (by (3.5))}. \end{aligned}$$

Since $h^2(L) = h^3(L) = 0$ by Proposition 2.1 (4), we have $h^0(L) = \chi(L) + h^1(L) \geq 10$. This contradicts part (3) of Proposition 2.1.

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