Normal forms for real quadratic forms

BERNHARD KRÖTZ AND HENRIK SCHLICHTKRULL

Abstract. We investigate the non-diagonal normal forms of a quadratic form on $\mathbb{R}^n$, in particular for $n = 3$. For this case it is shown that the set of normal forms is the closure of a 5-dimensional submanifold in the 6-dimensional Grassmannian of 2-dimensional subspaces of $\mathbb{R}^5$.

Mathematics Subject Classification (2000). 11E10, 17B20.

Keywords. Real quadratic form, diagonalization, semisimple Lie algebra.

1. Introduction. According to the principal axes theorem every real quadratic form in n variables allows an orthogonal diagonalization with normal form

$$A_1x_1^2 + \cdots + A_nx_n^2,$$

where $A_1, \dots, A_n \in \mathbb{R}$. In this article we investigate (for the case $n = 3$) the existence of other normal forms.

To be more precise, let $q_1, \dots, q_n$ be quadratic forms on $\mathbb{R}^n$. If for every quadratic form q on n -dimensional Euclidean space there exists an orthonormal basis in which q takes the form

$$q(x) = A_1q_1(x) + \cdots + A_nq_n(x)$$

for some set of coefficients, we say that this expression is a *normal form* of q .

Passing to matrices, let us consider $V = \text{Sym}(n, \mathbb{R})$, the vector space of symmetric $n \times n$ -matrices. On V there is the natural action of the special orthogonal group $K := SO(n, \mathbb{R})$ by conjugation, say

$$k \cdot X := kXk^{-1} \quad (k \in K, X \in V).$$

If D denotes the space of diagonal matrices in V , then the principal axes theorem asserts that

$$V = K \cdot D := \{k \cdot d \mid k \in K, d \in D\}.$$

Furthermore if $d_1, d_2 \in D$, then $K \cdot d_2 = K \cdot d_1$ if and only if d_2 is obtained from d_1 by a permutation of coordinates (the set of eigenvalues is unique).

The question we address is, *for which n -dimensional subspaces W in V is $V = K \cdot W$?* It would be tempting to assert that the unique property of D (and its conjugates by K), which causes the principal axes theorem, is that it is *abelian* with respect to matrix multiplication. However this is not correct, in fact there exist non-abelian n -dimensional subspaces W with $V = K \cdot W$ (see the example below).

There is some redundancy in the problem, namely the center of V on which K acts trivially. Let us remove that and define $\mathfrak{p} := V_{\text{tr}=0}$ to be the space of zero-trace elements in V . Likewise we set $\mathfrak{a} := D_{\text{tr}=0}$. The principal axes theorem now reads as

$$\mathfrak{p} = K \cdot \mathfrak{a},$$

and the problem we pose (and solve for $n = 3$) is to determine the set of subspaces

$$\mathcal{X} := \{W \subset \mathfrak{p} \mid \dim W = 2, K \cdot W = \mathfrak{p}\}.$$

The following example shows that there are elements in $\mathcal{X}$ which are not conjugate to $\mathfrak{a}$.

Example 1.1. Let $\mathfrak{p} = \text{Sym}(3, \mathbb{R})_{\text{tr}=0}$ and $K = \text{SO}(3, \mathbb{R})$. Define

$$W := \left\{ X_{\mu\lambda} := \begin{pmatrix} \mu & 0 & \lambda \\ 0 & -\mu & 0 \\ \lambda & 0 & 0 \end{pmatrix} \mid \mu, \lambda \in \mathbb{R} \right\}.$$

Then $K \cdot W = \mathfrak{p}$, and W is not abelian.

Proof. A more general result will be established later. Here we can give a simple proof.

Let $A \in \mathfrak{p}$ be given, and let $\nu_1 \geq \nu_2 \geq \nu_3$ be its eigenvalues. Then $\nu_1 + \nu_2 + \nu_3 = 0$, and hence $\nu_1 \geq 0 \geq \nu_3$. Let

$$\mu = -\nu_2 = \nu_1 + \nu_3, \quad \lambda = \sqrt{-\nu_1 \nu_3}.$$

The matrix $X_{\mu\lambda}$ has the characteristic polynomial

$$\begin{aligned} \det \begin{pmatrix} \mu - x & 0 & \lambda \\ 0 & -\mu - x & 0 \\ \lambda & 0 & -x \end{pmatrix} &= (-\mu - x)(-(\mu - x)x - \lambda^2) \\ &= (\nu_2 - x)(x - \nu_1)(x - \nu_3) \end{aligned}$$

Hence A and $X_{\mu\lambda}$ have the same eigenvalues, and thus they are conjugate. It is easily seen that $X_{10}X_{01} \neq X_{01}X_{10}$. $\square$

Corollary 1.2. *Every trace free real quadratic form in three variables allows a normal form of the type*

$$A(x^2 - y^2) + Bxz$$

for $A, B \in \mathbb{R}$.

The problem can be stated in a more general setting. Let us consider a real semi-simple Lie algebra $\mathfrak{g}$ with Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. The space $\mathfrak{p} = \text{Sym}(n, \mathbb{R})_{\text{tr}=0}$ is obtained in the special case $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace and $K = e^{\text{ad } \mathfrak{k}}$. According to standard structure theory of semi-simple Lie algebras (see for example [1, Theorem 6.51]) the following generalization of the principal axes theorem holds:

- $\mathfrak{p} = K \cdot \mathfrak{a}$.
- $K \cdot X = K \cdot Y$ for $X, Y \in \mathfrak{a}$ if and only if $\mathcal{W} \cdot X = \mathcal{W} \cdot Y$ where $\mathcal{W} = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ is the Weyl group.

Let $r := \dim \mathfrak{a}$ be the real rank of $\mathfrak{g}$. We consider $\text{Gr}_r(\mathfrak{p})$ the Grassmannian of r -dimensional subspaces in $\mathfrak{p}$. Inside of $\text{Gr}_r(\mathfrak{p})$ we consider the subset

$$\mathcal{X} := \{W \in \text{Gr}_r(\mathfrak{p}) \mid K \cdot W = \mathfrak{p}\}.$$

Let

$$\mathcal{X}_{\text{ab}} := \{W \in \text{Gr}_r(\mathfrak{p}) \mid W \text{ abelian}\}.$$

Then the following are immediate (by the same reference as before):

- $\mathcal{X} \supset \mathcal{X}_{\text{ab}} \simeq K/N$ where $N = N_K(\mathfrak{a})$.
- If $r = 1$ then $\mathcal{X} = \mathcal{X}_{\text{ab}} = \text{Gr}_r(\mathfrak{p})$.

If $r \geq 2$ and $\mathfrak{g}$ simple, then $\mathcal{X} \subsetneq \text{Gr}_r(\mathfrak{p})$, and we expect $\mathcal{X}_{\text{ab}} \subsetneq \mathcal{X}$, as in Example 1.1.

Here we describe the set $\mathcal{X}$ for $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{R})$, in which case $r = 2$, $\dim \text{Gr}_2(\mathfrak{p}) = 6$ and $\dim \mathcal{X}_{\text{ab}} = 2$. It turns out that $\mathcal{X}$ is dominated by a real algebraic variety of dimension 5: There exists a surjective algebraic map

$$\Phi : K \times_{N_0} \mathbb{P}(\mathbb{R}^3) \rightarrow \mathcal{X}$$

with $N_0 \simeq (\mathbb{Z}/4\mathbb{Z}) \rtimes \mathbb{Z}/2\mathbb{Z}$ and generically trivial fibers (see Theorem 3.2 in Section 3 below).

It would be of interest to obtain a similar description for the general case. In particular this would provide alternative parametrizations of the K -orbits in $\mathfrak{p}$. In an extended version of this article (see arXiv:0805.0180), we describe an approach by tools from algebraic geometry, which could possibly be of use in such an investigation.

2. Description by invariants. Let $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{R})$ and $\mathfrak{a} = \text{diag}(3, \mathbb{R})_{\text{tr}=0}$. We give the following description of $\mathcal{X}$, which will lead to the classification in the following section.

Proposition 2.1. *The two dimensional subspace $W \in \text{Gr}_2(\mathfrak{p})$ belongs to $\mathcal{X}$ if and only if it contains a non-zero matrix X with two equal eigenvalues.*

For example, with the notation in Example 1.1, the matrix $X_{\mu\lambda}$ with $\mu = -1$ and $\lambda = \sqrt{2}$ has eigenvalues $1, 1, -2$. Hence the space W in this example belongs to $\mathcal{X}$.

Proof. That this is a necessary condition is clear, since $W \in \mathcal{X}$ means that W contains at least one element from *every* K -orbit on $\mathfrak{p}$.

In order to describe the K -orbits, we recall some basic invariant theory. Let

$$\begin{aligned} u_1(X) &= \operatorname{tr} X^2 \\ u_2(X) &= \det X \end{aligned}$$

for $X \in \mathfrak{p}$. Then $u_1, u_2 \in \mathbb{C}[\mathfrak{p}]^K$, the ring of K -invariant polynomials on $\mathfrak{p}$. In fact, it is a well-known fact that

$$\mathbb{C}[\mathfrak{p}]^K = \mathbb{C}[u_1, u_2],$$

but we shall not use this here.

Lemma 2.2. *The level sets for $u = (u_1, u_2)$ are single K -orbits.*

Proof. Each K -orbit is uniquely determined by a set of eigenvalues (with multiplicities). It is easily seen that the characteristic polynomial of a trace free 3×3 -matrix X is

$$-x^3 + \frac{1}{2}u_1(X)x + u_2(X).$$

The lemma follows immediately. $\square$

It follows that W belongs to $\mathcal{X}$ if and only if it has a non-trivial intersection with each level set. Notice that $u_1(X)$ is the square of the trace norm of X , for X symmetric. In particular, $u_1(X) > 0$ for $X \neq 0$. Since u_1 and u_2 are homogeneous, it suffices to consider level sets of the form $\{u_1 = 1, u_2 = c_2\}$.

We thus consider for each $W \in \operatorname{Gr}_2(\mathfrak{p})$ the unit sphere

$$W_1 = \{X \in W \mid u_1(X) = 1\},$$

and we define

$$J := \{u_2(X) \mid X \in W_1\}.$$

Since W_1 is connected, J is an interval. Moreover, it is symmetric around 0, since u_2 has odd degree. In particular, we denote by

$$I := \{u_2(X) \mid X \in \mathfrak{a}_1\}$$

the interval corresponding to the unit sphere in $\mathfrak{a}$. We now show:

Lemma 2.3. *The interval I is given by $I = [-c, c]$, where $c = 54^{-1/2}$. Furthermore, the extreme values $\pm c$ are obtained precisely in those elements $X \in \mathfrak{a}_1$, which have two equal eigenvalues.*

Proof. Let us introduce coordinates for $\mathfrak{a}$, namely

$$\mathfrak{a} = \{D_{xy} := \operatorname{diag}(x, y, -x - y) \mid x, y \in \mathbb{R}\}.$$

Furthermore, let us introduce two functions:

$$\begin{aligned} f_1(x, y) &:= u_1(D_{xy}) = 2(x^2 + y^2 + xy) \\ f_2(x, y) &:= u_2(D_{xy}) = -xy(x + y). \end{aligned}$$

We wish to maximize/minimize f_2 under the condition of $f_1 = 1$. For that we perform the method of Lagrange: $df_1 = 2(2x + y, 2y + x)$ and $df_2 = -(y(2x + y), x(2y + x))$ have to be collinear. This can only happen in three cases: $x = y$, $2x + y = 0$ or $2y + x = 0$. Notice that these are exactly the cases in which two of the diagonal entries of D_{xy} are equal.

We start with $x = y$. Here $f_1(x, x) = 6x^2 = 1$ means that $x = \pm 6^{-1/2}$. Hence $f_2(x, x) = -2x^3 = \pm 54^{-1/2}$. Secondly, if $2x + y = 0$, then $f_1(x, -2x) = 6x^2 = 1$, so again $x = \pm 6^{-1/2}$. Hence $f_2(x, -2x) = -2x^3 = \pm 54^{-1/2}$. Finally, the case $2y + x = 0$ is similar. $\square$

In order to complete the proof of Proposition 2.1, we only have to note that, as $\mathfrak{p} = K \cdot \mathfrak{a}$ we have $J \subset I$ and equality $J = I$ holds if and only if $W \in \mathcal{X}$. $\square$

Remark 2.4. (a) It follows from Proposition 2.1 that not all 2-dimensional subspaces $W \subset \mathfrak{p}$ belong to $\mathcal{X}$. An extreme case is

$$W = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda & \mu \\ 0 & \mu & -\lambda \end{pmatrix} \mid \lambda, \mu \in \mathbb{R} \right\}.$$

for which $u_2(W) = \{0\}$ and hence $W \notin \mathcal{X}$.

(b) Let us define a continuous function on $\text{Gr}_2(\mathfrak{p})$ by

$$f : \text{Gr}_2(\mathfrak{p}) \rightarrow \mathbb{R}_{\geq 0}, \quad W \mapsto \max_{X \in W_1} u_2(X)$$

Then we get

$$\mathcal{X} = \{W \in \text{Gr}_2(\mathfrak{p}) \mid f(W) = 54^{-1/2}\}$$

by our previous result. In particular, $\mathcal{X}$ is a closed subset of $\text{Gr}_2(\mathfrak{p})$.

3. K -orbits on $\mathcal{X}$. We aim to describe $\mathcal{X}$ explicitly. Our starting point is the following observation. Let

$$X_0 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

and observe that the two dimensional subspace $W \in \text{Gr}_2(\mathfrak{p})$ belongs to $\mathcal{X}$ if and only if it contains a vector in the K -orbit of X_0 .

In fact this is an immediate consequence of our discussion in the previous section. Since we consider trace free matrices, X has two equal eigenvalues if and only if its eigenvalues are $\nu, \nu, -2\nu$ for some $\nu \in \mathbb{R}$, that is, X is conjugate to νX_0 .

Let Ω denote the 3-dimensional variety

$$\Omega = \{W \in \mathrm{Gr}_2(\mathfrak{p}) \mid X_0 \in W\},$$

in $\mathrm{Gr}_2(\mathfrak{p})$. The stabilizer $H \subset K$ of the line $\mathbb{R}X_0$ acts on Ω . We have proved the following result:

Proposition 3.1. *The map $(k, W) \mapsto k \cdot W$ from $K \times_H \Omega$ to $\mathcal{X}$ is surjective.*

Notice that $H = MT$ where T is the maximal torus

$$T := \begin{pmatrix} \mathrm{SO}(2, \mathbb{R}) & 0 \\ 0 & 1 \end{pmatrix}$$

and $M \simeq [\mathbb{Z}/2\mathbb{Z}]^2$ is the diagonal group group generated by

$$m_1 := \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad m_2 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

in K .

For $Y \notin \mathbb{R}X_0$ we set

$$W_Y := \mathrm{span}_{\mathbb{R}}\{X_0, Y\},$$

then $\Omega = \{W_Y \mid Y \notin \mathbb{R}X_0\}$. Since X_0 is fixed under the maximal torus T , we have that W_Y and $W_{t \cdot Y}$ belong to the same K -orbit for $t \in T$. Thus it suffices to consider elements Y of the following shape:

$$Y = Y_{\alpha, \delta, \epsilon} = \begin{pmatrix} \alpha & 0 & \delta \\ 0 & -\alpha & \epsilon \\ \delta & \epsilon & 0 \end{pmatrix}.$$

Let $\mathcal{Y}$ denote the 2-dimensional projective space of these lines and consider the algebraic mapping

$$(3.1) \quad K \times \mathcal{Y} \rightarrow \mathcal{X}, \quad (k, [Y]) \mapsto k \cdot W_Y.$$

The group H does not act on $\mathcal{Y}$. However, let N_0 denote the subgroup of order 8, generated by

$$s_0 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

and m_2 (note that $m_1 = s_0^2$). It follows from the relations

$$(3.2) \quad s_0 \cdot Y_{\alpha, \delta, \epsilon} = Y_{-\alpha, -\epsilon, \delta}, \quad m_2 \cdot Y_{\alpha, \delta, \epsilon} = Y_{\alpha, -\delta, \epsilon}.$$

that N_0 acts on $\mathcal{Y}$. Conversely, if $k \in H$ and $k \cdot Y \in \mathcal{Y}$ for some $Y = Y_{\alpha, \delta, \epsilon} \in \mathcal{Y}$ with $\alpha \neq 0$, then $k \in N_0$.

Since furthermore $k \cdot W_Y = W_{k \cdot Y}$ for $Y \in \mathcal{Y}$ and $k \in N_0$, the above map (3.1) factorizes to an algebraic map

$$K \times_{N_0} \mathcal{Y} \rightarrow \mathcal{X}.$$

We define the following open dense subset of $\mathcal{Y}$:

$$(3.3) \quad \mathcal{Y}' = \{[Y_{\alpha, \delta, \epsilon}] \mid \alpha \neq 0, \delta \neq 0, \epsilon \neq 0\}$$

and note that it is preserved by N_0 .

Theorem 3.2. *The algebraic map*

$$K \times_{N_0} \mathcal{Y} \rightarrow \mathcal{X}, \quad [k, [Y]] \mapsto k \cdot W_Y$$

is K -equivariant, continuous and onto. Moreover, the restriction

$$K \times_{N_0} \mathcal{Y}' \rightarrow \mathcal{X}$$

is injective. Its image $\mathcal{X}' = \{k \cdot W_Y \mid k \in K, [Y] \in \mathcal{Y}'\}$ is dense in $\mathcal{X}$ and carries a natural structure of a smooth 5-dimensional manifold.

The first statement was seen already. In order to obtain the statements about $\mathcal{X}'$, we study the intersection of the K -orbit of X_0 with W_Y . We first prove:

Lemma 3.3. *Assume*

$$X := \begin{pmatrix} \lambda & 0 & \delta \\ 0 & \mu & \epsilon \\ \delta & \epsilon & -(\lambda + \mu) \end{pmatrix} \in K \cdot X_0.$$

Then $\delta = 0$ or $\epsilon = 0$.

Proof. It follows from Lemma 2.2 that

$$(3.4) \quad u_1(X) = 2(\lambda^2 + \mu^2 + \lambda\mu + \delta^2 + \epsilon^2) = u_1(X_0) = 6$$

$$(3.5) \quad u_2(X) = -\lambda\mu(\lambda + \mu) - \epsilon^2\lambda - \delta^2\mu = u_2(X_0) = -2.$$

In particular, it follows from (3.4) that $\lambda^2 + \mu^2 + \lambda\mu \leq 3$. Since $\lambda^2 + \mu^2 + \lambda\mu = (\lambda + \frac{1}{2}\mu)^2 + \frac{3}{4}\mu^2$ this implies $|\mu| \leq 2$.

Multiplying by $\frac{1}{2}\mu$ in (3.4) and adding (3.5) we obtain

$$\mu^3 + \epsilon^2\mu - \epsilon^2\lambda = 3\mu - 2,$$

or equivalently

$$(3.6) \quad \epsilon^2(\lambda - \mu) = (\mu + 2)(\mu - 1)^2.$$

In particular it follows that $\epsilon = 0$ or $\lambda \geq \mu$.

Since μ and λ appear symmetrically in (3.4) and (3.5) we obtain similarly $|\lambda| \leq 2$,

$$(3.7) \quad \delta^2(\mu - \lambda) = (\lambda + 2)(\lambda - 1)^2,$$

and conclude that $\delta = 0$ or $\mu \geq \lambda$.

Notice finally that if $\lambda = \mu$, then $\lambda = \mu = -2$ or $\lambda = \mu = 1$ by (3.6), and from (3.4) it then follows that $\lambda = \mu = 1$ and $\delta = \epsilon = 0$. $\square$

We can now prove Theorem 3.2.

Proof. It remains to be seen that $k \cdot W_Y = W_{Y'}$ implies $k \in N_0$ and $[Y'] = [k \cdot Y]$ for $Y, Y' \in \mathcal{Y}'$. In particular, it follows from $k \cdot W_Y = W_{Y'}$ that $k \cdot X_0 = aX_0 + bY'$ for some $a, b \in \mathbb{R}$. Since $Y' \in \mathcal{Y}'$ it follows from Lemma 3.3 that $b = 0$, and hence $k \in H$. Now $k \cdot Y$ must be a multiple of Y' , by orthogonality with X_0 with respect to the trace form $\langle A, B \rangle = \text{tr}(AB)$. It follows that $k \in N_0$. $\square$

In order to give a complete classification of $\mathcal{X}$, one needs to describe the fibers in $K \times \mathcal{Y}$ above the elements outside of $\mathcal{X}'$. We omit the details, but mention that in general the fibers will not be finite.

References

- [1] A. W. KNAPP, Lie groups beyond an introduction, Birkhäuser, 2002.

BERNHARD KRÖTZ, Max-Planck-Institut für Mathematik, Vivatsgasse 7, D-53111 Bonn, Germany

e-mail: kroetz@mpim-bonn.mpg.de

HENRIK SCHLICHTKRULL, Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, DK-2100 Copenhagen, Denmark

e-mail: schlicht@math.ku.dk

Received: 27 June 2008