Fixed elements in 2-groups revisited

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Abstract. We provide an elementary proof of the criterion for the triviality of the action of an odd-order group on a 2-group.

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1. Introduction. In [1], the authors provide a character theoretic proof of the following result:

Theorem 1 (Theorem B). *Let K be a group of odd order that acts on a 2-group P and assume that K fixes all elements of order 2 in P and all real elements of order 4. Then K acts trivially on P .*

Recall that a group element g is said to be *real* if it is conjugate to its inverse g^{-1} . The authors wrote: “it is not clear how one might prove results of this type without characters”. The aim of this note is to provide an elementary proof of the above result.

2. Proof of Theorem 1. We may assume that the group K is cyclic, $K = \langle x \rangle$, of odd prime order q . Suppose that the statement is not true and that P is a minimal group for which $[P, K] > 1$.

From the minimality of P it follows by standard arguments that P' is central, P/P' is elementary abelian and P has exponent at most 4. For readers' convenience, we quote the argument from [1]. By minimality of P , for any proper K -invariant subgroup $Q < P$ holds $[Q, K] = 1$. We apply this observation first to $Q = [P, K]$: if $[P, K] < P$ then $[P, K, K] = 1$. From Lemma 4.29 of [2] we then get $[P, K] = [P, K, K] = 1$, a contradiction. Therefore $[P, K] = P$. When we set $Q = P'$, we get $[P', K] = 1$. Therefore $[P', P, K] = 1 = [K, P', P]$ and thus $[P, P'] = [P, K, P'] = 1$, i.e., P' is central. Again, by minimality of P , P/P' has no proper K -invariant subgroups, hence P/P' is elementary

abelian. Finally, for any $a, b \in P$ we have $[a, b]^2 = [a, b^2] = 1$, as $b^2 \in P'$ and $P' \leq Z(P)$. Therefore the abelian group P' is generated by elements of order 2, hence it is an elementary abelian group as well. In consequence, the exponent of P is at most 4.

Take any $a \in P$ and define $a_i = a^{x^i}$ for $0 \leq i < q$. Let $b_i = a_i a_{i+1}^{-1}$ with subscripts mod q . Notice that $b_{i+1}^2 = (b_i^2)^x = b_i^2$ as $b_i^2 \in P'$ and $[P', K] = 1$.

We clearly have $b_0 b_1 \cdots b_{q-1} = 1$. Consider $c = b_0 \cdots b_{q-3}$. Then $c^{b_{q-2}} = b_{q-2}^{-1} b_0 \cdots b_{q-3} b_{q-2} = b_{q-2}^{-1} (b_0 \cdots b_{q-3} b_{q-2} b_{q-1}) b_{q-1}^{-1} = b_{q-2}^{-1} b_{q-1}^{-1}$.

However, from $b_{q-2}^{-2} = b_{q-2}^2 = b_{q-1}^2$ it follows that $b_{q-2}^{-1} b_{q-1}^{-1} = b_{q-2} b_{q-1}$, so

$$c^{b_{q-2}} = b_{q-2}^{-1} b_{q-1}^{-1} = b_{q-2} b_{q-1} = c^{-1}.$$

It follows that either $c^2 = 1$ or c is a real element of order 4. In either case, by assumption, $c \in C_P(K)$. Hence

$$a_0 a_{q-2}^{-1} = b_0 \cdots b_{q-3} = c \in C_P(K).$$

It follows that $[x^{q-2}, a^{-1}] = a^{x^{q-2}} a^{-1} = c^{-1} \in C_P(K)$. Since x^{q-2} generates K and $a \in P$ was arbitrary, we get $[K, P] \leq C_P(K)$ and hence $[K, P] = [P, K] = [P, K, K] = 1$ —a contradiction. $\square$

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