Porosity and the L^p -conjecture

SZYMON GŁĄB AND FILIP STROBIN

Abstract. Recently, Abtahi, Nasr-Isfahani, and Rejali [1] have shown that if G is a locally compact but not compact topological group and $p > 2$, then there are two functions, f and g , such that the convolution $f \star g$ is equal to ∞ on some set of positive measure. In the paper we show the nonexistence of $f \star g$ on a set of positive measure for (f, g) from a complement of a σ -porous set.

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1. Introduction. Assume that G is a locally compact group and let μ be a left-invariant Haar measure on G (cf. [3] for details). If $p \geq 1$, then by $L^p(G)$ (in short L^p), we denote the Banach space of all measurable functions with

$$\|f\|_p = \left(\int_G |f|^p d\mu \right)^{\frac{1}{p}} < \infty.$$

If f, g are two measurable functions, $x \in G$, and an appropriate function is integrable, then we define the convolution of f and g in the point x by:

$$f \star g(x) = \int_G f(y)g(y^{-1}x) d\mu(y).$$

Assume that $p > 1$. The famous L^p -conjecture, stated by Żelazko and Rajagopalan in the 1960's, asserts that if for all $f, g \in L^p$, $f \star g \in L^p$ (that is, $f \star g$ is defined almost everywhere on G and belongs to L^p), then G is compact. During the next 30 years this conjecture was established in special

cases, and, finally, in 1990 Saeki [4] proved the L^p -conjecture in its general form. In [1] the conjecture was strengthened for $p > 2$. Namely, it was shown that if G is not compact, then for every compact, symmetric neighbourhood of the neutral element K , there exist functions $f, g \in L^p$ such that for any $x \in K$, $f \star g(x) = \infty$. In our paper we show that if $p, q > 1$ are such that $\frac{1}{p} + \frac{1}{q} < 1$, then for every compact set K , the set of those pairs of functions $(f, g) \in L^p \times L^q$ for which $f \star g$ is well defined for some $x \in K$ (i.e., $f \star g(x)$ is finite or equal to ∞ or $-\infty$), forms a σ -porous subset of $L^p \times L^q$.

Before we state our result, we will give some notation. Let X be a metric space. $B(x, R)$ stands for the open ball with a radius R centered at a point x . Let $c \in (0, 1]$. We say that $M \subset X$ is c -lower porous [6], if

$$\forall x \in M \forall \beta \in (0, c/2) \exists R_0 > 0 \forall R \in (0, R_0) \exists z \in X B(z, \beta R) \subset B(x, R) \setminus M.$$

Note that the above definition is equivalent to the one presented in [6].

If M is a countable union of c -lower porous sets (with the same constant $c > 0$), then we say that M is σ - c -lower porous. If M is a countable union of sets such that each of them is c -lower porous for some $c > 0$, then we say that M is σ -lower porous (cf. [6, Proposition 2.2]). It can be easily seen that the c -lower porosity implies the nowhere density and hence the σ -lower porosity implies the meagerness. Thus if X is a complete space, then σ -lower porous sets are small subsets of X . Moreover, it can be proved that the notion of the σ -lower porosity is more restrictive than the meagerness, and that the notion of the σ - c -lower porosity is more restrictive than the σ -lower porosity.

For more information on porosity, we refer the reader to survey papers [5] and [6] on porosity on the real line, metric spaces and normed linear spaces.

If $p, q \geq 1$, then we consider the product $L^p \times L^q$ as a Banach space with a norm defined as the maximum of norms of coordinates, that is

$$\|(f, g)\| = \max\{\|f\|_p, \|g\|_q\}.$$

Our main result is the following:

Theorem 1. Assume that G is locally compact but not compact topological group and μ is a left-invariant Haar measure on G . If $p, q > 1$ are such that $\frac{1}{p} + \frac{1}{q} < 1$, then we have:

- (i) for every compact subset $K \subset G$, the set

$$E_K = \{(f, g) \in L^p \times L^q : \exists x \in K \text{ } f \star g(x) \text{ is well defined}\}$$

is σ - c -lower porous for some $c > 0$;

- (ii) if G is σ -compact, then the set

$$E = \{(f, g) \in L^p \times L^q : \exists x \in G \text{ } f \star g(x) \text{ is well defined}\}$$

is σ -lower porous.

If $p > 2$, then $\frac{1}{p} + \frac{1}{p} < 1$. Since $L^p \times L^p$ is complete, by Theorem 1, there are $f, g \in L^p$ such that $f \star g$ does not exist.

2. Proof of Theorem 1. By Δ we denote the modular function of G , that is the function $\Delta : G \rightarrow \mathbb{R}$ such that for every $x \in G$ and every measurable $A \subset G$,

$$\mu(Ax) = \Delta(x)\mu(A).$$

A modular function is a continuous homomorphism from G to $\mathbb{R}_+$. Moreover, for every measurable $A \subset G$,

$$\mu(A^{-1}) = \int_A \Delta(x^{-1})d\mu.$$

A group G is called unimodular if its modular function is identically equal to 1. Note that Abelian groups and discrete groups are unimodular (those basic properties of modular functions can be found in [3]).

In the proof of Theorem 1 we mix some ideas and techniques from [1] and from our paper [2].

Note that the sets investigated in this paper will be porous in a bit stronger sense. However, we do not want to define any new notion of porosity, so in the formulations of theorems we deal only with c -lower porosity.

Proof. Part (ii) follows immediately from Part (i). To prove (i), fix K , any compact subset of G . By taking, if needed, the set $K \cup K^{-1} \cup K'$, where K' is any compact symmetric neighbourhood of the neutral element of G , we can assume that K is a symmetric compact neighbourhood of the neutral element (in particular, $\mu(K) > 0$). For $u > 0$, put (Σ) denotes the family of measurable subsets of G :

$$E_u^+ = \left\{ (f, g) \in L^p \times L^q : \exists x \in K \forall A \in \Sigma \int_A f(y)g(y^{-1}x)dy < u \right\}$$

and

$$E_u^- = \left\{ (f, g) \in L^p \times L^q : \exists x \in K \forall A \in \Sigma \int_A f(y)g(y^{-1}x)dy > -u \right\}.$$

It can be easily seen that $E_K \subset \bigcup_{u \in \mathbb{N}} E_u^+ \cup \bigcup_{u \in \mathbb{N}} E_u^-$, hence it is sufficient to show that each E_u^+ and E_u^- is c -lower porous. Since, moreover, for every $(f, g) \in L^p \times L^q$ and every $u > 0$, we have that $(f, g) \in E_u^+$ iff $(f, -g) \in E_u^-$ (in particular, it means that E_u^+ and E_u^- are isometrically isomorphic), we only have to show that for every $u > 0$, the set E_u^+ is c -lower porous for some $c > 0$ (which does not depend on u).

Let $S = \sup_{x \in K} \Delta(x)$ and $c \in (0, 1)$ be such that

$$\left(\frac{c}{1-c} \right)^p + \left(\frac{c}{1-c} \right)^q S \frac{\mu(K^2)}{\mu(K)} = 1.$$

Let $u \in \mathbb{N}$, $(f, g) \in E_u^+$, $0 < \delta < c$ and $R > 0$. Then, clearly

$$\left(\frac{\delta}{1-\delta} \right)^p + \left(\frac{\delta}{1-\delta} \right)^q S \frac{\mu(K^2)}{\mu(K)} < 1,$$

and hence there exist $0 < \eta < 1 - \delta$ and $C \in (0, 1)$ such that also

$$P = 1 - \left(\frac{\delta}{\eta(1-C)} \right)^p - \left(\frac{\delta}{\eta(1-C)} \right)^q S \frac{\mu(K^2)}{\mu(K)} > 0.$$

Now, by induction, we will define a sequence $(a_n) \subset G$ such that for any $m, n \in \mathbb{N}$, $m \neq n$,

$$a_m K^2 \cap a_n K^2 = \emptyset, \quad K a_m^{-1} \cap K a_n^{-1} = \emptyset \quad \text{and} \quad \Delta(a_n) \leq 1. \quad (1)$$

Take $a_1 \in G$ with $\Delta(a_1) \leq 1$. Assume that we have already defined $a_1, \dots, a_n$. Then the set $\bigcup_{k=1}^n (a_k K^4 \cup K^4 a_k^{-1})$ is symmetric and compact, so we can take $a_{n+1} \in G \setminus \bigcup_{k=1}^n (a_k K^4 \cup K^4 a_k^{-1})$ such that $\Delta(a_{n+1}) \leq 1$. It is trivial to verify that (a_n) have the desired properties.

Since $f \in L^p$ and $g \in L^q$, we have that

$$\int_{\bigcup_{n \in \mathbb{N}} K a_n^{-1}} |f|^p < \infty \quad \text{and} \quad \int_{\bigcup_{n \in \mathbb{N}} a_n K^2} |g|^q < \infty.$$

Hence there exists $n_0 \in \mathbb{N}$ such that

$$\left(\int_{\bigcup_{n=n_0}^{\infty} K a_n^{-1}} |f|^p \right)^{\frac{1}{p}} < (1 - \delta - \eta)R \quad (2)$$

and

$$\left(\int_{\bigcup_{n=n_0}^{\infty} a_n K^2} |g|^q \right)^{\frac{1}{q}} < (1 - \delta - \eta)R. \quad (3)$$

Now, let $n_1 > n_0$ be such that (recall the definition of the number P)

$$(\mu(K)(n_1 - n_0 + 1))^{1 - \frac{1}{p} - \frac{1}{q}} > u \left[C^2 \eta^2 R^2 S^{\frac{1}{p}-1} \left(\frac{\mu(K)}{\mu(K^2)} \right)^{\frac{1}{q}} P \right]^{-1}. \quad (4)$$

Set $A = \bigcup_{n=n_0}^{n_1} K a_n^{-1}$ and $B = \bigcup_{n=n_0}^{n_1} a_n K^2$. By (1), we have

$$\mu(A^{-1}) = \mu \left(\bigcup_{n=n_0}^{n_1} a_n K \right) = \sum_{n=n_0}^{n_1} \mu(a_n K) = (n_1 - n_0 + 1) \mu(K). \quad (5)$$

Since by (1), $\mu(B) = (n_1 - n_0 + 1) \mu(K^2)$, we have that

$$\frac{\mu(B)}{\mu(A^{-1})} = \frac{\mu(K^2)}{\mu(K)}. \quad (6)$$

Next, let M_1, M_2 be such that

$$M_1 (\mu(A^{-1}))^{\frac{1}{p}} = \eta R \quad \text{and} \quad M_2 (\mu(B))^{\frac{1}{q}} = \eta R. \quad (7)$$

Let $\tilde{f}$ and $\tilde{g}$ be defined in the following way:

$$\tilde{f}(x) = \begin{cases} M_1 \Delta(x^{-1})^{\frac{1}{p}}, & x \in A; \\ f(x), & x \notin A, \end{cases} \quad \text{and} \quad \tilde{g}(x) = \begin{cases} M_2, & x \in B; \\ g(x), & x \notin B. \end{cases}$$

By (2) and (7), we have

$$\begin{aligned} \|\tilde{f} - f\|_p &= \left(\int_A |M_1 \Delta(x^{-1})^{\frac{1}{p}} - f|^p d\mu \right)^{\frac{1}{p}} \\ &\leq \left(\int_A \Delta(x^{-1}) M_1^p d\mu \right)^{\frac{1}{p}} + \left(\int_A |f|^p d\mu \right)^{\frac{1}{p}} \\ &\leq M_1 \left(\int_A \Delta(x^{-1}) d\mu \right)^{\frac{1}{p}} + (1 - \delta - \eta) R \\ &\leq \eta R + (1 - \delta - \eta) R \\ &= R - \delta R. \end{aligned}$$

Similarly by (3) and (7), $\|\tilde{g} - g\|_q \leq R - \delta R$. Hence $B((\tilde{f}, \tilde{g}), \delta R) \subset B((f, g), R)$. Thus we only have to prove that $B((\tilde{f}, \tilde{g}), \delta R) \cap E_u^+ = \emptyset$.

Fix any $(h, s) \in B((\tilde{f}, \tilde{g}), \delta R)$ and let $A_1 = \{x \in A : h(x) \geq CM_1 \Delta(x^{-1})^{\frac{1}{p}}\}$, $A_2 = A \setminus A_1$, $B_1 = \{x \in B : s(x) \geq CM_2\}$, $B_2 = B \setminus B_1$. Then we have

$$\begin{aligned} \delta R \geq \|h - \tilde{f}\|_p &\geq \left(\int_{A_2} |h - M_1 \Delta(x^{-1})^{\frac{1}{p}}|^p d\mu \right)^{\frac{1}{p}} \\ &\geq \left(\int_{A_2} (M_1 \Delta(x^{-1})^{\frac{1}{p}} (1 - C))^p d\mu \right)^{\frac{1}{p}} \\ &= M_1 (1 - C) \left(\int_{A_2} \Delta(x^{-1}) d\mu \right)^{\frac{1}{p}} = M_1 (1 - C) \mu(A_2^{-1})^{\frac{1}{p}}. \end{aligned}$$

Hence by (7),

$$\mu(A_2^{-1}) \leq \left(\frac{\delta R}{M_1(1 - C)} \right)^p = \mu(A^{-1}) \left(\frac{\delta}{\eta(1 - C)} \right)^p. \quad (8)$$

In the same way, by (6) and (7), we obtain

$$\mu(B_2) \leq \left(\frac{\delta R}{M_2(1 - C)} \right)^q = \mu(A^{-1}) \frac{\mu(K^2)}{\mu(K)} \left(\frac{\delta}{\eta(1 - C)} \right)^q. \quad (9)$$

Now let $x_0 \in K$ and define the set $F = (A_1^{-1}x_0) \cap B_1$ and $H = x_0 F^{-1}$. Since for every $n \in \mathbb{N}$, $(Ka_n^{-1})^{-1}x_0 \subset a_n K^2$, we have that $A^{-1}x_0 \subset B$, and thus

$A_1^{-1}x_0 \subset B$. Hence by (8) and (9),

$$\begin{aligned}\mu(H^{-1}) &= \mu(Fx_0^{-1}) = \mu(A_1^{-1}) - \mu(A_1^{-1} \setminus (B_1x_0^{-1})) \\ &\geq \mu(A_1^{-1}) - \mu((B \setminus B_1)x_0^{-1}) = \mu(A_1^{-1}) - \mu(B_2x_0^{-1}) \\ &= \mu(A^{-1}) - \mu(A_2^{-1}) - \Delta(x_0^{-1})\mu(B_2) \stackrel{(8)(9)}{\geq} \mu(A^{-1})P\end{aligned}$$

Also, $H \subset A_1$, $F \subset B_1$ and $H^{-1}x_0 = F$. Moreover, if $y \in A$, then $y = xa_n^{-1}$ for some $n \in N$ and $x \in K$. Hence by (1), $\Delta(y^{-1}) = \Delta(a_n)\Delta(x^{-1}) \leq S$.

Finally, by (4), (5), (6), and (7), we have

$$\begin{aligned}\int_H h(y)s(y^{-1}x_0)d\mu(y) &\geq C^2M_1M_2 \int_H \Delta(y^{-1})^{\frac{1}{p}}d\mu(y) \\ &= C^2M_1M_2 \int_H \Delta(y^{-1})^{\frac{1}{p}-1}\Delta(y^{-1})d\mu(y) \\ &\geq C^2M_1M_2 \int_H S^{\frac{1}{p}-1}\Delta(y^{-1})d\mu(y) \\ &= C^2M_1M_2S^{\frac{1}{p}-1} \int_H \Delta(y^{-1})d\mu(y) \\ &= C^2M_1M_2S^{\frac{1}{p}-1}\mu(H^{-1}) \geq C^2M_1M_2S^{\frac{1}{p}-1}\mu(A^{-1})P \stackrel{(6)(7)}{=} \\ &= C^2\eta^2R^2 \left(\frac{\mu(K)}{\mu(K^2)} \right)^{\frac{1}{q}} S^{\frac{1}{p}-1}\mu(A^{-1})^{(1-\frac{1}{p}-\frac{1}{q})}P \stackrel{(4)(5)}{>} u.\end{aligned}$$

Thus $(h, s) \notin E_u^+$ and hence the result follows. $\square$

3. Corollaries and other results. If $p, q \geq 1$, then by $E_{p,q}$ we denote the set:

$$E_{p,q} = \{(f, g) \in L^p \times L^q : f \star g \text{ is defined a.e. on } G\}.$$

Theorem 1, together with a well known fact that if G is compact and $p, q \geq 1$, then for every $(f, g) \in L^p \times L^q$, $f \star g \in L^1$, give us the following

Corollary 2. *The following conditions are equivalent:*

- (i) *for any $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} < 1$, the set $E_{p,q}$ is σ -c-lower porous for some $c > 0$;*
- (ii) *for any $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} < 1$, the set $E_{p,q}$ is not equal to $L^p \times L^q$;*
- (iii) *there are $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} < 1$ and the set $E_{p,q}$ is σ -c-lower porous for some $c > 0$;*
- (iv) *there are $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} < 1$ and the set $E_{p,q}$ is not equal to $L^p \times L^q$;*
- (v) *G is not compact.*

Finally, we will generalize the second result from [1] (cf. Remark from [1] and note that any discrete topological group is unimodular). This result is another strengthening of the L^p -conjecture for $p > 2$ and unimodular groups.

Theorem 3. *Assume that G is a noncompact unimodular group and $p > 2$.*

(i) For any compact set $K \subset G$, the set

$$E_K = \{f \in L^p : \exists x \in K \text{ } f \star f(x) \text{ is well defined}\}$$

is σ - c -lower porous subset of L^p for some $c > 0$.

(ii) If G is σ -compact, then the set

$$E = \{f \in L^p : \exists x \in G \text{ } f \star f(x) \text{ is well defined}\}$$

is σ -lower porous.

Proof. The proof is very similar to the proof of Theorem 1, so we will skip some computations. Again, we can assume that K is a symmetric compact neighborhood of the identity element, and we only have to show that for any $u > 0$, the set

$$E_u^+ = \left\{ f \in L^p : \exists x \in K \forall A \subset G \int_A f(y)f(y^{-1}x)d\mu(y) < u \right\}$$

is c -lower porous for some $c > 0$ which does not depend on u . Let $c \in (0, 1)$ be such that

$$\left(\frac{c}{1-c}\right)^p \left(1 + \left(\frac{\mu(K^2)}{\mu(K)}\right)^{\frac{1}{p}}\right)^p + \left(\frac{c}{1-c}\right)^p \left(\frac{1 + \left(\frac{\mu(K^2)}{\mu(K)}\right)^{\frac{1}{p}}}{\left(\frac{\mu(K^2)}{\mu(K)}\right)^{\frac{1}{p}}}\right)^p \frac{\mu(K^2)}{\mu(K)} = 1,$$

and let $u > 0$, $f \in L^p$, $R > 0$ and $\delta < c$. Fix $\eta < 1 - \delta$ and $C \in (0, 1)$ so that

$$\begin{aligned} P &= 1 - \left(\frac{\delta}{\eta(1-C)}\right)^p \left(1 + \left(\frac{\mu(K^2)}{\mu(K)}\right)^{\frac{1}{p}}\right)^p \\ &\quad - \left(\frac{\delta}{\eta(1-C)}\right)^p \left(\frac{1 + \left(\frac{\mu(K^2)}{\mu(K)}\right)^{\frac{1}{p}}}{\left(\frac{\mu(K^2)}{\mu(K)}\right)^{\frac{1}{p}}}\right)^p \frac{\mu(K^2)}{\mu(K)} > 0. \end{aligned}$$

Now take a sequence $(a_n) \subset G$ so that $a_m K^2 \cap a_n K^2 = \emptyset = K a_m^{-1} \cap K a_n^{-1}$ for every $m \neq n$. Let $n_1 > n_0$ be such that

$$\left(\int_{\bigcup_{n=n_0}^{n_1} K a_n^{-1}} |f|^p\right)^{\frac{1}{p}} < \frac{1-\delta-\eta}{2} R,$$

$$\left(\int_{\bigcup_{n=n_0}^{n_1} a_n K^2} |f|^p\right)^{\frac{1}{p}} < \frac{1-\delta-\eta}{2} R,$$

and

$$(\mu(K)(n_1 - n_0 + 1))^{1 - \frac{2}{p}} > u \left(C^2 \eta^2 R^2 \left(\frac{1}{1 + \left(\frac{\mu(K^2)}{\mu(K)} \right)^{\frac{1}{p}}} \right)^2 P \right)^{-1}.$$

Now set $A = \bigcup_{n=n_0}^{n_1} K a_n^{-1}$ and $B = \bigcup_{n=n_0}^{n_1} a_n K^2$. Then $\mu(A) = (n_1 - n_0 + 1) \mu(K)$ and $\mu(B) = \frac{\mu(K^2)}{\mu(K)} \mu(A)$. Next, let $M > 0$ be such that

$$M \mu(A)^{\frac{1}{p}} = \frac{\eta R}{1 + \left(\frac{\mu(K^2)}{\mu(K)} \right)^{\frac{1}{p}}}.$$

Then

$$M \mu(B)^{\frac{1}{p}} = \eta R \frac{\left(\frac{\mu(K^2)}{\mu(K)} \right)^{\frac{1}{p}}}{1 + \left(\frac{\mu(K^2)}{\mu(K)} \right)^{\frac{1}{p}}},$$

and hence $M \mu(A)^{\frac{1}{p}} + M \mu(B)^{\frac{1}{p}} = \eta R$. Now let $\tilde{f}$ be defined in the following way:

$$\tilde{f}(x) = \begin{cases} M, & x \in A \cup B; \\ f(x), & x \notin A \cup B. \end{cases}$$

We have

$$\begin{aligned} \|f - \tilde{f}\|_p &= \left(\int_{A \cup B} |f - M|^p d\mu \right)^{\frac{1}{p}} \leq \left(\int_A |f - M|^p d\mu \right)^{\frac{1}{p}} \\ &\quad + \left(\int_{B \setminus A} |f - M|^p d\mu \right)^{\frac{1}{p}} \leq \left(\int_A |f - M|^p d\mu \right)^{\frac{1}{p}} \\ &\quad + \left(\int_B |f - M|^p d\mu \right)^{\frac{1}{p}} \leq M \mu(A)^{\frac{1}{p}} \\ &\quad + \left(\int_A |f|^p \right)^{\frac{1}{p}} + M \mu(B)^{\frac{1}{p}} + \left(\int_B |f|^p \right)^{\frac{1}{p}} \\ &\leq \eta R + 2 \frac{1 - \delta - \eta}{2} R = (1 - \delta) R, \end{aligned}$$

and hence $B(\tilde{f}, \delta R) \subset B(f, R)$. Now let $g \in B(\tilde{f}, \delta R)$. We only have to show that $g \notin E_u^+$.

Set $A_1 = \{x \in A : g(x) \geq CM\}$, $A_2 = A \setminus A_1$, $B_1 = \{x \in B : g(x) \geq CM\}$, and $B_2 = B \setminus B_1$. Then

$$\delta R \geq \left(\int_{A_2} |\tilde{f} - g|^p d\mu \right)^{\frac{1}{p}} \geq (1 - C)M\mu(A_2)^{\frac{1}{p}},$$

hence

$$\mu(A_2) \leq \left(\frac{\delta R}{(1 - C)M} \right)^p \leq \mu(A) \left(\frac{\delta}{\eta(1 - C)} \right)^p \left(1 + \left(\frac{\mu(K^2)}{\mu(K)} \right)^{\frac{1}{p}} \right)^p.$$

In the same way we obtain

$$\mu(B_2) \leq \mu(A) \left(\frac{\delta}{\eta(1 - C)} \right)^p \left(\frac{1 + \left(\frac{\mu(K^2)}{\mu(K)} \right)^{\frac{1}{p}}}{\left(\frac{\mu(K^2)}{\mu(K)} \right)^{\frac{1}{p}}} \right)^p \frac{\mu(K^2)}{\mu(K)}.$$

Now fix any $x_0 \in K$ and define $F = (A_1^{-1}x_0) \cap B_1$ and $H = x_0F^{-1}$. Then $H \subset A_1$ and $A_1^{-1} \subset Bx_0^{-1}$, so

$$\begin{aligned} \mu(H) &= \mu(F) = \mu((A_1^{-1}x_0) \cap B_1) = \mu(A_1^{-1} \cap (B_1x_0^{-1})) = \mu(A_1^{-1}) \\ &\quad - \mu(A_1^{-1} \setminus (B_1x_0^{-1})) \geq \mu(A_1^{-1}) - \mu(B_2x_0^{-1}) \\ &= \mu(A) - \mu(A_2) - \mu(B_2) \geq \mu(A)P. \end{aligned}$$

Hence

$$\begin{aligned} \int_H g(y)g(y^{-1}x_0)d\mu(y) &\geq C^2M^2\mu(H) \geq C^2M^2\mu(A)^{\frac{1}{p}}\mu(A)^{\frac{1}{p}}\mu(A)^{(1-\frac{2}{p})}P \\ &= C^2\eta^2R^2 \left(1 + \left(\frac{\mu(K^2)}{\mu(K)} \right)^{\frac{1}{p}} \right)^{-2} \mu(A)^{(1-\frac{2}{p})}P > u \end{aligned}$$

so $g \notin E_u^+$ and the result follows. $\square$

4. Final remarks and open questions. In [1] it was observed that:

- if G is unimodular and $f, g \in L^2(G)$, then $f \star g$ exists;
- if G is discrete, $p \in (1, 2]$ and $f, g \in L^p(G)$, then $f \star g$ exists.

Hence the following questions arise:

- Assume that $f, g \in L^2(G)$ and G is not unimodular. Does $f \star g$ always exist?
- Assume that $f, g \in L^p(G)$, G is not discrete and $p \in (1, 2]$. Does $f \star g$ always exist?

If the answer is negative for one of these questions, one can ask if it is typical in $L^p \times L^p$ (for suitable p).

Since the L^p -conjecture states that for any $p > 1$, there are $f, g \in L^p$ such that $f \star g \notin L^p$, one can ask if the set $\{(f, g) \in L^p \times L^p : f \star g \in L^p\}$ is small (meager or σ -porous) in $L^p \times L^p$. Clearly, by Theorem 1, the set $\{(f, g) \in L^p \times L^p : f \star g \in L^p\}$ is σ - c -lower porous in $L^p \times L^p$ for $p > 2$. Hence this question is open for $p \in (1, 2]$.

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SZYMON GŁĄB
Institute of Mathematics,
Technical University of Łódź,
Wólczańska 215,
93-005 Łódź,
Poland
e-mail: szymon_glab@yahoo.com

FILIP STROBIN
Institute of Mathematics,
Technical University of Łódź,
Wólczańska 215,
93-005 Łódź,
Poland

and

Institute of Mathematics,
Polish Academy of Sciences,
Śniadeckich 8,
00-956 Warszawa,
Poland
e-mail: F.Strobin@impan.gov.pl

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