New characterizations of the Takagi function via functional equations

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Abstract. We provide two new characterizations of the Takagi function as the unique bounded solution of some systems of two functional equations. The results are independent of those obtained by Kairies (Wyż Szkoł Ped Krakow Rocznik Nauk Dydakt Prace Mat 196:73–82, 1998), Kairies (Aequ Math 53:207–241, 1997), Kairies (Aequ Math 58:183–191, 1999) and Kairies et al. (Rad Mat 4:361–374, 1989; Errata, Rad Mat 5:179–180, 1989).

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Introduction

The *Takagi function* $T : \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$T(x) := \sum_{n=0}^{\infty} \frac{d(2^n x)}{2^n}, \quad (\text{T})$$

where $d(y)$ is a distance from y to the closest integer. Below is the approximation of how the Takagi function looks like on the interval $[0, 1]$.

This is a continuous function that is not differentiable at any point in $\mathbb{R}$ (see [4]).

The following fact is known due to H.-H. Kairies. It concerns equations fundamental for this presentation.

Remark. ([1, Theorem 3]). If $f : [0, 1] \rightarrow \mathbb{R}$ is a solution of any two of the functional equations

$$f\left(\frac{x}{2}\right) + f\left(\frac{x+1}{2}\right) = f(x) + \frac{1}{2}, \quad (1)$$

$$f\left(\frac{x}{2}\right) = \frac{1}{2}f(x) + \frac{x}{2}, \quad (2)$$

$$f\left(\frac{x+1}{2}\right) = \frac{1}{2}f(x) - \frac{x}{2} + \frac{1}{2}, \quad (3)$$

$$f\left(\frac{x}{2}\right) = f\left(\frac{x+1}{2}\right) + x - \frac{1}{2}, \quad (4)$$

then f satisfies any other two.

The starting point for us are the following two results proved by H.-H. Kairies, W.F. Darsow and M.J. Frank (Theorem DFK) and by H.-H. Kairies (Theorem K).

Theorem DFK. ([4], [5, Theorem 8.2]). If $f : [0, 1] \rightarrow \mathbb{R}$ is a bounded solution of at least two of the functional Eqs. (1)–(4), then $f = T|_{[0,1]}$.

Theorem K. ([1, Theorem 3]). If $f : \mathbb{R} \rightarrow \mathbb{R}$ is a bounded solution of the functional equation

$$2f(x) = f(2x) + 2d(x), \quad (5)$$

then f is the Takagi function.

A trivial calculation shows that the function $T_{[0,1]}$ satisfies all of Eqs. (1)–(4). T also satisfies (5) as well as the symmetry equation

$$f(x) = f(1 - x) \quad (6)$$

for every $x \in \mathbb{R}$.

Our aim is to analyse which of the systems (1)+(6), (2)+(6), (3)+(6) and (4)+(6) characterizes the Takagi function in the class of bounded functions. As we will see this is the case for systems (2)+(6) and (3)+(6) (see Theorem 3) but not if we think about (1)+(6) and (4)+(6).

1. Characterization of the Takagi function on the interval $[0,1]$

We start with the following auxiliary result.

Proposition 1. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function with bounded $f|_{[0,1]}$. If equalities (5) and (6) hold for every $x \in [0, 1]$, then $f|_{[0,1]} = T|_{[0,1]}$.*

Proof. Assume that equalities (5) and (6) hold for every $x \in [0, 1]$. Putting $x = 0$ in (6) we get $f(1) = f(0)$. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be the 1-periodic extension of $f|_{[0,1]}$. Of course the function g is bounded. We shall verify that it satisfies Eq. (5). Fix a number x and represent it in the form $x = n + t$, where $n \in \mathbb{Z}$ and $t \in [0, 1)$. If $t \in [0, \frac{1}{2})$, then $2t \in [0, 1)$ and thus, by the periodicity of g ,

$$\begin{aligned} 2g(x) &= 2g(n+t) = 2g(t) = 2f(t) = f(2t) + 2d(t) = g(2t) + 2d(t) \\ &= g(2t+2n) + 2d(t+n) = g(2x) + 2d(x). \end{aligned}$$

On the other hand, if $t \in [\frac{1}{2}, 1)$, then $2(1-t) \in [0, 1)$ and whence, making use of the symmetry condition, we get

$$\begin{aligned} 2g(x) &= 2g(n+t) = 2f(t) = 2f(1-t) = f(2(1-t)) + 2d(1-t) \\ &= f(1-2(1-t)) + 2d(t) = f(2t-1) + 2d(t) \\ &= g(2t+2n) + 2d(t+n) = g(2x) + 2d(x). \end{aligned}$$

In both cases we obtain the equality $2g(x) = g(2x) + 2d(x)$.

By Theorem K we infer that g is the Takagi function. In particular, $f|_{[0,1]} = g|_{[0,1]} = T|_{[0,1]}$. $\square$

There exist functions that do not satisfy the conditions of Theorem K but satisfy those of Proposition 1.

Remark 2. The function $f : \mathbb{R} \rightarrow \mathbb{R}$, defined by

$$f(x) = \begin{cases} T(x), & x \in [0, 2], \\ 0, & x \notin [0, 2], \end{cases} \quad (7)$$

is bounded and, clearly, equality (6) holds for every $x \in [0, 1]$. Moreover, if $x \in [0, 1]$ then $2x \in [0, 2]$, and thus, as T satisfies Eq. (5),

$$2f(x) = 2T(x) = T(2x) + 2d(x) = f(2x) + 2d(x),$$

i.e. (5) is fulfilled for each $x \in [0, 1]$. Consequently, all assumptions of Proposition 1 are satisfied. On the other hand f is not the Takagi function, so Theorem K is not applicable.

The next theorem is the main result of the note.

Theorem 3. *If $f : [0, 1] \rightarrow \mathbb{R}$ is a bounded function satisfying (6) and at least one of Eqs. (2) and (3), then $f = T_{|[0,1]}$.*

Proof. If Eq. (2) holds, then putting $x = 0$ in (6) and (2) we see that $f(1) = f(0) = 0$. Similarly, if Eq. (3) is satisfied, then setting $x = 0$ in (6) and $x = 1$ in (3) we again come to $f(1) = f(0) = 0$.

Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be the 1-periodic extension of f . Of course the function g is bounded. Moreover, for every $x \in \mathbb{R}$, taking $n \in \mathbb{Z}$ and $t \in [0, 1]$ with $x = n + t$ and making use of the symmetry condition (6), we have

$$g(-x) = g(-n - t) = g(1 - t) = f(1 - t) = f(t) = g(t) = g(n + t) = g(x).$$

This means that the function g is even.

We shall check that

$$2g(x) = g(2x) + 2d(x) \tag{8}$$

for every $x \in [0, 1]$. First assume that f satisfies (2). If $x \in [0, \frac{1}{2}]$, then

$$2g(x) = 2f(x) = 2f\left(\frac{1}{2}2x\right) = f(2x) + 2x = g(2x) + 2d(x),$$

so we come to (8). Now take an arbitrary $x \in [\frac{1}{2}, 1]$. Then $1 - x \in [0, \frac{1}{2}]$, and thus

$$\begin{aligned} 2g(x) &= 2f(x) = 2f(1 - x) = f(2 - 2x) + 2(1 - x) = g(2 - 2x) + 2d(x) \\ &= g(-2x) + 2d(x) = g(2x) + 2d(x), \end{aligned}$$

which means that equality (8) holds again. Now assume that f satisfies (3). If $x \in [\frac{1}{2}, 1]$, then $2x - 1 \in [0, 1]$, and thus

$$\begin{aligned} 2g(x) &= 2f(x) = 2f\left(\frac{2x - 1 + 1}{2}\right) = 2\left(\frac{1}{2}f(2x - 1) - \frac{2x - 1}{2} + \frac{1}{2}\right) \\ &= g(2x - 1) - 2x + 2 = g(2x) - 2x + 2 = g(2x) + 2d(x), \end{aligned}$$

which is (8). Now take an arbitrary $x \in [0, \frac{1}{2}]$. Then $1 - x \in [\frac{1}{2}, 1]$, whence

$$\begin{aligned} 2g(x) &= 2g(-x) = 2g(1 - x) = g(2 - 2x) + 2d(1 - x) \\ &= g(-2x) + 2d(1 - x) = g(2x) + 2d(x), \end{aligned}$$

which gives (8) again.

Applying Proposition 1 we see that $g_{|[0,1]} = T_{|[0,1]}$ that is $f = T_{|[0,1]}$. $\square$

Remark 4. Of course the function $f : [0, 1] \rightarrow \mathbb{R}$, given by $f(x) = \frac{1}{2}$, is bounded and satisfies (6) and (1). Thus neither (2) nor (3) can be replaced by Eq. (1) in Theorem 3.

Even adding the condition $f(0) = f(1) = 0$ does not help, which can be seen if we define $f : [0, 1] \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & x \in \{0, 1\}, \\ \frac{1}{2}, & x \in (0, 1). \end{cases} \quad (9)$$

Clearly it is bounded. Observe also that if $x \in (0, 1)$ then $1 - x, \frac{x}{2}, \frac{x+1}{2} \in (0, 1)$, so equalities (1) and (6) hold. Moreover, if $x = 0$ then

$$f(x) = f(0) = f(1) = f(1 - x)$$

and

$$f\left(\frac{x}{2}\right) + f\left(\frac{x+1}{2}\right) = f(0) + f\left(\frac{1}{2}\right) = \frac{1}{2} = f(0) + \frac{1}{2} = f(x) + \frac{1}{2}.$$

Similarly, if $x = 1$ then

$$f\left(\frac{x}{2}\right) + f\left(\frac{x+1}{2}\right) = f\left(\frac{1}{2}\right) + f(1) = \frac{1}{2} = f(1) + \frac{1}{2} = f(x) + \frac{1}{2}.$$

Consequently, f satisfies (6) and (1).

Remark 5. The function $f : [0, 1] \rightarrow \mathbb{R}$, defined by $f(x) = 2x(1 - x)$ is bounded and clearly satisfies (6). Moreover, if $x \in [0, 1]$ then

$$\begin{aligned} f\left(\frac{x+1}{2}\right) + x - \frac{1}{2} &= 2\frac{x+1}{2}\left(1 - \frac{x+1}{2}\right) + x - \frac{1}{2} \\ &= (x+1)\left(\left(1 - \frac{x}{2}\right) - \frac{1}{2}\right) + x - \frac{1}{2} \\ &= x\left(1 - \frac{x}{2}\right) + \left(1 - \frac{x}{2}\right) - \frac{x+1}{2} + x - \frac{1}{2} \\ &= 2\frac{x}{2}\left(1 - \frac{x}{2}\right) = f\left(\frac{x}{2}\right), \end{aligned}$$

so equality (4) holds. This shows that neither (2) nor (3) can be replaced by Eq. (4) in Theorem 3.

2. Characterization of the Takagi function as a subadditive solution of some functional equations

We start this section with recalling the notion of subadditivity on an arbitrary interval. We will say that a real function f defined on an interval $I \subset \mathbb{R}$ is *subadditive* if

$$x, y, x + y \in I \Rightarrow f(x + y) \leq f(x) + f(y).$$

Remark 6. Clearly $d_{|[0,1]}$ is a concave function, and thus, by [6, Theorem 5.2], it is subadditive. So from [6, Theorem 5.4] we see that d , as a periodic extension of $d_{|[0,1]}$, is subadditive. By scaling the argument of a function we do not lose its subadditivity. Moreover, any linear combination of subadditive functions, with positive coefficients, is still subadditive. Consequently, T , as the limit of a sequence of subadditive functions, is subadditive.

Notice the following two auxiliary facts.

Lemma 7. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a subadditive function such that $f(-1) \leq 0$ and $f(1) \leq 0$. Then f is 1-periodic.*

Proof. It is enough to observe that

$$f(x) = f(x+1-1) \leq f(x+1) + f(-1) \leq f(x+1) \leq f(x) + f(1) \leq f(x)$$

for every $x \in \mathbb{R}$. □

Lemma 8. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a subadditive function such that $f(-1) \leq 0$. If $f_{|[0,1]} = T_{|[0,1]}$, then $f = T$.*

Proof. We have $f(1) = T(1) = 0$, and thus, since f is subadditive and $f(-1) \leq 0$, it follows from Lemma 7 that f is 1-periodic. Consequently, $f = T$. □

As an immediate consequence of Theorem 3 and Lemma 8 we obtain the next result.

Corollary 9. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a subadditive function with bounded $f_{|[0,1]}$ and such that $f(-1) \leq 0$ and equality (6) holds for every $x \in [0, 1]$. If f satisfies (2) or (3) for every $x \in [0, 1]$, then $f = T$.*

Corollary 10. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a subadditive and even function with bounded $f_{|[0,1]}$ and such that $f(-1) \leq 0$. If f satisfies (2) or (3) for every $x \in [0, 1]$, then $f = T$.*

Proof. Since f is even, it follows that $f(1) = f(-1) \leq 0$. By Lemma 7 we deduce that f is 1-periodic, whence $f(x) = f(-x) = f(1-x)$ for every $x \in [0, 1]$, so f satisfies (6). Now from Corollary 9 we see that $f = T$. □

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