On functional equation stemming from utility theory and psychophysics

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Abstract. We deal with the functional equations

$$f(\sigma(y)x + (1 - \sigma(y))y) = \tau(y)f(x) + (1 - \tau(y))f(y)$$

for $x, y \in [0, \infty)$, $x \geq y$, where $f : [0, \infty) \rightarrow \mathbb{R}$ and $\sigma, \tau : [0, \infty) \rightarrow [0, 1]$; and

$$F(\sigma(y)x + (1 - \sigma(y))y) = \tau(y)F(x) + (1 - \tau(y))F(y)$$

for $x, y \in \mathcal{C}$, $x - y \in \mathcal{C}$, where $\mathcal{C}$ is a convex cone in a real linear space, $F : \mathcal{C} \rightarrow \mathbb{R}$ and $\sigma, \tau : \mathcal{C} \rightarrow [0, 1]$. We determine the solutions of these equations satisfying some natural regularity assumptions. In this way we generalize the result of J. Aczél and R. D. Luce.

Keywords. Composite equation, convex cone, continuity on rays.

1. Introduction

In 2002 Professors Janos Aczél and R. Duncan Luce published the paper [2] where two functional equations stemming from some problems in utility theory and psychophysics, questions concerning pairs of homomorphic homogeneous functions of degree 1 and the link of the latter questions to quasilinear means were considered. One of the equations investigated in [2] has the form

$$f(\sigma(y)x + (1 - \sigma(y))y) = \tau(y)f(x) + (1 - \tau(y))f(y) \quad \text{for } x, y \in [0, \infty), x \geq y, \quad (1)$$

where $f : [0, \infty) \rightarrow [0, \infty)$ and $\sigma, \tau : [0, \infty) \rightarrow [0, 1]$. The following result concerning the solutions of (1) was proved in [2].

Theorem 1.1. [2, Theorem 1] *If f is strictly increasing and continuously differentiable, σ and τ are continuous and $\sigma(y) \in (0, 1)$ for $y \in (0, \infty)$ then the triple*

(f, σ, τ) satisfies equation (1) if and only if $\sigma = \tau$ and there exist $a \in (0, \infty)$ and $b \in [0, \infty)$ such that

$$f(x) = ax + b \quad \text{for } x \in [0, \infty). \quad (2)$$

Furthermore (cf. [2, Note 3]) Professor Zsolt Páles succeeded in eliminating the differentiability assumption from Theorem 1.1. Unfortunately, to the best of our knowledge, this result has never been published. The aim of this paper is to generalize Theorem 1.1 in various directions. First, we show that in Theorem 1.1 not only the differentiability of f can be replaced by continuity, but also the monotonicity of f and continuity of τ can be eliminated. Then we study a more general version of equation (1), namely

$$F(\sigma(y)x + (1 - \sigma(y))y) = \tau(y)F(x) + (1 - \tau(y))F(y) \quad \text{for } x, y \in \mathcal{C}, x - y \in \mathcal{C}, \quad (3)$$

where $\mathcal{C}$ is a convex cone in a real linear space, $F : \mathcal{C} \rightarrow \mathbb{R}$ and $\sigma, \tau : \mathcal{C} \rightarrow [0, 1]$.

2. Results

We begin with the following generalization of Theorem 1.1.

Theorem 2.1. *Let $f : [0, \infty) \rightarrow \mathbb{R}$ and $\sigma, \tau : [0, \infty) \rightarrow [0, 1]$. Assume that f and σ are continuous, f is nonconstant and $\sigma(y) \in (0, 1)$ for $y \in (0, \infty)$. Then the triple (f, σ, τ) satisfies equation (1) if and only if $\sigma = \tau$ and there exist $a \in \mathbb{R} \setminus \{0\}$ and $b \in \mathbb{R}$ such that f is of the form (2).*

Proof. Assume that the triple (f, σ, τ) satisfies equation (1). First consider the case where σ is constant, say $\sigma(x) = c$ for $x \in [0, \infty)$ with some $c \in (0, 1)$. Then (1) becomes

$$f(cx + (1 - c)y) = \tau(y)f(x) + (1 - \tau(y))f(y) \quad \text{for } x, y \in [0, \infty), x \geq y. \quad (4)$$

Setting in (4) $x = \frac{u}{c}$ and $y = \frac{v}{1-c}$, we obtain

$$f(u + v) = m(v)n(u) + l(v) \quad \text{for } u, v \in [0, \infty), u \geq \frac{c}{1-c}v, \quad (5)$$

where $m : [0, \infty) \rightarrow [0, 1]$ and $n, l : [0, \infty) \rightarrow \mathbb{R}$ are given by

$$m(v) = \tau\left(\frac{v}{1-c}\right) \quad \text{for } v \in [0, \infty), \quad (6)$$

$$n(u) = f\left(\frac{u}{c}\right) \quad \text{for } u \in [0, \infty) \quad (7)$$

and

$$l(v) = \left(1 - \tau\left(\frac{v}{1-c}\right)\right) f\left(\frac{v}{1-c}\right) \quad \text{for } v \in [0, \infty). \quad (8)$$

In particular, (5) is valid on the set $\{(u, v) \in (0, \infty)^2 : u > \frac{c}{1-c}v\}$, which is nonempty, open and connected. Since f is nonconstant and continuous, applying [3, Theorem 1], we obtain that either

$$\begin{aligned} f(u) &= \alpha z + \gamma + \delta \quad \text{for } u \in (0, \infty), \\ m(v) &= \beta \quad \text{for } v \in (0, \infty), \end{aligned} \quad (9)$$

$$\begin{aligned} n(u) &= \frac{1}{\beta}(\alpha u + \delta) \quad \text{for } u \in (0, \infty), \\ l(v) &= \alpha v + \gamma \quad \text{for } v \in (0, \infty) \end{aligned} \quad (10)$$

with some $\alpha, \beta \in \mathbb{R} \setminus \{0\}$ and $\gamma, \delta \in \mathbb{R}$, or

$$\begin{aligned} f(u) &= \alpha \beta e^{\gamma u} + \delta \quad \text{for } u \in (0, \infty), \\ m(v) &= \alpha e^{\gamma v} \quad \text{for } v \in (0, \infty), \end{aligned} \quad (11)$$

$$\begin{aligned} n(u) &= \beta e^{\gamma u} + \mu \quad \text{for } u \in (0, \infty), \\ l(v) &= -\alpha \mu e^{\gamma v} + \delta \quad \text{for } v \in (0, \infty) \end{aligned} \quad (12)$$

with some $\alpha, \beta, \gamma \in \mathbb{R} \setminus \{0\}$ and $\delta, \mu \in \mathbb{R}$.

In the first case (6) and (9) imply that $\tau(y) = \beta$ for $y \in (0, \infty)$ and so, in view of (8) and (10), we get $f(x) = \frac{\alpha(1-c)}{1-\beta}x + \frac{\gamma}{1-\beta}$ for $x \in (0, \infty)$. Thus, as f is continuous, (2) holds with $a := \frac{\alpha(1-c)}{1-\beta} \neq 0$ and $b := \frac{\gamma}{1-\beta}$. Moreover, setting into (4) f of the form (2), after straightforward calculations, we get $a(c - \tau(y))(x - y) = 0$ for $x, y \in [0, \infty)$ with $x \geq y$. Since $a \neq 0$, this implies that $\tau(y) = c$ for $y \in [0, \infty)$ and so $\tau = \sigma$. In the second case from (7), (11) and (12) we derive that $\alpha \beta e^{\gamma u} + \delta = \beta e^{\gamma c u} + \mu$ for $u \in (0, \infty)$. Differentiating both sides of the last equality with respect to u , we obtain $e^{\gamma(1-c)u} = \frac{c}{\alpha}$ for $u \in (0, \infty)$. As $c \in (0, 1)$ and $\gamma \neq 0$, this yields a contradiction.

Assume that σ is nonconstant. First we show that

$$\tau(y) \neq 1 \quad \text{for } y \in (0, \infty). \quad (13)$$

Suppose that (13) does not hold, that is $\tau(y_1) = 1$ for some $y_1 \in (0, \infty)$. Then, in view of (1), we get

$$f(\sigma(y_1)x + (1 - \sigma(y_1))y_1) = f(x) \quad \text{for } x \geq y_1. \quad (14)$$

Hence, as

$$\sigma(y_1)x + (1 - \sigma(y_1))y_1 = \sigma(y_1)(x - y_1) + y_1 \geq y_1 \quad \text{for } x \geq y_1,$$

we get by replacing x by $\sigma(y_1)x + (1 - \sigma(y_1))y_1$ in (14)

$$f(\sigma(y_1)^2(x - y_1) + y_1) = f(x) \quad \text{for } x \geq y_1.$$

Iterating this process, we obtain

$$f(\sigma(y_1)^n(x - y_1) + y_1) = f(x) \quad \text{for } x \geq y_1, n \in \mathbb{N}.$$

Since f is continuous and $\sigma(y_1) \in (0, 1)$, letting in the last equality $n \rightarrow \infty$, we conclude that

$$f(x) = f(y_1) \quad \text{for } x \geq y_1. \quad (15)$$

Furthermore, as f is nonconstant and continuous, there is $y_2 \in (0, y_1)$ such that $f(y_2) \neq f(y_1)$. If $\tau(y_2) = 1$ then arguing as previously we get $f(x) = f(y_2)$ for $x \geq y_2$. In particular $f(y_1) = f(y_2)$, which yields a contradiction. Thus $\tau(y_2) \neq 1$. Moreover, for sufficiently large $x \in (y_1, \infty)$, we have $\sigma(y_2)x + (1 - \sigma(y_2))y_2 \geq y_1$, whence taking (1) and (15) into account, we obtain $(1 - \tau(y_2))(f(y_2) - f(y_1)) = 0$, which again gives a contradiction. In this way we have proved (13).

Next we show that, for every $z \in (0, \infty)$, a function $f|_{(z, \infty)}$ is nonconstant. Suppose that this is not true. Then there is $z \in (0, \infty)$ such that $f(x) = c$ for $x \in (z, \infty)$ with some $c \in \mathbb{R}$. Let $y \in (0, \infty)$. As $\sigma(y)x + (1 - \sigma(y))y \geq z$ for sufficiently large $x \in (y, \infty)$, making use of (1), we get $(1 - \tau(y))(f(y) - c) = 0$ and so, in view of (13), $f(y) = c$. Since f is continuous this means that f is constant, which yields a contradiction.

Now, note that from the fact that σ is nonconstant and continuous it follows that there exists $y_0 \in (0, \infty)$ such that $\sigma|_{(y_0, y'_0)}$ is nonconstant for every $y'_0 > y_0$. Let $f^{(y_0)} : [0, \infty) \rightarrow \mathbb{R}$ be given by

$$f^{(y_0)}(x) = f(x + y_0) - f(y_0) \quad \text{for } x \in [0, \infty). \quad (16)$$

Then $f^{(y_0)}$ is nonconstant because $f|_{(y_0, \infty)}$ is nonconstant. Moreover, applying (1) with $y = y_0$, we obtain $f^{(y_0)}(\sigma(y_0)(x - y_0)) = \tau(y_0)f^{(y_0)}(x - y_0)$ for $x \in [y_0, \infty)$ and so

$$f^{(y_0)}(\sigma(y_0)z) = \tau(y_0)f^{(y_0)}(z) \quad \text{for } z \in [0, \infty). \quad (17)$$

Since $f^{(y_0)}$ is nonconstant, in view of (13) and (17), we get $\tau(y_0) \in (0, 1)$. Furthermore, from (17) we deduce that the function $p : \mathbb{R} \rightarrow \mathbb{R}$ defined by $p(z) = f^{(y_0)}(\sigma(y_0)^z)\tau(y_0)^{-z}$ for $z \in \mathbb{R}$, is continuous and 1-periodic, that is $p(z + 1) = p(z)$ for $z \in \mathbb{R}$. Taking (16) into account, we also have

$$f(x) = (x - y_0)^s p(\log_{\sigma(y_0)}(x - y_0)) + f(y_0) \quad \text{for } x \in (y_0, \infty), \quad (18)$$

where $s := \log_{\sigma(y_0)} \tau(y_0)$. Note that as $\sigma(y_0) \in (0, 1)$ and $\tau(y_0) \in (0, 1)$, we get $s > 0$. Moreover, setting into (1) f of the form (18), for every $x, y \in (y_0, \infty)$ such that $x \geq y$, we obtain

$$\begin{aligned} & (\sigma(y)x + (1 - \sigma(y))y - y_0)^s p(\log_{\sigma(y_0)}(\sigma(y)x + (1 - \sigma(y))y - y_0)) \\ &= \tau(y)(x - y_0)^s p(\log_{\sigma(y_0)}(x - y_0)) + (1 - \tau(y))(y - y_0)^s p(\log_{\sigma(y_0)}(y - y_0)). \end{aligned} \quad (19)$$

Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be given by $\phi(x) = \sigma(y_0)x + (1 - \sigma(y_0))y_0$ for $x \in \mathbb{R}$. Then ϕ is strictly increasing and the interval (y_0, ∞) is strongly invariant under ϕ , that is $\phi((y_0, \infty)) = (y_0, \infty)$. Furthermore

$$\sigma(y_0)^n \phi^{-n}(x) = (x - y_0) + \sigma(y_0)^n y_0 \quad \text{for } x \in \mathbb{R}, n \in \mathbb{N}. \quad (20)$$

Therefore, inserting into (19) $\phi^{-n}(x)$ instead of x and multiplying both sides of the equality by $\sigma(y_0)^{ns}$, for every $x, y \in (y_0, \infty)$ with $x \geq y$ and every $n \in \mathbb{N}$, we get

$$\begin{aligned} & [\sigma(y)(x - y_0) + \sigma(y_0)^n(y - y_0)(1 - \sigma(y))]^s \cdot \\ & \cdot p(\log_{\sigma(y_0)}(\sigma(y)\phi^{-n}(x) + (1 - \sigma(y))y - y_0)) \\ & = \tau(y)(x - y_0)^s p(\log_{\sigma(y_0)}(\phi^{-n}(x) - y_0)) \\ & \quad + \sigma(y_0)^{ns}(1 - \tau(y))(y - y_0)^s p(\log_{\sigma(y_0)}(y - y_0)). \end{aligned} \quad (21)$$

On the other hand, as p is 1-periodic, making use of (20), for every $x, y \in (y_0, \infty)$ and $n \in \mathbb{N}$, we obtain

$$\begin{aligned} & p(\log_{\sigma(y_0)}(\sigma(y)\phi^{-n}(x) + (1 - \sigma(y))y - y_0)) \\ & = p\left(\log_{\sigma(y_0)}\left(\frac{\sigma(y)\sigma(y_0)^n\phi^{-n}(x) + \sigma(y_0)^n(1 - \sigma(y))y - \sigma(y_0)^ny_0}{\sigma(y_0)^n}\right)\right) \\ & = p(\log_{\sigma(y_0)}(\sigma(y)(x - y_0) + \sigma(y_0)^n(y - y_0)(1 - \sigma(y))) - n) \\ & = p(\log_{\sigma(y_0)}(\sigma(y)(x - y_0) + \sigma(y_0)^n(y - y_0)(1 - \sigma(y)))). \end{aligned}$$

In the same way, we get $p(\log_{\sigma(y_0)}(\phi^{-n}(x) - y_0)) = p(\log_{\sigma(y_0)}(x - y_0))$ for $x \in (y_0, \infty)$, $n \in \mathbb{N}$. Therefore, taking (21) into account, for every $x, y \in (y_0, \infty)$ with $x \geq y$ and every $n \in \mathbb{N}$, we obtain

$$\begin{aligned} & [\sigma(y)(x - y_0) + \sigma(y_0)^n(y - y_0)(1 - \sigma(y))]^s \cdot \\ & \cdot p(\log_{\sigma(y_0)}(\sigma(y)(x - y_0) + \sigma(y_0)^n(y - y_0)(1 - \sigma(y)))) \\ & = \tau(y)(x - y_0)^s p(\log_{\sigma(y_0)}(x - y_0)) \\ & \quad + \sigma(y_0)^{ns}(1 - \tau(y))(y - y_0)^s p(\log_{\sigma(y_0)}(y - y_0)). \end{aligned}$$

As $\sigma(y_0) \in (0, 1)$, $s > 0$ and p is continuous, letting in the last equality $n \rightarrow \infty$, for every $x, y \in (y_0, \infty)$ with $x \geq y$, we get

$$[\sigma(y)(x - y_0)]^s p(\log_{\sigma(y_0)}(\sigma(y)(x - y_0))) = \tau(y)(x - y_0)^s p(\log_{\sigma(y_0)}(x - y_0)).$$

Hence, in view of (18), we obtain

$$f(\sigma(y)(x - y_0) + y_0) - f(y_0) = \tau(y)(f(x) - f(y_0)) \quad \text{for } x, y \in (y_0, \infty), x \geq y.$$

Thus, according to (16), we have

$$f^{(y_0)}(\sigma(y)z) = \tau(y)f^{(y_0)}(z) \quad \text{for } z \in (0, \infty), y \in (y_0, y_0 + z]. \quad (22)$$

Since $f^{(y_0)}$ is nonconstant, taking $z_0 \in (0, \infty)$ such that $f^{(y_0)}(z_0) \neq 0$ and applying (22) with $z = z_0$, we obtain $f^{(y_0)}(\sigma(y)z_0) = \tau(y)f^{(y_0)}(z_0)$ for $y \in (y_0, y_0 + z_0]$. Hence $\tau(y) = H(\sigma(y))$ for $y \in (y_0, y_0 + z_0]$, where $H(u) =$

$\frac{f^{(y_0)}(uz_0)}{f^{(y_0)}(z_0)}$ for $u \in (0, 1)$. Consequently, in view of (22), we get $f^{(y_0)}(\sigma(y)z) = H(\sigma(y))f^{(y_0)}(z)$ for $z \in (z_0, \infty)$, $y \in (y_0, y_0 + z_0]$. Thus

$$f^{(y_0)}(uz) = H(u)f^{(y_0)}(z) \quad \text{for } z \in (z_0, \infty), u \in \sigma((y_0, y_0 + z_0]). \quad (23)$$

In particular, (23) holds for $(z, u) \in (z_0, \infty) \times \text{int}[\sigma((y_0, y_0 + z_0])]$. Since σ is continuous and $\sigma|_{(y_0, y_0 + z_0]}$ is nonconstant, the latter set is a nonempty open and connected subset of $(0, \infty)^2$. Therefore, as $f^{(y_0)}$ is nonconstant, according to [5, Corollary 2], there exist a multiplicative function $m : (0, \infty) \rightarrow \mathbb{R} \setminus \{0\}$ and $a \in \mathbb{R} \setminus \{0\}$ such that $f^{(y_0)}(z) = am(z)$ for $z \in (z_0, \infty)$. Hence, in view of (16), we get

$$f(x) = am(x - y_0) + f(y_0) \quad \text{for } x \in (y_0 + z_0, \infty). \quad (24)$$

Furthermore, as f is continuous, m is continuous at every point of the interval (z_0, ∞) . Thus, according to [1, Proposition 6, p. 30], there is $r \in \mathbb{R}$ such that $m(x) = x^r$ for $x \in (0, \infty)$, which together with (24) gives

$$f(x) = a(x - y_0)^r + f(y_0) \quad \text{for } x \in (y_0 + z_0, \infty). \quad (25)$$

Note also that $r \neq 0$ because otherwise, in view of (25), $f|_{(y_0 + z_0, \infty)}$ would be constant. Moreover, for $y \in (0, \infty)$ and sufficiently large $x \in (0, \infty)$, we have

$$\sigma(y)x + (1 - \sigma(y))y > y_0 + z_0. \quad (26)$$

Hence, making use of (1) and (25), for sufficiently large $x \in (y_0 + z_0, \infty)$ and $y \in (0, x]$, we get

$$a(\sigma(y)x + (1 - \sigma(y))y - y_0)^r = a\tau(y)(x - y_0)^r + (1 - \tau(y))(f(y) - f(y_0)).$$

Differentiating both sides of the last equality with respect to x , for sufficiently large $x \in (y_0 + z_0, \infty)$ and $y \in (0, x]$, we obtain

$$\sigma(y)(\sigma(y)x + (1 - \sigma(y))y - y_0)^{r-1} = \tau(y)(x - y_0)^{r-1}.$$

Suppose that $r \neq 1$. Then from the above equality it follows that

$$\frac{(1 - \sigma(y))(y - y_0)}{x - y_0} = \left(\frac{\tau(y)}{\sigma(y)} \right)^{\frac{1}{r-1}} - \sigma(y)$$

for sufficiently large $x \in (y_0 + z_0, \infty)$ and $y \in (0, x]$. Differentiating both sides of this equality with respect to x , we obtain that $\frac{(1 - \sigma(y))(y - y_0)}{(x - y_0)^2} = 0$ for sufficiently large $x \in (y_0 + z_0, \infty)$ and $y \in (0, x]$. As $\sigma(y) \in (0, 1)$ for $y \in (0, \infty)$, this yields a contradiction. Thus $r = 1$, which in view of (25), implies that

$$f(x) = ax + b \quad \text{for } x \in (y_0 + z_0, \infty), \quad (27)$$

where $b := f(y_0) - ay_0$. Fix a $y \in (0, \infty)$. Then (26) holds for sufficiently large $x \in (y_0 + z_0, \infty)$, so taking (1) and (27) into account, after straightforward calculations we get

$$a(\sigma(y) - \tau(y))x = (1 - \tau(y))(f(y) - b) - a(1 - \sigma(y))y$$

for sufficiently large $x \in (\max\{y, y_0 + z_0\}, \infty)$. Thus $\sigma(y) = \tau(y)$ and

$$(1 - \tau(y))(f(y) - ay - b) = 0,$$

which in view of (13) gives $f(y) = ay + b$. Therefore $\sigma = \tau$ and, as f is continuous, we get (2).

The converse is easy to check. $\square$

Now, we are going to determine the solutions of equation (3). In order to formulate the results, let us recall that if X is a real linear space then a nonempty subset $\mathcal{C}$ of X is called a *convex cone* provided $\alpha x + \beta y \in \mathcal{C}$ for $x, y \in \mathcal{C}$ and $\alpha, \beta \in [0, \infty)$. Furthermore, a function $F : \mathcal{C} \rightarrow \mathbb{R}$ is said to be *continuous on rays* provided, for every $x \in \mathcal{C}$, a function $F_x : [0, \infty) \rightarrow \mathbb{R}$ given by $F_x(t) = F(tx)$ for $t \in [0, \infty)$, is continuous.

Theorem 2.2. *Let X be a real linear space, $\mathcal{C} \subseteq X$ be a convex cone, $F : \mathcal{C} \rightarrow \mathbb{R}$ and $\sigma, \tau : \mathcal{C} \rightarrow [0, 1]$. Assume that F and σ are continuous on rays, F is nonconstant and $\sigma(y) \in (0, 1)$ for $y \in \mathcal{C} \setminus \{0\}$. Then the triple (F, σ, τ) satisfies equation (3) if and only if $\sigma = \tau$ and there exist a nontrivial linear functional $L : \text{Lin } \mathcal{C} \rightarrow \mathbb{R}$ and $b \in \mathbb{R}$ such that*

$$F(x) = L(x) + b \quad \text{for } x \in \mathcal{C}. \quad (28)$$

Proof. Assume that the triple (F, σ, τ) satisfies equation (3). Let $\tilde{F} := F - F(0)$. Then, in view of (3), we get

$$\tilde{F}(\sigma(y)x + (1 - \sigma(y))y) = \tau(y)\tilde{F}(x) + (1 - \tau(y))\tilde{F}(y) \quad \text{for } x, y \in \mathcal{C}, x - y \in \mathcal{C}. \quad (29)$$

Furthermore, for every $x \in \mathcal{C}$, $\tilde{F}_x$ is continuous, $\tilde{F}_x(0) = \tilde{F}(0) = 0$ and, in view of (29), the triple $(\tilde{F}_x, \sigma_x, \tau_x)$ satisfies equation (1). Thus, according to Theorem 2.1, for every $x \in \mathcal{C}$ there exists $a(x) \in \mathbb{R}$ such that

$$\tilde{F}_x(t) = a(x)t \quad \text{for } t \in [0, \infty) \quad (30)$$

and

$$\sigma_x = \tau_x \quad \text{whenever } \tilde{F}_x \neq 0. \quad (31)$$

Note that, in view of (30), for every $x \in \mathcal{C}$ and $\alpha \in [0, \infty)$, we have

$$a(\alpha x)t = \tilde{F}_{\alpha x}(t) = \tilde{F}(\alpha tx) = \tilde{F}_x(\alpha t) = a(x)\alpha t \quad \text{for } t \in [0, \infty).$$

Hence $a(\alpha x) = \alpha a(x)$ for $x \in \mathcal{C}$, $\alpha \in [0, \infty)$. Since, according to (30), $\tilde{F}(x) = \tilde{F}_x(1) = a(x)$ for $x \in \mathcal{C}$, this implies that

$$\tilde{F}(\alpha x) = \alpha \tilde{F}(x) \quad \text{for } x \in \mathcal{C}, \alpha \in [0, \infty). \quad (32)$$

The remaining part of the proof is divided into three steps.

Step 1. We show that if $x, y \in \mathcal{C}$ are such that $x - y \in \mathcal{C}$ and either $\tilde{F}(x) = \tilde{F}(y) = 0$ or $\tilde{F}(y) \neq 0$, then $\tilde{F}(x + y) = \tilde{F}(x) + \tilde{F}(y)$.

To this end fix $x, y \in \mathcal{C}$ with $x - y \in \mathcal{C}$. If $x = 0$ or $y = 0$ then the equality follows from the fact that $\tilde{F}(0) = 0$. Assume that $x, y \neq 0$. First suppose

that $\tilde{F}(x) = \tilde{F}(y) = 0$. Since σ is continuous on rays, σ_y is continuous on $[0, \infty)$. Furthermore, we have $(1 - \sigma_y(t))t > 0$ for $t \in (0, \infty)$ and $\lim_{t \rightarrow 0^+} (1 - \sigma_y(t))t = 0$. Thus there exist $T \in (0, 1)$ and $n \in \mathbb{N}$ such that $(1 - \sigma(Ty))T = (1 - \sigma_y(T))T = \frac{1}{n}$. Obviously $\frac{x}{\sigma(Ty)}, Ty \in \mathcal{C}$. Moreover, as $\sigma(Ty) \in (0, 1)$, we have $\frac{1}{\sigma(Ty)} > T$ and so $\frac{x}{\sigma(Ty)} - Ty = T(x - y) + \left(\frac{1}{\sigma(Ty)} - T\right)x \in \mathcal{C}$. Therefore, in view of (29) and (32), we get

$$\begin{aligned} \tilde{F}\left(x + \frac{1}{n}y\right) &= \tilde{F}\left(\sigma(Ty)\frac{x}{\sigma(Ty)} + (1 - \sigma(Ty))Ty\right) \\ &= \tau(Ty)\tilde{F}\left(\frac{x}{\sigma(Ty)}\right) + (1 - \tau(Ty))\tilde{F}(Ty) \\ &= \frac{\tau(Ty)}{\sigma(Ty)}\tilde{F}(x) + (1 - \tau(Ty))T\tilde{F}(y) = 0. \end{aligned}$$

Hence, as $\frac{x + \frac{1}{n}y}{\sigma(Ty)}, Ty \in \mathcal{C}$ and $\frac{x + \frac{1}{n}y}{\sigma(Ty)} - Ty = T(x - y) + \frac{T}{n}y + \left(\frac{1}{\sigma(Ty)} - T\right)(x + \frac{1}{n}y) \in \mathcal{C}$, applying again (29) and (32), we obtain

$$\begin{aligned} \tilde{F}\left(x + \frac{2}{n}y\right) &= \tilde{F}\left(\left(x + \frac{1}{n}y\right) + \frac{1}{n}y\right) \\ &= \tilde{F}\left(\sigma(Ty)\frac{x + \frac{1}{n}y}{\sigma(Ty)} + (1 - \sigma(Ty))Ty\right) \\ &= \frac{\tau(Ty)}{\sigma(Ty)}\tilde{F}\left(x + \frac{1}{n}y\right) + (1 - \tau(Ty))T\tilde{F}(y) = 0. \end{aligned}$$

Repeating this procedure, after n steps we conclude that $\tilde{F}(x + y) = 0 = \tilde{F}(x) + \tilde{F}(y)$.

Now, suppose that $\tilde{F}(y) \neq 0$. As previously, let $T \in (0, 1)$ and $n \in \mathbb{N}$ be such that $(1 - \sigma(Ty))T = \frac{1}{n}$. Since $\frac{x}{\sigma(Ty)}, Ty, \frac{x}{\sigma(Ty)} - Ty \in \mathcal{C}$ and, in view of (31), $\tau(Ty) = \sigma(Ty)$, making use of (29) and (32), we obtain

$$\begin{aligned} \tilde{F}\left(x + \frac{1}{n}y\right) &= \tilde{F}\left(\sigma(Ty)\frac{x}{\sigma(Ty)} + (1 - \sigma(Ty))Ty\right) = \sigma(Ty)\tilde{F}\left(\frac{x}{\sigma(Ty)}\right) \\ &\quad + (1 - \sigma(Ty))\tilde{F}(Ty) = \tilde{F}(x) + (1 - \sigma(Ty))T\tilde{F}(y) = \tilde{F}(x) + \frac{1}{n}\tilde{F}(y). \end{aligned}$$

Repeating this procedure, as in the previous case, after n steps we get $\tilde{F}(x + y) = \tilde{F}(x) + \tilde{F}(y)$.

Step 2. We show that

$$\tilde{F}(x + y) = \tilde{F}(x) + \tilde{F}(y) \quad \text{for } x, y \in \mathcal{C}. \quad (33)$$

Fix $x, y \in \mathcal{C}$. Since $\tilde{F}(0) = 0$, the case where $x = 0$ or $y = 0$ is trivial. So, assume that $x, y \neq 0$. First we prove that

$$\tilde{F}(u + v) = \tilde{F}(u) + \tilde{F}(v) \quad \text{for } u, v \in \mathcal{C}(x, y), \quad (34)$$

where $\mathcal{C}(x, y) := \{\alpha x + \beta y : \alpha, \beta \in (0, \infty)\}$. Note that $\mathcal{C}(x, y) \subseteq \mathcal{C}$ is a convex cone. Let $u, v \in \mathcal{C}(x, y)$. Without loss of generality we may assume that either $\tilde{F}(u) = \tilde{F}(v) = 0$ or $\tilde{F}(v) \neq 0$. Furthermore $u = \alpha_1 x + \beta_1 y$ and $v = \alpha_2 x + \beta_2 y$ with some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in (0, \infty)$. Thus there is $n \in \mathbb{N}$ such that $u - \frac{1}{n}v = (\alpha_1 - \frac{1}{n}\alpha_2)x + (\beta_1 - \frac{1}{n}\beta_2)y \in \mathcal{C}(x, y) \subseteq \mathcal{C}$. Moreover, in view of (32), we have $\tilde{F}(u) = \tilde{F}(\frac{1}{n}v) = 0$ whenever $\tilde{F}(u) = \tilde{F}(v) = 0$, and $\tilde{F}(\frac{1}{n}v) \neq 0$ whenever $\tilde{F}(v) \neq 0$. Therefore, applying the result of Step 1 and taking (32) into account, we get

$$\tilde{F}\left(u + \frac{1}{n}v\right) = \tilde{F}(u) + \frac{1}{n}\tilde{F}(v). \quad (35)$$

In particular $\tilde{F}(u + \frac{1}{n}v) = 0$ whenever $\tilde{F}(u) = \tilde{F}(v) = 0$. Thus, as $u + \frac{1}{n}v \in \mathcal{C}$, $\frac{1}{n}v \in \mathcal{C}$ and $(u + \frac{1}{n}v) - \frac{1}{n}v = u \in \mathcal{C}$, applying again the result of Step 1, in view of (32), we obtain $\tilde{F}(u + \frac{2}{n}v) = \tilde{F}((u + \frac{1}{n}v) + \frac{1}{n}v) = \tilde{F}(u + \frac{1}{n}v) + \frac{1}{n}\tilde{F}(v)$, which together with (35) gives $\tilde{F}(u + \frac{2}{n}v) = \tilde{F}(u) + \frac{2}{n}\tilde{F}(v)$. Repeating this arguments we conclude that $\tilde{F}(u + v) = \tilde{F}(u) + \tilde{F}(v)$ and so (34) is proved.

Now, as $\mathcal{C}(x, y) \subseteq \mathcal{C}$ is a convex cone, according to [4, Theorem 4.4.1], from (32) and (34) it follows that there exists a unique linear functional $\tilde{L} : \text{Lin } \mathcal{C}(x, y) \rightarrow \mathbb{R}$ such that

$$\tilde{F}(u) = \tilde{L}(u) \quad \text{for } u \in \mathcal{C}(x, y). \quad (36)$$

Since $y \neq 0$, we get $\sigma(y) \in (0, 1)$ and so

$$\sigma(y)w + (1 - \sigma(y))y \in \mathcal{C}(x, y) \quad \text{for } w \in \mathcal{C}(x, y). \quad (37)$$

If $\mathcal{C}(x, y) \subseteq \ker \tilde{L}$ then $\alpha\tilde{L}(x) + \beta\tilde{L}(y) = 0$ for $\alpha, \beta \in (0, \infty)$, whence $\tilde{L}(x) = \tilde{L}(y) = 0$. Moreover, taking $w \in \mathcal{C}(x, y)$, we get $w = \alpha_w x + \beta_w y$ with some $\alpha_w, \beta_w \in (0, \infty)$. Thus $sw - y = s\alpha_w x + (s\beta_w - 1)y \in \mathcal{C}(x, y)$ for $s \in (\frac{1}{\beta_w}, \infty)$ and so, making use of (29), (36) and (37), for every $s \in (\frac{1}{\beta_w}, \infty)$, we get

$$\begin{aligned} 0 &= \tilde{L}(\sigma(y)sw + (1 - \sigma(y))y) = \tilde{F}(\sigma(y)sw + (1 - \sigma(y))y) \\ &= \tau(y)\tilde{F}(sw) + (1 - \tau(y))\tilde{F}(y) = (1 - \tau(y))\tilde{F}(y). \end{aligned}$$

Hence $\tilde{F}(y) = 0$ because otherwise, according to (31), we would have

$$\tau(y) = \tau_y(1) = \sigma_y(1) = \sigma(y) \in (0, 1)$$

and so $(1 - \tau(y))\tilde{F}(y) \neq 0$. Similarly, we get $\tilde{F}(x) = 0$. Consequently, as $x + y \in \mathcal{C}(x, y)$, taking (36) into account, we obtain $\tilde{F}(x + y) = \tilde{L}(x + y) = 0 = \tilde{F}(x) + \tilde{F}(y)$.

If $\tilde{L}(w) \neq 0$ for some $w \in \mathcal{C}(x, y)$ then, as previously, we get $sw - y \in \mathcal{C}(x, y)$ for sufficiently large $s \in (0, \infty)$. Hence, in view of (29), (36) and (37), after straightforward calculations, we get

$$(\sigma(y) - \tau(y))s\tilde{L}(w) = (1 - \tau(y))\tilde{F}(y) - (1 - \sigma(y))\tilde{L}(y)$$

for sufficiently large $s \in (0, \infty)$. Since $\tilde{L}(w) \neq 0$, this means that $\sigma(y) = \tau(y)$ and so $(1 - \sigma(y))(\tilde{F}(y) - \tilde{L}(y)) = 0$. Hence, as $\sigma(y) \in (0, 1)$, we get $\tilde{F}(y) = \tilde{L}(y)$. In the same way we obtain that $\tilde{F}(x) = \tilde{L}(x)$. Therefore, in view of (36), we have $\tilde{F}(x + y) = \tilde{L}(x + y) = \tilde{L}(x) + \tilde{L}(y) = \tilde{F}(x) + \tilde{F}(y)$. So, (33) is proved.

Step 3. Since $\tilde{F}$ is nonconstant, applying [4, Theorem 4.4.1], from (32) and (33) we derive that there exists a unique nontrivial linear functional $L : \text{Lin } \mathcal{C} \rightarrow \mathbb{R}$ such that $\tilde{F}(x) = L(x)$ for $x \in \mathcal{C}$. Hence $F(x) = L(x) + F(0)$ for $x \in \mathcal{C}$, so (28) holds with $b := F(0)$. Furthermore, inserting into (3) F of the form (28), we obtain

$$(\sigma(y) - \tau(y))L(x - y) = 0 \quad \text{for } x, y \in \mathcal{C}, x - y \in \mathcal{C}. \quad (38)$$

Since $\tilde{F}$ is nonconstant, we have $L(z) = \tilde{F}(z) \neq 0$ for some $z \in \mathcal{C}$. Therefore, taking $y \in \mathcal{C}$ and $x = y + z \in \mathcal{C}$, in view of (38), we get $\sigma(y) = \tau(y)$. In this way we have proved that $\sigma = \tau$.

The converse is easy to check. $\square$

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