Orthogonality equation with two unknown functions

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Dedicated to Professor Roman Ger on his 70th birthday

Abstract. The orthogonality equation involving two unknown functions

$$\langle f(x)|g(y) \rangle = \langle x|y \rangle$$

and, among related problems, the orthogonality preserving property

$$x \perp y \Rightarrow f(x) \perp g(y)$$

are considered.

Mathematics Subject Classification. Primary 39B52; Secondary 15A86 · 46C05.

Keywords. Orthogonality equation, orthogonality preserving mappings, linear isometries, Hilbert spaces.

1. Orthogonality equation

Let X and Y be (real or complex) inner product spaces. By *orthogonality equation* we mean the one with a single unknown function $f: X \rightarrow Y$

$$\langle f(x)|f(y) \rangle = \langle x|y \rangle, \quad x, y \in X. \quad (1.1)$$

It is not difficult to observe that f is a solution of (1.1) or, in other words, preserves the inner product if and only if, it is a linear isometry, hence injective but not necessarily surjective. Several aspects of this functional equation, including its approximate solutions and stability, have been intensively studied (cf. for example [1, 5] or a survey [3]). In the present paper we generalize equation (1.1) introducing two unknown functions $f, g: X \rightarrow Y$ so that we have:

$$\langle f(x)|g(y) \rangle = \langle x|y \rangle, \quad x, y \in X. \quad (1.2)$$

Remark 1.1. It is obvious that a pair (f, g) satisfies (1.2) whenever (g, f) does. Therefore, f and g must share their necessary properties. If (f, g) is a solution

of (1.2), then both mappings f, g are injective. Indeed, $f(x_1) = f(x_2)$ yields, for an arbitrary $y \in X$, $\langle x_1|y \rangle = \langle f(x_1)|g(y) \rangle = \langle f(x_2)|g(y) \rangle = \langle x_2|y \rangle$. Thus $x_1 = x_2$ and f (hence also g) is injective.

Simple examples below show, in particular, that the solutions of (1.2) need not be surjective nor linear and they need not satisfy (1.1).

Example 1.2. Let $X = Y = l^2$, $x = (x_1, x_2, \dots) \in l^2$. For the following mappings f, g the pair (f, g) is a solution of (1.2).

1. $f(x) = (0, 1, x_1, x_2, \dots)$, $g(x) = (1, 0, x_1, x_2, \dots)$, $x \in X$; both mappings are nonlinear and nonsurjective.
2. $f(x) = (0, x_1, x_2, \dots)$, $g(x) = (1, x_1, x_2, \dots)$, $x \in X$; f is linear whereas g is not, none of them is surjective.
3. $f(x) = (x_1, 0, x_2, 0, \dots)$, $g(x) = (x_1, x_1, x_2, x_2, \dots)$, $x \in X$; f, g are linear, f satisfies (1.1) but g does not.
4. $f(x) = 2x$, $g(x) = \frac{1}{2}x$, $x \in X$; both f and g are linear and surjective, none of them satisfies (1.1).

On the other hand, a pair of linear isometries need not be a solution of (1.2).

Example 1.3. Let $X = Y = l^2$, $x = (x_1, x_2, \dots) \in l^2$.

1. $f(x) = (0, x_1, x_2, \dots)$, $g(x) = (x_1, 0, x_2, \dots)$, $x \in X$; both f and g are linear and satisfy (1.1), however, (f, g) is not a solution of (1.2).
2. $f(x) = -x$, $g(x) = x$, $x \in X$; both f and g are linear, surjective and satisfy (1.1) but (f, g) is not a solution of (1.2).

Remark 1.4. A pair (f, g) consisting of two different solutions of (1.1) cannot be a solution of (1.2). Indeed, for $x \in X$ we have $\|f(x)\| = \|g(x)\| = \|x\|$, whence $\|f(x) - g(x)\|^2 = \|f(x)\|^2 - 2\operatorname{Re} \langle f(x)|g(x) \rangle + \|g(x)\|^2 = 0$. Similarly, if f or g is a surjective solution of (1.1), then (1.2) does not hold unless $g = f$. Indeed, for $x, y \in X$

$$\begin{aligned} \langle x|y \rangle &= \langle f(x)|g(y) \rangle = \langle f(x)|f(y) \rangle + \langle f(x)|g(y) - f(y) \rangle \\ &= \langle x|y \rangle + \langle f(x)|(g - f)(y) \rangle, \end{aligned}$$

whence $\langle f(x)|(g - f)(y) \rangle = 0$ and, by the surjectivity of f , $g = f$ follows.

In the remaining part of this section we employ surjectivity assumption and its consequences for solutions of (1.2). According to Remark 1.1, the roles of f and g can be interchanged.

Lemma 1.5. *Suppose that the pair (f, g) solves (1.2) and $f(X)^\perp = \{0\}$. Then g is linear.*

Proof. For arbitrary $x, y, z \in X$, $\lambda, \mu \in \mathbb{K}$ we have

$$\begin{aligned} \langle f(z)|g(\lambda x + \mu y) - \lambda g(x) - \mu g(y) \rangle &= \langle f(z)|g(\lambda x + \mu y) \rangle \\ &\quad - \bar{\lambda} \langle f(z)|g(x) \rangle - \bar{\mu} \langle f(z)|g(y) \rangle \\ &= \langle z|\lambda x + \mu y \rangle - \bar{\lambda} \langle z|x \rangle - \bar{\mu} \langle z|y \rangle \\ &= 0. \end{aligned}$$

Thus $g(\lambda x + \mu y) - \lambda g(x) - \mu g(y) \in f(X)^\perp = \{0\}$ and we are done. $\square$

Consequently, if both f and g are surjective, then both must be linear. However, the surjectivity of merely one of them suffices.

Lemma 1.6. *Suppose (f, g) solves (1.2) and f is surjective. Then both f and g are linear; moreover $g(X)^\perp = \{0\}$.*

Proof. It follows from Lemma 1.5 that g is linear. Since f is surjective, there exists $u \in X$ such that $f(u) = 0$. But $\|u\|^2 = \langle f(u)|g(u) \rangle = \langle 0|g(u) \rangle = 0$, whence $u = 0$. Therefore $f(x) = 0 \Leftrightarrow x = 0$. Let $z \in Y$ and $z \perp g(X)$. The surjectivity of f yields that there exists $v \in X$ such that $f(v) = z$. Then we have $\|v\|^2 = \langle f(v)|g(v) \rangle = \langle z|g(v) \rangle = 0$, whence $v = 0$ and $z = f(0) = 0$. Thus $g(X)^\perp = \{0\}$, as claimed, and Lemma 1.5 gives the linearity of f . $\square$

It follows that if X or Y is finite dimensional, (f, g) is a solution of (1.2) and at least one of the mappings f, g is surjective, then both are linear, continuous bijections. This appears to be true in arbitrary Hilbert spaces.

Theorem 1.7. *Let X and Y be Hilbert spaces, $f, g: X \rightarrow Y$ and let (f, g) satisfy (1.2). If f is surjective, then both mappings f and g are linear, continuous and bijective.*

Proof. The injectivity and linearity of f and g have already been proved (Remark 1.1 and Lemma 1.6). Choose arbitrarily x from a closed unit ball B in X and a functional $\varphi \in Y^*$. There exists $z \in Y$ such that $\varphi(u) = \langle u|z \rangle$, $u \in Y$ and by the surjectivity of f , $z = f(y)$ for some $y \in X$. Observe that

$$|\varphi(g(x))| = |\langle g(x)|f(y) \rangle| = |\langle x|y \rangle| \leq \|y\|$$

and y depends only on φ , not on x . Thus the set $g(B)$ is weakly bounded, hence bounded, and g is a bounded linear operator.

Now, we prove the continuity of f . For $x \in X$ we have

$$\|x\|^2 = |\langle f(x)|g(x) \rangle| \leq \|f(x)\| \|g(x)\| \leq \|f(x)\| \|g\| \|x\|,$$

whence

$$\|x\| \leq \|g\| \|f(x)\|, \quad x \in X.$$

Inserting $f^{-1}(y)$ in place of x , one gets

$$\|f^{-1}(y)\| \leq \|g\| \|y\|, \quad y \in Y.$$

Hence f^{-1} is a continuous linear bijection, thus also f is continuous.

Similarly as before, for $x \in X$ one gets

$$\|x\|^2 = |\langle f(x)|g(x) \rangle| \leq \|f(x)\| \|g(x)\| \leq \|f\| \|x\| \|g(x)\|,$$

whence

$$\|x\| \leq \|f\| \|g(x)\|, \quad x \in X. \quad (1.3)$$

Let $g(x_n) \rightarrow y$ for some sequence $(g(x_n))$ in $g(X)$ and some $y \in Y$. It follows from (1.3) that (x_n) is a Cauchy sequence in X , whence $x_n \rightarrow x_0$ for some $x_0 \in X$. The continuity of g yields $y = \lim_{n \rightarrow \infty} g(x_n) = g(x_0) \in g(X)$, which proves that $g(X)$ is closed in Y . On the other hand, Lemma 1.6 implies that $g(X)^\perp = \{0\}$ whence $g(X)$ is dense in Y . Finally, $g(X) = Y$, as required. $\square$

Example 1.8. Let $X = Y = \mathbb{K}^{(\mathbb{N})} := \{x = (x_1, x_2, \dots) : x_n \in \mathbb{K}; \exists n_0 \forall n \geq n_0 : x_n = 0\}$. With a norm $\|x\| = \sqrt{\sum_{i=1}^{\infty} |x_i|^2}$ it is a noncomplete subspace of l^2 . Let

$$f(x) := (x_1, 2x_2, 3x_3, 4x_4, \dots) \quad \text{and} \quad g(x) := (x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \frac{1}{4}x_4, \dots).$$

Then (f, g) satisfies (1.2), both mappings are linear bijections but only one of them is continuous. Similarly, with

$$f(x) := \left(x_1, \frac{1}{2}x_2, 3x_3, \frac{1}{4}x_4, \dots\right) \quad \text{and} \quad g(x) := \left(x_1, 2x_2, \frac{1}{3}x_3, 4x_4, \dots\right)$$

(f, g) satisfies (1.2), both mappings are linear bijections and none of them is continuous. Thus completeness is essential in Theorem 1.7.

Example 1.9. Let $X = Y = l^2$ and let

$$f(x) := \left(\sum_{i=1}^{\infty} |x_i|^2, x_1, x_2, \dots\right), \quad g(x) := (0, x_1, x_2, \dots).$$

Then (f, g) satisfies (1.2), and $f(X)^\perp = \{0\}$ but neither f nor g is surjective (actually, $g(X)^\perp \neq \{0\}$) and f is nonlinear. Thus the surjectivity assumption in Theorem 1.7 cannot be relaxed to $f(X)^\perp = \{0\}$.

Problem 1.10. Find an example of (f, g) (in a noncomplete inner product space) satisfying (1.2) such that f is surjective whereas g is not.

Corollary 1.11. *Let X be a Hilbert space, $f, g: X \rightarrow X$ and suppose that the pair (f, g) satisfies (1.2). If one of the mappings f, g is surjective, then both are bijective linear operators and g is adjoint to f^{-1} .*

Proof. It follows from Theorem 1.7 that f and g are bounded linear bijections. Moreover, for any $x, y \in X$, one has $\langle x|y \rangle = \langle f(x)|g(y) \rangle = \langle x|f^*g(y) \rangle$, whence $f^*g = \text{id}$. $\square$

2. Complementary mappings

The aim of this section is to answer a question whether for a given mapping f , one can find a (complementary) mapping g such that the pair (f, g) satisfies the Eq. (1.2). We start with the simplest case.

Proposition 2.1. *Let X, Y be inner product spaces and let X be finite dimensional. Then for an arbitrary linear injection $f: X \rightarrow Y$ there exists a linear injection $g: X \rightarrow Y$ such that (1.2) holds true. Moreover, if additionally $\dim Y = \dim X$, then such a mapping g is unique.*

Proof. Let $\{e_1, e_2, \dots, e_n\}$ be an orthonormal basis of X . Since $f: X \rightarrow Y$ is injective, vectors $f(e_1), \dots, f(e_n)$ are linearly independent and there exist vectors $c_1, \dots, c_n$ in Y such that

$$\langle f(e_i) | c_j \rangle = \delta_{ij}, \quad i, j = 1, \dots, n.$$

If $\dim Y = \dim X = n$, then such a sequence $c_1, \dots, c_n$ is unique. Let $g: X \rightarrow Y$ be a linear mapping defined by $g(e_i) := c_i$, $i = 1, \dots, n$. For arbitrary

$$x = \xi_1 e_1 + \dots + \xi_n e_n, \quad y = \eta_1 e_1 + \dots + \eta_n e_n \in X$$

one has

$$\begin{aligned} \langle f(x) | g(y) \rangle &= \left\langle \sum_{i=1}^n \xi_i f(e_i) \middle| \sum_{j=1}^n \eta_j g(e_j) \right\rangle = \sum_{i,j=1}^n \xi_i \overline{\eta_j} \langle f(e_i) | c_j \rangle \\ &= \sum_{i=1}^n \xi_i \overline{\eta_i} = \langle x | y \rangle. \end{aligned}$$

For any linear mapping $g: X \rightarrow Y$ such that (f, g) satisfies (1.2) we have

$$\langle f(e_i) | g(e_j) \rangle = \langle e_i | e_j \rangle = \delta_{ij}.$$

Thus $g(e_j) = c_j$, which proves the uniqueness if $\dim Y = \dim X$. $\square$

If $\dim Y > \dim X$, g need not be unique. Let us consider $f: \mathbb{R}^2 \rightarrow \mathbb{R}^3$, $f(x) = f(x_1, x_2) = (x_1, x_2, 0)$. Then any mapping of the form $g(x) = (x_1, x_2, \alpha x_1 + \beta x_2)$ ($\alpha, \beta \in \mathbb{R}$) is linear and satisfies, with f , equation (1.2). We proceed to a more general case, starting with a necessary condition.

Proposition 2.2. *Let X and Y be inner product spaces. Suppose that $\mathcal{B} = (e_i)_{i \in I}$ is an orthonormal Hilbert basis of X . Let $f, g: X \rightarrow Y$ satisfy (1.2). Then*

$$\forall i \in I \quad f(e_i) \notin Y_i := \overline{\text{Lin}(f(\mathcal{B}) \setminus \{f(e_i)\})}. \quad (2.1)$$

Proof. Suppose that the assertion is not true, i.e., for some $i_0 \in I$

$$f(e_{i_0}) = \lim_{n \rightarrow \infty} z_n, \quad \text{where} \quad z_n = \sum_{i \neq i_0} \xi_i^n f(e_i) \text{ (finite sum).}$$

Then for an arbitrary $x \in X$ one has

$$\begin{aligned} \langle e_{i_0} | x \rangle &= \langle f(e_{i_0}) | g(x) \rangle = \lim_{n \rightarrow \infty} \langle z_n | g(x) \rangle = \lim_{n \rightarrow \infty} \sum_{i \neq i_0} \xi_i^n \langle f(e_i) | g(x) \rangle \\ &= \lim_{n \rightarrow \infty} \sum_{i \neq i_0} \xi_i^n \langle e_i | x \rangle = \left\langle \lim_{n \rightarrow \infty} \sum_{i \neq i_0} \xi_i^n e_i | x \right\rangle. \end{aligned}$$

Thus $e_{i_0} = \lim_{n \rightarrow \infty} \sum_{i \neq i_0} \xi_i^n e_i$. Since $e_{i_0} \perp e_i$, $i \neq i_0$, it follows that $e_{i_0} \perp e_{i_0}$ whence $e_{i_0} = 0$, a contradiction. $\square$

Condition (2.1) may be sufficient for our aim.

Theorem 2.3. *Let X be an inner product space and Y a Hilbert space and suppose that there exists an orthonormal algebraic basis $\mathcal{B} = (e_i)_{i \in I}$ of X . Let $f: X \rightarrow Y$ be a linear mapping satisfying (2.1). Then there exists a linear mapping $g: X \rightarrow Y$ such that (1.2) holds. Moreover, if*

$$Y_0 := \overline{\text{Lin } f(\mathcal{B})} = Y, \quad (2.2)$$

then such a mapping g is unique.

Proof. Fix $i_0 \in I$. Since $f(e_{i_0}) \notin Y_{i_0}$, due to the Hahn–Banach theorem, there exists $\varphi_{i_0} \in Y^*$ such that $\varphi_{i_0}(f(e_{i_0})) = 1$ and $\varphi_{i_0}|_{Y_{i_0}} \equiv 0$. Notice, that if $Y_0 = Y$, then such a functional φ_{i_0} is unique. Further, there exists a unique $c_{i_0} \in Y$ such that $\varphi_{i_0}(y) = \langle y | c_{i_0} \rangle$ for all $y \in Y$. Thus, in particular

$$\langle f(e_{i_0}) | c_{i_0} \rangle = 1 \quad \text{and} \quad \langle f(e_i) | c_{i_0} \rangle = 0, \quad i \neq i_0.$$

We have proved the existence of a (unique if $Y_0 = Y$) system $(c_i)_{i \in I}$ such that

$$\langle f(e_i) | c_j \rangle = \delta_{ij}, \quad i, j \in I.$$

Since $\mathcal{B}$ is an algebraic basis of X we can uniquely define a linear mapping $g: X \rightarrow Y$ by setting $g(e_i) := c_i$, $i \in I$. Now, considering arbitrary $x = \sum_{i \in I} \xi_i e_i$, $y = \sum_{i \in I} \eta_i e_i \in X$ (finite sums) one gets

$$\langle f(x) | g(y) \rangle = \sum_{i, j \in I} \xi_i \bar{\eta}_j \langle f(e_i) | g(e_j) \rangle = \sum_{i \in I} \xi_i \bar{\eta}_i = \langle x | y \rangle.$$

$\square$

Remark 2.4. Let $\mathcal{B}$ be a Hilbert basis for X . If $f: X \rightarrow Y$ is a continuous linear surjection, then (2.2) holds true. Indeed, take any $y \in Y$ and choose an $x \in X$ such that $y = f(x)$. We have $x = \lim_{n \rightarrow \infty} x_n$ with $x_n \in \text{Lin } \mathcal{B}$. Then the continuity of f yields $y = f(x) = \lim_{n \rightarrow \infty} f(x_n) \in Y_0$.

Proposition 2.5. *Let X be a Hilbert space and $\mathcal{B} = (e_i)_{i \in I}$ its Hilbert basis. Let $f: X \rightarrow Y$ be a linear, injective and continuous mapping. Then (2.1) holds true.*

Proof. Assume that for some $i_0 \in I$, $f(e_{i_0}) \in Y_{i_0}$, i.e.,

$$f(e_{i_0}) = \lim_{n \rightarrow \infty} \sum_{i \neq i_0} \xi_i^n f(e_i)$$

with some ξ_i^n . Then, by the linearity and continuity of f , we would have

$$f(e_{i_0}) = f\left(\lim_{n \rightarrow \infty} \sum_{i \neq i_0} \xi_i^n e_i\right)$$

and, by the injectivity of f , $e_{i_0} = \lim_{n \rightarrow \infty} \sum_{i \neq i_0} \xi_i^n e_i$. Since $\langle e_{i_0} | e_i \rangle = 0$ for $i \neq i_0$, we get $\langle e_{i_0} | e_{i_0} \rangle = 0$, whence $e_{i_0} = 0$, a contradiction. $\square$

Corollary 2.6. *If f is a linear, bijective and continuous mapping between two Hilbert spaces, then both conditions (2.1) and (2.2) hold true.*

It can be derived from Theorem 2.3 that for a linear bijection f between Hilbert spaces X and Y , a unique complementary mapping g exists. This can be, however, proved directly. Let $U_X: X \ni x \mapsto \langle \cdot | x \rangle \in X^*$ (and similarly U_Y) denote the isomorphism which follows from the Riesz-Fréchet theorem.

Theorem 2.7. *Let X and Y be Hilbert spaces and let $f: X \rightarrow Y$ be a linear, continuous and bijective mapping. Then the mapping $g := U_Y^{-1} \circ (f^{-1})^* \circ U_X$ is the unique mapping $g: X \rightarrow Y$ (also linear, continuous and bijective) such that*

$$\langle f(x) | g(y) \rangle = \langle x | y \rangle, \quad x, y \in X.$$

Proof. For arbitrary $x, y \in X$ we have

$$\begin{aligned} \langle f(x) | g(y) \rangle &= \langle f(x) | U_Y^{-1}((f^{-1})^*(U_X(y))) \rangle = (f^{-1})^*(U_X(y)(f(x))) \\ &= U_X(y)(f^{-1}(f(x))) = U_X(y)(x) = \langle x | y \rangle. \end{aligned}$$

To prove the uniqueness, let g' satisfy the assertion. Then $\langle f(x) | g(y) \rangle = \langle f(x) | g'(y) \rangle$ for all $x, y \in X$ and the surjectivity of f yields $g = g'$. $\square$

3. Preservation of orthogonality

Let X and Y be inner product spaces ($\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ denotes the scalar field). By $\perp$ we denote the standard orthogonality relation: $x \perp y \Leftrightarrow \langle x | y \rangle = 0$. A mapping $f: X \rightarrow Y$ is called *orthogonality preserving* if

$$\forall x, y \in X \quad x \perp y \Rightarrow f(x) \perp f(y).$$

This class of mappings has been widely studied in the realm of inner product spaces, as well as in more general structures (cf. e.g. [2] or a survey [4]). Such mappings need not be linear. Actually, they may be very far from linear and continuous mappings (cf. [2]). On the other hand, for linear orthogonality preserving mappings we have a simple characterization.

Theorem 3.1. ([2], Th. 1) *For a non-vanishing mapping $f: X \rightarrow Y$ the following conditions are equivalent:*

1. f is linear and $\exists \gamma > 0 \forall x \in X \|f(x)\| = \gamma \|x\|$;
2. $\exists \gamma > 0 \forall x, y \in X \langle f(x)|f(y) \rangle = \gamma^2 \langle x|y \rangle$;
3. f is linear and orthogonality preserving.

Now, we assume that two linear mappings $f, g: X \rightarrow Y$ are given and we prove a generalization of the above theorem.

Theorem 3.2. *Suppose that linear mappings f, g satisfy the condition*

$$\forall x, y \in X \quad x \perp y \Rightarrow f(x) \perp g(y). \quad (3.1)$$

Then, there exists $\gamma \in \mathbb{K}$ such that

$$\langle f(x)|g(y) \rangle = \gamma \langle x|y \rangle, \quad x, y \in X. \quad (3.2)$$

Proof. Let $x, y \in X \setminus \{0\}$, $x \perp y$. Define $u := \frac{x}{\|x\|}$, $v := \frac{y}{\|y\|}$. Then $u \perp v$ and $u + v \perp u - v$ whence $f(u) \perp g(v)$, $f(v) \perp g(u)$ and $f(u) + f(v) \perp g(u) - g(v)$. Therefore

$$\begin{aligned} 0 &= \langle f(u) + f(v)|g(u) - g(v) \rangle \\ &= \langle f(u)|g(u) \rangle - \langle f(u)|g(v) \rangle + \langle f(v)|g(u) \rangle - \langle f(v)|g(v) \rangle \\ &= \langle f(u)|g(u) \rangle - \langle f(v)|g(v) \rangle. \end{aligned}$$

Thus $\left\langle f\left(\frac{x}{\|x\|}\right) \middle| g\left(\frac{x}{\|x\|}\right) \right\rangle = \left\langle f\left(\frac{y}{\|y\|}\right) \middle| g\left(\frac{y}{\|y\|}\right) \right\rangle$ and finally:

$$\langle f(x)|g(x) \rangle = \frac{\langle f(y)|g(y) \rangle}{\|y\|^2} \cdot \|x\|^2, \quad x, y \in X \setminus \{0\}, \quad x \perp y.$$

Fix arbitrarily $y_0 \in X \setminus \{0\}$ and set $\gamma := \frac{\langle f(y_0)|g(y_0) \rangle}{\|y_0\|^2}$. Then we have

$$\langle f(x)|g(x) \rangle = \gamma \cdot \|x\|^2, \quad x \in X, \quad x \perp y_0. \quad (3.3)$$

Now fix $x, y \in X$. Let $X_0 = \text{Lin}\{x, y, y_0\}$. There is

$$x = x_1 + \lambda y_0 \quad \text{for some } x_1 \in X_0 \quad \text{such that} \quad x_1 \perp y_0, \quad \lambda \in \mathbb{K}.$$

Moreover, for some $y_1 \in X_0$ such that $y_1 \perp \text{Lin}\{x_1, y_0\}$ and for some $\alpha, \beta \in \mathbb{K}$:

$$y = y_1 + \alpha x_1 + \beta y_0.$$

Now, we have

$$\begin{aligned} \langle x|y \rangle &= \langle x_1 + \lambda y_0|y_1 + \alpha x_1 + \beta y_0 \rangle \\ &= \langle x_1|y_1 \rangle + \bar{\alpha} \langle x_1|x_1 \rangle + \bar{\beta} \langle x_1|y_0 \rangle \\ &\quad + \lambda \langle y_0|y_1 \rangle + \lambda \bar{\alpha} \langle y_0|x_1 \rangle + \lambda \bar{\beta} \langle y_0|y_0 \rangle \\ &= \bar{\alpha} \|x_1\|^2 + \lambda \bar{\beta} \|y_0\|^2. \end{aligned}$$

On the other hand

$$\begin{aligned}\langle f(x)|g(y)\rangle &= \langle f(x_1) + \lambda f(y_0)|g(y_1) + \alpha g(x_1) + \beta g(y_0)\rangle \\ &= \langle f(x_1)|g(y_1)\rangle + \bar{\alpha} \langle f(x_1)|g(x_1)\rangle + \bar{\beta} \langle f(x_1)|g(y_0)\rangle \\ &\quad + \lambda \langle f(y_0)|g(y_1)\rangle + \lambda \bar{\alpha} \langle f(y_0)|g(x_1)\rangle + \lambda \bar{\beta} \langle f(y_0)|g(y_0)\rangle \\ &= \bar{\alpha} \langle f(x_1)|g(x_1)\rangle + \lambda \bar{\beta} \langle f(y_0)|g(y_0)\rangle.\end{aligned}$$

Now, it follows from (3.3) and the definition of γ that

$$\langle f(x)|g(y)\rangle = \bar{\alpha}\gamma\|x_1\|^2 + \lambda\bar{\beta}\gamma\|y_0\|^2 = \gamma\langle x|y\rangle.$$

□

Remark 3.3. Notice that if there exists $y_0 \neq 0$ such that $f(y_0) \perp g(y_0)$, then $\gamma = 0$ and consequently $\langle f(x)|g(y)\rangle = 0$ for all $x, y \in X$. Thus $f(X) \perp g(X)$.

Putting $x = y$ in (3.2) we derive a consequence of Theorem 3.2.

Corollary 3.4. *For linear mappings f, g satisfying (3.1) there exists $\gamma \in \mathbb{K}$ such that*

$$\langle f(x)|g(x)\rangle = \gamma \cdot \|x\|^2, \quad x \in X. \quad (3.4)$$

Remark 3.5. For $f = g$ we get from (3.4), $\|f(x)\|^2 = \gamma\|x\|^2$, $x \in X$, whence $\gamma \geq 0$. However, in the general case, γ can be quite arbitrary. Let us consider an arbitrary $\gamma \in \mathbb{K}$ and linear mappings $f(x) = \gamma x$, $g(x) = x$, $x \in X$. Then $\langle f(x)|g(y)\rangle = \gamma\langle x|y\rangle$, $x, y \in X$ and, in particular, f, g satisfy (3.1).

For $X = Y$ we obtain yet another corollary from Theorem 3.2.

Corollary 3.6. *If a linear mapping $f: X \rightarrow X$ acting on an inner product space X satisfies*

$$\forall x, y \in X \quad x \perp y \Rightarrow f(x) \perp y,$$

then, with some $\gamma \in \mathbb{K}$,

$$f(x) = \gamma x, \quad x \in X.$$

Proof. Applying Theorem 3.2 to f and the identity mapping $g(x) = x$ one obtains, with some $\gamma \in \mathbb{K}$,

$$\langle f(x) - \gamma x|y\rangle = 0, \quad x, y \in X,$$

and the assertion follows. □

Theorem 3.7. *Assume that X and Y are complex inner product spaces. Let $f, g: X \rightarrow Y$ be linear mappings such that, for some $\gamma \in \mathbb{C}$, (3.4) holds. Then (3.2) follows.*

Proof. For $x, y \in X$ one has

$$\begin{aligned}
 \gamma \operatorname{Re} \langle x|y \rangle &= \frac{1}{4} \gamma (\|x + y\|^2 - \|x - y\|^2) \\
 &= \frac{1}{4} (\langle f(x) + f(y)|g(x) + g(y) \rangle - \langle f(x) - f(y)|g(x) - g(y) \rangle) \\
 &= \frac{\langle f(x)|g(y) \rangle + \langle f(y)|g(x) \rangle}{2}, \\
 i\gamma \operatorname{Im} \langle x|y \rangle &= i\frac{1}{4} \gamma (\|ix - y\|^2 - \|ix + y\|^2) \\
 &= \frac{\langle f(x)|g(y) \rangle - \langle f(y)|g(x) \rangle}{2},
 \end{aligned}$$

whence (adding) $\gamma \langle x|y \rangle = \langle f(x)|g(y) \rangle$. □

It is essential that the considered spaces are over the field of complex numbers. The above result is no longer valid in real spaces.

Example 3.8. Let $f, g: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be linear mappings defined by

$$f(x) = (x_1 - x_2, x_1 + x_2), \quad g(x) = x, \quad x = (x_1, x_2) \in \mathbb{R}^2.$$

Then we have, for $x \in \mathbb{R}^2$

$$\langle f(x)|g(x) \rangle = \langle (x_1 - x_2, x_1 + x_2)|(x_1, x_2) \rangle = x_1^2 + x_2^2 = \|x\|^2,$$

i.e., (3.4) is satisfied with $\gamma = 1$. On the other hand, for $e_1 = (1, 0)$ and $e_2 = (0, 1)$ one has

$$\langle f(e_1)|g(e_2) \rangle = \langle (1, 1)|(0, 1) \rangle = 1 \quad \text{and} \quad \langle e_1|e_2 \rangle = 0,$$

whence (3.2) does not hold.

Theorem 3.9. *Let X, Y be inner product spaces and $f, g: X \rightarrow Y$ linear mappings. The following conditions are equivalent, with some $\gamma \in \mathbb{K}$:*

1. $\forall x, y \in X \quad x \perp y \Rightarrow f(x) \perp g(y);$
2. $\forall x, y \in X \quad \langle f(x)|g(y) \rangle = \gamma \langle x|y \rangle.$

Moreover, in the complex case, each of the above conditions is equivalent to

3. $\forall x \in X \quad \langle f(x)|g(x) \rangle = \gamma \|x\|^2.$

Proof. Implication 1 $\Rightarrow$ 2 follows from Theorem 3.2 and the reverse is obvious. Implication 2 $\Rightarrow$ 3 is obvious and its reverse, in the complex case only, follows from Theorem 3.7. □

4. Conjugate orthogonality equation and Wigner's equation

Assume now that X and Y are complex inner product spaces. By *conjugate orthogonality equation* we mean:

$$\langle f(x)|f(y)\rangle = \overline{\langle x|y\rangle}, \quad x, y \in X. \quad (4.1)$$

A mapping f is a solution of (4.1) if and only if, it is a conjugate-linear (i.e., $f(\lambda x + \mu y) = \bar{\lambda}f(x) + \bar{\mu}f(y)$ for each $x, y \in X$, $\lambda, \mu \in \mathbb{C}$) isometry, hence injective but not necessarily surjective. Similarly as before, we generalize the equation by involving two unknown functions $f, g: X \rightarrow Y$:

$$\langle f(x)|g(y)\rangle = \overline{\langle x|y\rangle}, \quad x, y \in X. \quad (4.2)$$

Let us formulate analogous results to those proved in previous sections.

Theorem 4.1. *Let X, Y be complex Hilbert spaces and let (f, g) satisfy (4.2). If f or g is surjective, then both mappings f and g are conjugate-linear, continuous bijections.*

Theorem 4.2. *Let X be a complex inner product space, Y a complex Hilbert space and suppose that there exists an orthonormal algebraic basis $\mathcal{B} = (e_i)_{i \in I}$ of X . Let $f: X \rightarrow Y$ be a conjugate-linear mapping. Then there exists a conjugate-linear mapping $g: X \rightarrow Y$ such that (4.2) holds true if and only if f satisfies (2.1).*

Theorem 4.3. *Let X, Y be complex inner product spaces and $f, g: X \rightarrow Y$ conjugate-linear mappings. The following conditions are equivalent, with some $\gamma \in \mathbb{C}$:*

1. $\forall x, y \in X \quad x \perp y \Rightarrow f(x) \perp g(y);$
2. $\forall x, y \in X \quad \langle f(x)|g(y)\rangle = \gamma \overline{\langle x|y\rangle};$
3. $\forall x \in X \quad \langle f(x)|g(x)\rangle = \gamma \|x\|^2.$

Let X, Y be inner product spaces and $f: X \rightarrow Y$. By a *Wigner equation* we mean:

$$|\langle f(x)|f(y)\rangle| = |\langle x|y\rangle|, \quad x, y \in X. \quad (4.3)$$

Wigner's theorem [8] (cf. also e.g. [6]) states that $f: X \rightarrow Y$ is a solution of (4.3) if and only if it is phase-equivalent to a linear or conjugate-linear isometry. It means that there exists an operator $g: X \rightarrow Y$ satisfying (1.1) or (4.1) and a scalar-valued function $\sigma: X \rightarrow \mathbb{S} := \{z \in \mathbb{C} : |z| = 1\}$ such that $f(x) = \sigma(x)g(x)$, $x \in X$. Recently, an elegant and short proof of Wigner's theorem has been given by Turnšek [7].

The following equation can be regarded as a natural generalization of Wigner's equation:

$$|\langle f(x)|g(y)\rangle| = |\langle x|y\rangle|, \quad x, y \in X. \quad (4.4)$$

Clearly, if (f, g) satisfies (1.2) or (4.2), then (f, g) satisfies (4.4). Bearing Wigner's theorem in mind, one could conjecture that for $f, g: X \rightarrow Y$ satisfying (4.4) there exist $U, V: X \rightarrow Y$ satisfying (1.2) or (4.2) and $\sigma_1, \sigma_2: X \rightarrow \mathbb{S}$ such that

$$f(x) = \sigma_1(x)U(x), \quad g(x) = \sigma_2(x)V(x), \quad x \in X.$$

The validity of the above conjecture is an open problem. Let us make an observation concerning a particular case.

Theorem 4.4. *Let X, Y be inner product spaces and let $f, g: X \rightarrow Y$. The following statements are equivalent:*

1. *a pair (f, g) satisfies (4.4) and*

$$\|f(x)\| = \|g(x)\| = \|x\|, \quad x \in X; \quad (4.5)$$

2. *each of f and g satisfies (4.3) and they are phase-equivalent;*
3. *there exists a linear or conjugate-linear isometry $U: X \rightarrow Y$ such that both f and g are phase-equivalent with U .*

Proof. $(1 \Rightarrow 2)$ For an arbitrary $x \in X$ we have

$$|\langle f(x)|g(x) \rangle| = \|x\|^2 = \|f(x)\| \|g(x)\|$$

whence $f(x) = \lambda(x)g(x)$ for some $\lambda(x) \in \mathbb{C}$. Because of (4.5) we must have $|\lambda(x)| = 1$, hence f and g are phase-equivalent. Now, for $x, y \in X$

$$|\langle x|y \rangle| = |\langle \lambda(x)g(x)|g(y) \rangle| = |\langle g(x)|g(y) \rangle|,$$

i.e., g satisfies (4.3). Obviously, so does f as it is phase-equivalent with g .

$(2 \Rightarrow 3)$ It follows from Wigner's theorem that there exists a linear or conjugate-linear isometry U such that g is phase-equivalent with U . Since f is phase-equivalent with g , it has to be also phase-equivalent with U .

$(3 \Rightarrow 1)$ It is clear that

$$|\langle f(x)|g(y) \rangle| = |\langle U(x)|U(y) \rangle| = |\langle x|y \rangle|$$

and $\|f(x)\| = \|g(x)\| = \|U(x)\| = \|x\|$. □

Remark 4.5. In the proof of $(1) \Rightarrow (2)$ one can replace the condition (4.5) by a weaker one:

$$\|f(x)\| \leq \|x\|, \quad \|g(x)\| \leq \|x\|, \quad x \in X. \quad (4.6)$$

Indeed, from (4.4) and (4.6) we have for $x \in X$:

$$\|x\|^2 = |\langle f(x)|g(x) \rangle| \leq \|f(x)\| \|g(x)\| \leq \|x\| \|g(x)\|.$$

It follows that $\|x\| \leq \|g(x)\|$ whence $\|g(x)\| = \|x\|$. Similarly, $\|f(x)\| = \|x\|$ and (4.5) is satisfied.

It is easy to see that (4.4) does not imply (4.5) whereas (4.5) is necessary for 2. and 3. in the above theorem. Moreover, (4.5) cannot be replaced either by $\|f(x)\| = \|x\|$, $x \in X$ or by $\|f(x)\| = \|g(x)\|$, $x \in X$ only. See Example 1.2.

Finally, we derive the following result from Theorems 3.9 and 4.3.

Theorem 4.6. *Let X, Y be complex inner product spaces and let mappings $f, g: X \rightarrow Y$ be both linear or both conjugate-linear. Then the following conditions are equivalent, with some $\gamma \in \mathbb{C}$:*

1. $\forall x, y \in X : x \perp y \Rightarrow f(x) \perp g(y);$
2. $\forall x, y \in X \quad |\langle f(x)|g(y) \rangle| = |\gamma| |\langle x|y \rangle|;$
3. $\forall x \in X \quad \langle f(x)|g(x) \rangle = \gamma \|x\|^2.$

Acknowledgements

I would like to thank Dr. P. Wójcik and Dr. S. Siudut for their contribution to some parts of the manuscript.

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References

- [1] Badora, R., Chmieliński, J.: Decomposition of mappings approximately inner product preserving. *Nonlinear Anal.* **62**, 1015–1023 (2005)
- [2] Chmieliński, J.: Linear mappings approximately preserving orthogonality. *J. Math. Anal. Appl.* **304**, 158–169 (2005)
- [3] Chmieliński, J.: Stability of the Wigner equation and related topics. *Nonlinear Funct. Anal. Appl.* **11**, 859–879 (2006)
- [4] Chmieliński, J.: Orthogonality preserving property and its Ulam stability. Chapter 4. In: J. Brzdęk, Th.M. Rassias (eds.) *Functional Equations in Mathematical Analysis*. Springer Optimization and Its Applications 52, pp. 33–58, Springer, Berlin (2012)
- [5] Chmieliński, J., Ger, R.: On mappings preserving inner product modulo an ideal. *Arch. Math. (Basel)* **73**, 186–192 (1999)
- [6] Rätz, J.: On Wigner's theorem: remarks, complements, comments, and corollaries. *Aequationes Math.* **52**, 1–9 (1996)
- [7] Turnšek, A.: A variant of Wigner's functional equation. *Aequationes Math.* (to appear). doi:[10.1007/s00010-014-0296-0](https://doi.org/10.1007/s00010-014-0296-0)
- [8] Wigner, E.P.: *Gruppentheorie und ihre Anwendungen auf die Quantenmechanik der Atomspektren*. Friedr. Vieweg und Sohn Akt.-Ges. (1931)

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Received: January 26, 2015

Revised: May 12, 2015