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A 2-element antichain that is not contained in any finite retract

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Abstract. We give an example of an ordered set P which contains a 2-element antichain that is not contained in any finite retract of P.

1. Introduction

The question in [2, p. 259, Remark 8] asks if every finite subset of an (infinite) ordered set is actually contained in a finite retract. The question was motivated by the product problem for the fixed point property, but, as a possible structural property, it is interesting in its own right.

Moreover, by [1, Theorem 2], every isometric spanning fence is a retract, which means that in a chain-complete ordered set, every 2-antichain (that is, 2-element antichain) consisting of minimal or maximal elements is contained in a finite retract. The construction can be generalized to show that in an arbitrary ordered set, every 2-antichain in which one of the two elements is maximal or minimal is contained in a finite retract: Let $\{m, a\}$ be an antichain and without loss of generality let m be minimal. Let $m = f_0 < f_1 > f_2 < \cdots f_n = a$ be a shortest possible fence from m to a. Mapping the elements whose distance to m is $j < n$ to f_j and mapping the elements whose distance to m is $\geq n$ to $f_n = a$ is a retraction.

Given the generality and simplicity of the above construction, it is all the more surprising that there is an ordered set of height 2 with a 2-antichain that is *not* contained in any finite retract.

2. The construction

Lemma 1. *Let P be an ordered set, let $\{a, b\} \subseteq P$ be a 2-antichain, and let $F = \{a = f_0 < f_1 > \cdots f_n = b\}$ be a shortest possible fence from f_0 to f_n. If there is no other fence from a to b that is of length n or if any other fence F' from a to b that is of length n and has the property that $a = f'_0 < f'_1$, then any retract of P that contains $\{a, b\}$ must contain F.*

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Proof. Let $r\colon P \to P$ be a retraction such that $r(a) = a$ and $r(b) = b$. Now,

$$a = f_0 < f_1 > \cdots f_n = b$$

implies

$$a = r(a) = r(f_0) \leq r(f_1) \geq \cdots r(f_n) = r(b) = b.$$

Because the distance from a to b is n, the $r(f_j)$ must form a fence of length n from a to b. In particular, we have

$$a = r(a) = r(f_0) < r(f_1) > \cdots r(f_n) = r(b) = b.$$

By the conditions on P, the image of F under r must be F itself. □

To construct the ordered set P for our example, let a and b be two points. Let

$$U := \{a = u_0 < u_1 > u_2 < u_3 > \cdots > u_{90} = b\}$$

and

$$L := \{a = l_0 > l_1 < l_2 > l_3 < \cdots < l_{90} = b\}$$

be two fences that are disjoint, except for the endpoints. Let

$$G := \{u_{32} = g_0 < g_1 > g_2 < \cdots > g_{10} = l_{33}\},$$
$$H := \{u_{34} = h_0 < h_1 > h_2 < \cdots > h_{10} = l_{35}\},$$
$$I := \{u_{36} = i_0 < i_1 > i_2 < \cdots > i_{10} = l_{37}\},$$
$$J := \{u_{38} = j_0 < j_1 > j_2 < \cdots > j_{10} = l_{39}\},$$
$$K := \{u_{40} = k_0 < k_1 > k_2 < \cdots > k_{10} = l_{41}\},$$

be pairwise disjoint fences that only intersect U at their starting points and that only intersect L at their endpoints. (The numbers 10 and 90 are in no way optimal. They were chosen to make it obvious that the construction works as desired.) Let $Q_0 := \{g_4, h_4, i_4, j_4, k_4\}$. For $n \geq 1$, let Q_n be the set of two element subsets of Q_{n-1}, considered as an antichain. Let

$$P := U \cup L \cup G \cup H \cup I \cup J \cup K \cup \bigcup_{n=1}^{\infty} Q_n.$$

The order on P is the union of the orders on U, L, G, H, I, J, and K together with the element relation $\leq := \in$ on $Q_{2k} \cup Q_{2k+1}$ and the reverse element relation $\leq := \ni$ on $Q_{2k+1} \cup Q_{2k+2}$. The resulting ordered set has height 2.

Now let $r\colon P \to P$ be a retraction such that $\{a, b\} \subseteq r[P]$. Then, by Lemma 1, $U \subseteq r[P]$ and $L \subseteq r[P]$. Similarly, because G, H, I, J, and K are the shortest fences between their endpoints, we must have $G \subseteq r[P]$, $H \subseteq r[P]$, $I \subseteq r[P]$, $J \subseteq r[P]$, and $K \subseteq r[P]$. Thus, in particular, r is the identity on Q_0. Once more by Lemma 1, we must have that r is the identity on Q_1, because for every $\{x, y\} \in Q_1$, the fence $x \in \{x, y\} \ni y$ is the shortest fence from x to y.

We now prove inductively that r is the identity on $\bigcup_{n=2}^{\infty} Q_n$. Let $x \in Q_m$, assume that r is the identity on $\bigcup_{j=0}^{m-1} Q_j$, and, without loss of generality, assume that m is even. Then $x = \{g, h\}$ for two distinct elements $g, h \in Q_{m-1}$

and $x < g, h$. By definition of Q_{m-1}, as the set of two-element subsets of Q_{m-2}, there is at most one $v \in Q_{m-2}$ such that $v < g$ and $v < h$. If there is no such $v \in Q_{m-2}$, then $r(x) = x$, because g and h are fixed by r and x is their only common lower cover. In case there is such a $v \in Q_{m-2}$, suppose for a contradiction that $r(x) = v$. Let s, t, u, v, and w be 5 distinct elements of Q_{m-2}. We may assume that $g = \{u, v\}$ and $h = \{v, w\}$. Let $g_* := \{u, w\} \in Q_{m-1}$, $h_* := \{s, t\} \in Q_{m-1}$, and $x_* := \{g_*, h_*\} \in Q_m$. Then, because x_* is the only common lower cover of g_* and h_*, which are both fixed by r, we have $r(x_*) = x_*$. Now, $x, x_* < \{x, x_*\} \in Q_{m+1}$, so $r(x), r(x_*) < r(\{x, x_*\})$. But there is no element greater than both $v = r(x)$ and $x_* = r(x_*)$, a contradiction. Thus, $r(x) \neq v$. Because x and v are the only common lower covers of the elements g and h, which are fixed by r, we must thus have $r(x) = x$ in this case, too. This proves that r is the identity on Q_m. Hence, r is the identity on $\bigcup_{n=0}^{\infty} Q_n$, and thus it is the identity on P. So in fact, the only retract of P that contains $\{a, b\}$ is P itself.

If an example without infinite fences is desired, the union $\bigcup_{n=1}^{\infty} Q_n$ could be replaced with a disjoint union of sets $Z_n := \bigcup_{j=1}^{n} Q_j$ attached in the same fashion to Q_0. In this set, once more $\{a, b\}$ is not contained in any finite retract, but there are retracts other than P that contain $\{a, b\}$: First of all, because the induction above also used an element of Q_{m+1}, a retraction that fixes the elements of a union, $\bigcup_{j=1}^{n-1} Q_j$ in Z_n, could still not fix some elements of Q_n. Second, a retraction could map a set Z_n to a set Z_{n+j}. But because no retraction can map a set Z_{n+j} to a set Z_n, any retraction that fixes a and b must, for infinitely many n, fix the first $n-1$ stages of the set Z_n, which means that the retract must be infinite.

3. Concluding remarks

As we have noted, our set P has height 2. By [1, Theorem 2], every antichain consisting of 2 elements in a poset of height 1 is contained in a finite retract. This is no longer true for 3-element antichains. To see this, modify our construction by replacing L with

$$U' = \{a = u'_0 < u'_1 > u'_2 < \cdots > u'_{90} < u'_{91} > u'_{92} = b\},$$

and put $g_{10} = u'_{32}, h_{10} = u'_{34}, i_{10} = u'_{36}, j_{10} = u'_{38}, k_{10} = u'_{40}$. It is easy to see that the antichain $\{a, b, u'_{45}\}$ is not contained in any finite retract.

The original motivation for the question in [2] came from fixed point theory. Neither the set P nor any of its modifications considered in this note do have the fixed point property. Therefore, the question remains open whether every finite subset of a poset with the fixed point property is contained in a finite retract of this poset. Also, an answer for posets that do not contain infinite antichains could be interesting.

References

[1] Duffus, D., Rival, I., Simonovits, M.: Spanning retracts of a Partially Ordered Set. Discr. Math. **32**, 1–7 (1980)

[2] Schröder, B.: Ordered Sets—An Introduction. Birkhäuser, Boston–Basel–Berlin (2003)

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