On Sahoo–Riedel equations again

JOANNA GER

*Dedicated to Roman, my beloved husband, on the occasion of his 70th
 Birthday*

Abstract. We deal with the functional equations

$$\begin{aligned} f(x) - f(y) &= (x - y)[\varphi(x)\psi(y) + g(x) + g(y)], & x, y \in I, \\ xf(y) - yf(x) &= (x - y)[\varphi(x)\psi(y) + g(x) + g(y)], & x, y \in I, \end{aligned}$$

where $I \subset \mathbb{R}$ is an arbitrary interval with positive length, $f, g, \varphi, \psi : I \rightarrow \mathbb{R}$ are unknown functions. We are also looking for the conditions guaranteeing that the function h in the equation

$$f(x) - f(y) = (x - y)[h(\phi(x, y)) + g(x) + g(y)], \quad x, y \in I$$

be constant.

Mathematics Subject Classification. 39B22.

1. Introduction

Kuczma in the paper [2] gave general solutions of the equations

$$\begin{aligned} f(x) - f(y) &= (x - y)\varphi(x)\varphi(y), & x, y \in A, \\ xf(y) - yf(x) &= (x - y)\varphi(x)\varphi(y), & x, y \in A, \end{aligned}$$

where $A \subset F$ is a subset of a commutative field containing at least two elements and $f, \varphi : A \rightarrow F$. Sahoo and Riedel in their monograph [3] (see Lemma 3.3, p. 118) found solutions of the equation

$$f(x) - f(y) = (x - y)[\alpha xy + g(x) + g(y)], \quad x, y \in \mathbb{R}.$$

Also the present author dealt with the Sahoo–Riedel equation in [1]. There the equations were considered for functions defined in a real interval I . In the present paper we extend the above results to the case of more complicated equations and functions defined in subintervals of $\mathbb{R}$.

2. Preliminaries

In the sequel $I \subset \mathbb{R}$ denotes an arbitrary interval with positive length.

Assume that functions $f, g : I \rightarrow \mathbb{R}$, $\phi : I^2 \rightarrow J = \phi(I^2) \subset \mathbb{R}$, $h : J \rightarrow \mathbb{R}$ satisfy the following equation

$$f(x) - f(y) = (x - y)[h(\phi(x, y)) + g(x) + g(y)], \quad x, y \in I. \quad (2.1)$$

Remark 2.1. The functions f, g, ϕ, h satisfy Eq. (2.1) if and only if for every $p_0 \in \text{int } I$ the functions

$$F(x) := f(x + p_0), \quad G(x) := g(x + p_0), \quad \Phi(x, y) := \phi(x + p_0, y + p_0)$$

fulfil Eq. (2.1) for $x, y \in I - p_0$.

Functions F, G, Φ are then defined in a neighbourhood of zero. Therefore, we can assume without loss of generality that $0 \in \text{int } I$.

Remark 2.2. Let us observe that if the equation

$$xf(y) - yf(x) = (x - y)[h(\phi(x, y)) + g(x) + g(y)], \quad x, y \in I, \quad (2.2)$$

is satisfied then also the equation

$$xf(x) - yf(y) = (x - y)[-h(\phi(x, y)) + (f(x) - g(x)) + (f(y) - g(y))], \quad x, y \in I,$$

holds true. Similarly, if Eq. (2.1) is satisfied then also the equation

$$x \frac{f(y)}{y} - y \frac{f(x)}{x} = (x - y) \left[-h(\phi(x, y)) + \left(\frac{f(x)}{x} - g(x) \right) + \left(\frac{f(y)}{y} - g(y) \right) \right],$$

$$x, y \in I \setminus \{0\},$$

is satisfied.

Lemma 2.1. *Let $d \in \mathbb{R}$ be a constant. The functions $f, g : I \rightarrow \mathbb{R}$ satisfy the equation*

$$f(x) - f(y) = (x - y)[d + g(x) + g(y)], \quad x, y \in I, \quad (2.3)$$

if and only if

$$\begin{cases} f(x) = \alpha x^2 + (\beta + d)x + \gamma, \\ g(x) = \alpha x + \frac{\beta}{2}, \quad x \in I, \end{cases}$$

where α, β, γ are arbitrary constants.

Proof. Let

$$F(x) = f(x) - dx, \quad x \in I, \quad (2.4)$$

then from (2.3) we get

$$F(x) - F(y) = (x - y)[g(x) + g(y)], \quad x, y \in I. \quad (2.5)$$

Let us take an arbitrary $y_0 \in I$ and put $y = y_0$ in (2.5). We have

$$F(x) = F(y_0) + (x - y_0)[g(x) + g(y_0)], \quad x \in I. \quad (2.6)$$

Inserting (2.6) into (2.5) we obtain

$$\begin{aligned} (x - y_0)[g(x) + g(y_0)] - (y - y_0)[g(y) + g(y_0)] \\ = [(x - y_0) + (y_0 - y)][g(x) + g(y)], \quad x, y \in I. \end{aligned}$$

After a simple calculation we get for $\beta := 2(g(y_0) - \alpha y_0)$

$$g(x) = \alpha x + \frac{\beta}{2}, \quad x \in I,$$

and from (2.6) and (2.4)

$$f(x) = \alpha x^2 + (\beta + d)x + \gamma, \quad x \in I,$$

where α, β, γ are arbitrary constants. □

Using Remark 2.2 we immediately get the following.

Lemma 2.2. *Let $d \in \mathbb{R}$ be a constant.*

(I) *If $0 \in I$ then the functions $f, g : I \rightarrow \mathbb{R}$ satisfy the equation*

$$xf(y) - yf(x) = (x - y)[d + g(x) + g(y)], \quad x, y \in I, \quad (2.7)$$

if and only of

$$\begin{cases} f(x) = \alpha x + \beta - d, \\ g(x) = \frac{\beta}{2} - d, \quad x \in I. \end{cases}$$

(II) *If $0 \notin I$ then the functions $f, g : I \rightarrow \mathbb{R}$ satisfy Eq. (2.7) if and only if*

$$\begin{cases} f(x) = \alpha x + \beta - d + \frac{\gamma}{x}, \\ g(x) = \frac{\beta}{2} - d + \frac{\gamma}{x}, \quad x \in I. \end{cases}$$

In both cases α, β, γ are arbitrary constants.

Remark 2.3. The solution of Eq. (2.3) presented in Lemma 2.1 is exactly the one occurring when $I \in \mathbb{R}$ (see [3, Theorem 3.10]). An analogous situation occurs while dealing with Eq. (2.7) in the case where $0 \in I$.

Now let us consider the equation

$$f(x) - f(y) = (x - y)[\varphi(x)\psi(y) + g(x) + g(y)], \quad x, y \in I. \quad (2.8)$$

Remark 2.4. If functions $f, g, \varphi, \psi : I \rightarrow \mathbb{R}$ satisfy Eq. (2.8) then the following relation holds:

$$\varphi = 0 \text{ or } \psi = 0 \text{ or there exists } c \neq 0 \text{ such that } \psi = c\varphi.$$

Indeed, from (2.8) we easily get (after interchanging x and y) that

$$\varphi(x)\psi(y) = \varphi(y)\psi(x), \quad x, y \in I.$$

Suppose that neither $\varphi = 0$ nor $\psi = 0$. In particular then $\psi(y_0) \neq 0$ for some $y_0 \in I$. If $\varphi(y_0) = 0$ then

$$\varphi(x)\psi(y_0) = 0, \quad x \in I,$$

which would yield $\varphi = 0$. Thus we may assume that $\varphi(y_0) \neq 0$, and ultimately, taking $d = \frac{\varphi(y_0)}{\psi(y_0)} \neq 0$ we get

$$\varphi(x) = d\psi(x), \quad x \in I,$$

whence, with $c = \frac{1}{d}$,

$$\psi(x) = c\varphi(x), \quad x \in I,$$

easily follows.

3. Main results

Theorem 3.1. *If functions $f, g, \varphi, \psi : I \rightarrow \mathbb{R}$ satisfy Eq. (2.8) then they are of the form:*

$$\begin{cases} \varphi(x) = \beta, \quad \psi(x) = c\beta, \quad x \in I \setminus \{x_0\}, \quad \varphi(x_0) = a, \quad \psi(x_0) = ca, \\ g(x) = \gamma cx + \eta, \quad x \in I \setminus \{x_0\}, \quad g(x_0) = \gamma cx_0 + \eta + c\beta^2 - c\beta a, \\ f(x) = \gamma cx^2 + x(c\beta^2 + 2\eta) + \mu, \quad x \in I, \end{cases} \quad (\text{i})$$

where $x_0 \in I$ and $\beta, \gamma, \eta, \mu, a, c$ are arbitrary constants;

$$\begin{cases} \varphi(x) = \alpha x + \beta, \quad \psi(x) = c\alpha x + c\beta, \\ g(x) = c\alpha^2 x^2 + (c\gamma - \alpha\beta c)x + \eta, \\ f(x) = c\alpha^2 x^3 + c\gamma x^2 + (\beta^2 c + 2\eta)x + \mu, \quad x \in I, \end{cases} \quad (\text{ii})$$

for some constants $\alpha, \beta, \gamma, \eta, \mu, c$, $\alpha \neq 0$.

If $I \subset \mathbb{R}$ then for some $\alpha \notin I$ functions φ, ψ, f, g have the following form

$$\begin{cases} \varphi(x) = \frac{Abx}{x-\alpha} + \beta, \quad \psi(x) = \frac{cAbx}{x-\alpha} + c\beta, \\ g(x) = c\gamma x + \frac{Ab^2cx^2 - Acb(ab+\beta)x}{x-\alpha} + \eta, \\ f(x) = c\gamma x^2 + \frac{Acb^2(x^3 - ax^2)}{x-\alpha} + (c\beta^2 + 2\eta)x + \mu, \quad x \in I, \end{cases} \quad (\text{iii})$$

where $A := a - \alpha$, $a, c, b, \beta, \gamma, \eta, \mu$ are arbitrary constants. Conversely, functions given by the above formulas satisfy (2.8).

Proof. An easy though tedious calculation shows that the functions given by (i), (ii) and (iii) satisfy Eq. (2.8).

By Remark 2.4 we have $\varphi = 0$ or $\psi = 0$ or, if neither holds, there exists a $c \neq 0$ such that

$$\psi(x) = c\varphi(x), \quad x \in I. \quad (3.1)$$

If $\varphi = 0$ or $\psi = 0$ the problem is reduced to (2.3). From Remark 2.1 it follows that we can assume that $0 \in \text{int } I$. We can also assume that

$$\varphi(0) = g(0) = f(0) = 0.$$

Indeed; from (2.8) and (3.1) we have

$$\begin{aligned} & \left[\frac{f(x)-f(0)}{c} - x \left(\varphi(0)^2 + 2 \frac{g(0)}{c} \right) \right] - \left[\frac{f(y)-f(0)}{c} - y \left(\varphi(0)^2 + 2 \frac{g(0)}{c} \right) \right] \\ &= (x-y) [(\varphi(x) - \varphi(0))(\varphi(y) - \varphi(0)) \\ & \quad + \left(\frac{g(x)-g(0)}{c} + \varphi(x)\varphi(0) - \varphi(0)^2 \right) \\ & \quad + \left(\frac{g(y)-g(0)}{c} + \varphi(y)\varphi(0) - \varphi(0)^2 \right)] , \quad x, y \in I. \end{aligned} \quad (3.2)$$

Let

$$\begin{cases} \varphi_1(x) := \varphi(x) - \varphi(0), \\ g_1(x) := \frac{g(x)-g(0)}{c} + \varphi(x)\varphi(0) - \varphi(0)^2, \\ f_1(x) := \frac{f(x)-f(0)}{c} - x \left(\varphi(0)^2 + 2 \frac{g(0)}{c} \right), \end{cases} \quad x \in I. \quad (3.3)$$

Eq. (3.2) is now of the form

$$f_1(x) - f_1(y) = (x-y)[\varphi_1(x)\varphi_1(y) + g_1(x) + g_1(y)], \quad x, y \in I. \quad (3.4)$$

Putting $y = 0$ in (3.4) we get

$$f_1(x) = xg_1(x), \quad x \in I. \quad (3.5)$$

Substituting (3.5) into (3.4) we obtain

$$xg_1(x) - yg_1(y) = (x-y)[\varphi_1(x)\varphi_1(y) + g_1(x) + g_1(y)], \quad x, y \in I.$$

After some simple calculations we get

$$yg_1(x) - xg_1(y) = (x-y)\varphi_1(x)\varphi_1(y), \quad x, y \in I. \quad (3.6)$$

Dividing both sides of (3.6) by xy we have

$$\frac{g_1(x)}{x} - \frac{g_1(y)}{y} = (x-y) \frac{\varphi_1(x)}{x} \frac{\varphi_1(y)}{y}, \quad x, y \in I \setminus \{0\}. \quad (3.7)$$

Let

$$g_2(x) := \frac{g_1(x)}{x}, \quad \varphi_2(x) := \frac{\varphi_1(x)}{x}, \quad x \in I \setminus \{0\}. \quad (3.8)$$

Then (3.7) has now the following form

$$g_2(x) - g_2(y) = (x-y)\varphi_2(x)\varphi_2(y), \quad x, y \in I \setminus \{0\} \quad (3.9)$$

(see [2]). In particular, for a fixed $y_0 \in I \setminus \{0\}$ we obtain

$$g_2(x) = g_2(y_0) + (x-y_0)\varphi_2(x)\varphi_2(y_0), \quad x \in I \setminus \{0\}. \quad (3.10)$$

(A) Suppose that there exists a $y_0 \in I \setminus \{0\}$ such that

$$\varphi_2(y_0) = 0.$$

Then we get from (3.10) that

$$g_2(x) = g_2(y_0) = \text{const.} =: \gamma, \quad x \in I \setminus \{0\},$$

and from (3.9)

$$\varphi_2(x)\varphi_2(y) = 0, \quad x \neq y, \quad x, y \in I \setminus \{0\}.$$

Thus if there exists $x_0 \in I \setminus \{0\}$ such that $\varphi_2(x_0) \neq 0$, then $\varphi_2(x) = 0$ for $x \notin \{0, x_0\}$. Using (3.8) and (3.3) we get

$$\varphi_1(x) = 0, \quad x \in I \setminus \{0\}, \quad \varphi_1(x_0) = a, \quad g_1(x) = \gamma x, \quad f_1(x) = \gamma x^2, \quad x \in I,$$

where a and γ are some constants. From (3.3) we have

$$\begin{cases} \varphi(x) = \beta, & x \neq x_0, & \varphi(x_0) = a, & \psi(x) = c\beta, & x \neq x_0, & \psi(x_0) = ca, \\ g(x) = \gamma cx + g(0), & x \neq x_0, & g(x_0) = \gamma cx_0 + g(0) - ca\beta + \beta^2 c, \\ f(x) = c\gamma x^2 + cx \left(\beta^2 + \frac{2g(0)}{c} \right) + f(0), & x \in I, \end{cases}$$

and for $g(0) = \eta$, $f(0) = \mu$ we get assertion (i).

(B) We can assume that for every $y \in I \setminus \{0\}$

$$\varphi_2(y) \neq 0. \quad (3.11)$$

Putting (3.10) into (3.9) we get for $x, y \in I \setminus \{0\}$ and a fixed $y_0 \in I \setminus \{0\}$

$$(x - y_0)\varphi_2(x)[\varphi_2(y_0) - \varphi_2(y)] = (y - y_0)\varphi_2(y)[\varphi_2(y_0) - \varphi_2(x)]. \quad (3.12)$$

From (3.11) $\varphi_2(y_0) \neq 0$.

(I) If there exists $y_1 \in I \setminus \{0, y_0\}$ such that

$$\varphi_2(y_1) = \varphi_2(y_0) =: \alpha \neq 0$$

then from (3.12).

$$\varphi_2(x) = \alpha, \quad x \in I \setminus \{0\}.$$

It follows from (3.8) and (3.10) that for every $x \in I$

$$\begin{cases} \varphi_1(x) = \alpha x, \\ g_1(x) = \alpha^2 x^2 + \gamma x, \\ f_1(x) = \alpha^2 x^3 + \gamma x^2, \end{cases}$$

where $\gamma := g_2(y_0) - y_0\alpha^2$, and $\alpha \neq 0$ is an arbitrary constant. Using (3.3) we get (ii).

(II) We can assume that

$$\varphi_2(y) \neq \varphi_2(y_0), \quad y \in I \setminus \{0, y_0\}. \quad (3.13)$$

From (3.12) we get

$$\frac{\varphi_2(y)(y - y_0)}{\varphi_2(y_0) - \varphi_2(y)} = \text{const.} =: \gamma, \quad y \in I \setminus \{0, y_0\},$$

where $\gamma \neq 0$ is an arbitrary real number. It means that

$$\varphi_2(y) = \frac{\gamma\varphi_2(y_0)}{y - (y_0 - \gamma)}, \quad y \in I \setminus \{0, y_0, y_0 - \gamma\}.$$

If $I \subset \mathbb{R}$ and $\alpha := y_0 - \gamma \notin I$ then from (3.8) and (3.10) we obtain

$$\begin{cases} \varphi_1(x) = \frac{(a-\alpha)bx}{x-\alpha}, \\ g_1(x) = \gamma x + (x-a)\frac{(a-\alpha)b^2x}{x-\alpha}, \\ f_1(x) = \gamma x^2 + (x-a)\frac{(a-\alpha)b^2x^2}{x-\alpha}, \quad x \in I, \end{cases}$$

where $a := y_0, b := \varphi_2(y_0)$ are arbitrary constants. After simple calculations using (2.3) we get (iii). $\square$

Now we will present the results concerning the equation

$$xf(y) - yf(x) = (x-y)[\varphi(x)\psi(y) + g(x) + g(y)], \quad x, y \in I. \quad (3.14)$$

Theorem 3.2. (I) Let us assume that $0 \in I$. Functions $f, \varphi, \psi, g : I \rightarrow \mathbb{R}$ satisfy Eq. (3.14) if and only if

$$\begin{cases} \varphi(x) = \beta, \quad \psi(x) = -c\beta, \quad x \in I \setminus \{x_0\}, \quad \varphi(x_0) = a, \quad \psi(x_0) = -ca, \\ g(x) = c\beta^2 + \eta, \quad x \in I \setminus \{x_0\}, \quad g(x_0) = ca\beta + \eta, \\ f(x) = c\gamma x + c\beta^2 + 2\eta, \quad x \in I, \end{cases} \quad (i)$$

where $a, c, \beta, \gamma, \eta$ and $x_0 \in I$ are arbitrary constants;

or

$$\begin{cases} \varphi(x) = \alpha x + \beta, \quad \psi(x) = -c(\alpha x + \beta), \\ g(x) = \alpha\beta c x + \beta^2 c + \eta, \\ f(x) = c\alpha^2 x^2 + c\gamma x + \beta^2 c + 2\eta, \quad x \in I, \end{cases} \quad (ii)$$

where $\alpha, \beta, \gamma, \eta, c$ are arbitrary constants, $\alpha \neq 0$;

or in the case $I \subset \mathbb{R}$

$$\begin{cases} \varphi(x) = \frac{\alpha bx}{x-\alpha} + \beta, \quad \psi(x) = -c\left(\frac{\alpha bx}{x-\alpha} + \beta\right), \\ g(x) = \frac{A\beta c\beta x}{x-\alpha} + c\beta^2 + \eta, \\ f(x) = c\gamma x + \frac{Ab^2c(x^2-ax)}{x-\alpha} + c\beta^2 + 2\eta, \quad x \in I, \end{cases} \quad (iii)$$

where $\alpha \notin I, A := a - \alpha, a, c, \gamma, \beta$ are arbitrary constants.

(II) Let us assume that $0 \notin I$. The functions $f, \varphi, \psi, g : I \rightarrow \mathbb{R}$ satisfy Eq. (3.14) if and only if

$$\begin{cases} \varphi(x) = \beta, \quad \psi(x) = -c\beta, \quad x \in I \setminus \{x_0\}, \quad \varphi(x_0) = a, \quad \psi(x_0) = -ca, \\ g(x) = \beta^2 c + \eta + \frac{\mu}{x}, \quad x \in I \setminus \{x_0\}, \quad g(x_0) = ca\beta + \eta + \frac{\mu}{x_0}, \\ f(x) = c\gamma x + \beta^2 c + 2\eta + \frac{\mu}{x}, \quad x \in I, \end{cases} \quad (i)$$

where $a, c, \beta, \eta, \gamma, \mu$ and $x_0 \in I$ are arbitrary constants;

or

$$\begin{cases} \varphi(x) = \alpha x + \beta, & \psi(x) = -c(\alpha x + \beta), \\ g(x) = \alpha\beta cx + \beta^2 c + \eta + \frac{\mu}{x}, \\ f(x) = c\alpha^2 x^2 + c\gamma x + \beta^2 c + 2\eta + \frac{\mu}{x}, \end{cases} \quad x \in I, \quad (\text{ii})$$

where $c, \alpha, \beta, \gamma, \mu$ are arbitrary constants and $\alpha \neq 0$;

or in the case $I \subset \mathbb{R}$, for some $\alpha \notin I$

$$\begin{cases} \varphi(x) = \frac{Abx}{x-\alpha} + \beta, & \psi(x) = -c \left(\frac{Abx}{x-\alpha} + \beta \right), \\ g(x) = \frac{Abc\beta x}{x-\alpha} + \beta^2 c + \eta + \frac{\mu}{x}, \\ f(x) = c\gamma x + \frac{Ac b^2(x^2 - ax)}{x-\alpha} + \beta^2 c + 2\eta + \frac{\mu}{x}, \end{cases} \quad x \in I \quad (\text{iii})$$

where $A := a - \alpha$, $\beta, \gamma, \mu, \eta, a, b, c$ are arbitrary constants.

Proof. From Remark 2.2 we get the following form of Eq. (3.14) (for $\psi = -c\varphi$)

$$xf(x) - yf(y) = (x - y)[c\varphi(x)\varphi(y) + (f(x) - g(x)) + (f(y) - g(y))], \quad x, y \in I.$$

From Theorem 3.1 we have

$$xf(x) = c\alpha^2 x^3 + c\gamma x^2 + (\beta^2 c + 2\eta)x + \mu, \quad x \in I,$$

in the cases (i) and (ii). Let us note that actually (i) is (ii) with $\alpha = 0$.

We also have

$$xf(x) = c\gamma x^2 + \frac{Ac b^2(x^3 - ax^2)}{x - \alpha} + (\beta^2 c + 2\eta)x + \mu, \quad x \in I, \alpha \notin I$$

in the case (iii). In the case $0 \in I$ we get $\mu = 0$. □

Let us consider Eq. (2.1). Interchanging x and y we get

$$h(\phi(x, y)) = h(\phi(y, x)), \quad x, y \in I. \quad (3.15)$$

The following theorems guarantee that the function h is constant.

Theorem 3.3. Assume that the function $\phi : I^2 \xrightarrow{\text{onto}} J \subset \mathbb{R}$ is of class C^1 in $(\text{int } I)^2$ and

$$\det \begin{bmatrix} \frac{\partial \phi}{\partial x}(x, y) & \frac{\partial \phi}{\partial y}(x, y) \\ \frac{\partial \phi}{\partial y}(y, x) & \frac{\partial \phi}{\partial x}(y, x) \end{bmatrix} \neq 0, \quad x, y \in \text{int } I. \quad (\text{H})$$

If $h : J \rightarrow \mathbb{R}$ satisfies (3.15) then h is constant in $\text{int } J$.

Proof. It is obvious that if $\phi_1 := \phi$, $\phi_2 := \phi \circ S$, where $S(x, y) = (y, x)$, $(x, y) \in I^2$ and $\lambda(x) = \phi(x, x)$, then $\phi_1(I^2) = \phi_2(I^2) = J$ and $\lambda(I) \subset J$. Actually, $\lambda(I)$ is an interval but it may happen that $\lambda(I) = \{u_0\}$ for some $u_0 \in J$. Let us take arbitrary $x_0 \in \text{int } I$ and put

$$\phi(x_0, x_0) = u_0 = v_0.$$

Let us consider the following system of equations

$$\begin{cases} \phi(x, y) = u, \\ \phi(y, x) = v, \end{cases} \quad (3.16)$$

for $x, y \in I$, $u, v \in J$ and let us ask about its solutions for arbitrary $u, v \in J$. From the implicit function theorem applied to the function

$$F(u, v) = (u - \phi(x, y), v - \phi(y, x)), \quad (x, y) \in I^2, \quad (u, v) \in J^2, \quad (3.17)$$

we infer, taking into account (H), that there exist $\delta > 0, r > 0$ and exactly one differentiable function $(u, v) \rightarrow (x(u, v), y(u, v))$ such that for every $(u, v) \in (u_0 - r, u_0 + r)^2$ we have $(x(u, v), y(u, v)) \in (x_0 - \delta, x_0 + \delta)^2$, $x(u_0, v_0) = y(u_0, v_0) = x_0$ and

$$\begin{cases} \phi(x(u, v), y(u, v)) = u, \\ \phi(y(u, v), x(u, v)) = v. \end{cases} \quad (3.18)$$

Therefore

$$h(u) = h(v) =: c, \quad u, v \in (u_0 - r, u_0 + r). \quad (3.19)$$

The next step is the following: let

$$u_1 := u_0 + r \in \text{int } J. \quad (3.20)$$

There exists $(x_1, y_1) \in (\text{int } J)^2$ such that $u_1 = \phi(x_1, y_1)$. Let $v_1 = \phi(y_1, x_1)$. The points x_1, y_1, u_1, v_1 satisfy system (3.16). From the implicit function theorem applied to the function (3.17) we infer that there exist $\delta_1 > 0, r_1 > 0$ and exactly one differentiable function $(u, v) \rightarrow (x(u, v), y(u, v))$ such that for every $(u, v) \in (u_1 - r_1, u_1 + r_1) \times (v_1 - r_1, v_1 + r_1)$ we have $(x(u, v), y(u, v)) \in (x_1 - \delta_1, x_1 + \delta_1) \times (y_1 - \delta_1, y_1 + \delta_1)$, $x(u_1, v_1) = x_1, y(u_1, v_1) = x_2$ and (3.18) holds. From (3.20) we get

$$(u_0 - r, u_0 + r) \cap (u_0 + r - r_1, u_0 + r + r_1) \neq \emptyset$$

and (3.19) implies that

$$h(u) = c, \quad u \in (u_0 - r, u_0 + r + r_1).$$

Using condition (H) and the implicit function theorem we can extend equality (3.19) for every $u \in \text{int } J$. $\square$

An analogous proof shows the following

Corollary 3.1. *Let $\lambda(x) := \phi(x, x)$, $x \in I$. Assume that $\lambda(I) = J$. $\phi : I^2 \rightarrow J$ is of class C^1 in a neighbourhood of the diagonal $D = \{(x, x) \in \mathbb{R}^2 : x \in \text{int } I\}$,*

$$\left(\frac{\partial \phi}{\partial x}(x, x) \right)^2 \neq \left(\frac{\partial \phi}{\partial y}(x, x) \right)^2, \quad x \in \text{int } I.$$

If $h : J \rightarrow \mathbb{R}$ satisfies (3.15) then h is constant in $\text{int } J$.

Theorem 3.4. *Let us assume that $\phi : I^2 \rightarrow J$ satisfies the following conditions*

$$\left\{ \begin{array}{l} \text{there exists } x_0 \in \text{int} I \text{ such that } \phi \text{ is of class } C^1 \\ \text{in a neighbourhood of } (x_0, x_0) \text{ and} \\ \left(\frac{\partial \phi}{\partial x}(x_0, x_0) \right)^2 \neq \left(\frac{\partial \phi}{\partial y}(x_0, x_0) \right)^2, \\ \varphi_1 := \phi(\cdot, x_0), \varphi_2 := \phi(x_0, \cdot) \text{ are continuous and strictly} \\ \text{monotonic of the same kind and the sign of } (\varphi_1 - \varphi_2) \\ \text{is constant in each of the intervals } I \cap (-\infty, x_0) \text{ and} \\ I \cap (x_0, \infty), \varphi_1(I) = \varphi_2(I) = J. \end{array} \right. \quad (H')$$

If Eq. (3.15) holds then h is constant in $\text{int } J$.

Proof. Let us consider the system of Eq. (3.16) and let

$$\phi(x_0, x_0) = u_0 = v_0.$$

From the implicit function theorem applied to the function F given by the formula (3.17) we get that there exist $\delta > 0, r > 0$ and exactly one differentiable function $(u, v) \rightarrow (x(u, v), y(u, v))$ such that for every $(u, v) \in (u_0 - r, u_0 + r)^2$ we have $(x(u, v), y(u, v)) \in (x_0 - \delta, x_0 + \delta)^2$, $x(u_0, v_0) = x_0 = y(u_0, v_0)$ and (3.18) holds. Therefore, for every $u, v \in (u_0 - r, u_0 + r)$ we get

$$h(u) = h(v) = c. \quad (3.21)$$

Suppose that $U \subseteq J$ is a maximal open interval, containing u_0 and such that (3.21) holds for every $u, v \in U$. Further, suppose that $U \subset \text{Int } J$, e.g. $\sup U < \sup J$. Then $u_1 := \sup U \in J$, and $u_1 > u_0$. We may admit, without loss of generality, that φ_1 and φ_2 are both strictly increasing and moreover,

$$\varphi_1(x) < \varphi_2(x), \quad x \in I \cap (x_0, \infty).$$

Obviously, $u_1 = \varphi_1(x_1)$ for some $x_1 \in I \cap (x_0, \infty)$. Under our assumption we have $\varphi_2 \circ \varphi_1^{-1}(u_0) = u_0$, and

$$\varphi_2 \circ \varphi_1^{-1}(u) > u, \quad u \in J \cap (u_0, \infty).$$

In particular, we get $\varphi_2 \circ \varphi_1^{-1}(u_1) > u_1$ whence

$$V = \{v \in J : h(v) = c = h(u_0)\} \supset (u_0, \varphi_2 \circ \varphi_1^{-1}(u_1)) \not\supseteq (u_0, u_1).$$

This is in contradiction with our assumption, and proves the statement because the proof that $\inf U = \inf J$ is analogous. $\square$

Remark 3.1. If the system (3.16) is of the form

$$\begin{cases} \phi(x, y) = u, \\ \phi(y, x) = f(u), \quad x, y \in I \end{cases}$$

then f is an involutory function, and usually h is non-constant (see [1, 3], for suitable examples).

Open Access. This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

References

- [1] Ger, J.: On Sahoo–Riedel equations on a real interval. *Aequ. Math.* **63**, 168–179 (2002)
- [2] Kuczma, M.: On some functional equations with conic section as solutions. *Rocz. Naukowo Dydakt. WSP Krakowie* **159**, 197–213 (1993)
- [3] Sahoo, P.K., Riedel, T.: *Mean Value Theorems and Functional Equations*. World Scientific, Singapore (1998)

Joanna Ger
Institute of Mathematics
Silesian University
Bankowa 14
40-007 Katowice, Poland
e-mail: maciej.sablik@us.edu.pl

Received: April 21, 2015

Revised: June 8, 2015