On continuous on rays solutions of a composite-type equation

ELIZA JABŁOŃSKA

Abstract. Let X be a real linear space. We characterize solutions $f, g : X \rightarrow \mathbb{R}$ of the equation $f(x + g(x)y) = f(x)f(y)$, where f is continuous on rays. Our result refers to papers by Brzdęk (Acta Math Hungar 101:281–291, 2003), Chudziak (Aequat Math, doi:10.1007/s00010-013-0228-4, 2013) and Jabłońska (J Math Anal Appl 375:223–229, 2011).

Mathematics Subject Classification (2010). Primary 39B52; Secondary 26A15.

Keywords. Generalized Gołąb–Schinzel equation, continuous on rays solutions.

1. Introduction

The Gołąb–Schinzel equation

$$f(x + f(x)y) = f(x)f(y)$$

was introduced in [5] in connection with looking for subgroups of the centroaffine group of the field. This equation and its generalizations seem to be very important composite-type functional equations because of their applications, especially in algebra, in the theory of geometric objects, as well as in differential equations appearing in meteorology and fluid mechanics (the extensive bibliography concerning the Gołąb–Schinzel-type functional equations and their applications can be found in [2] and [8]).

In 2012 in [7] it was proved that solutions of the pexiderized Gołąb–Schinzel equation, i.e. the equation

$$f(x + g(x)y) = h(x)k(y)$$

with four unknown functions, can be described by solutions of the following equation:

$$f(x + g(x)y) = f(x)f(y) \tag{1.1}$$

with two unknown functions. That is why the equation (1.1) seems to be one of the most important Gołąb–Schinzel-type equations.

For the first time the equation (1.1) was studied in 2006 by J. Chudziak [3]. Among others, he determined all real solutions of this equation under the assumption that g is continuous. Seven years later, in [9], real solutions of (1.1) were studied under the assumption that the function f is continuous. It turned out that the following theorem holds (which is essential in our considerations):

Theorem 1.1. [9, Theorem 4] *Functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$ satisfy (1.1) and f is continuous if and only if one of the following conditions holds:*

- (i) g is arbitrary and either $f = 0$, or $f = 1$;
- (ii) there is $c \in \mathbb{R} \setminus \{0\}$ such that

$$\begin{cases} g = 1, \\ f(x) = e^{cx} \text{ for } x \in \mathbb{R}; \end{cases} \quad (1.2)$$

- (iii) there are $c \in \mathbb{R} \setminus \{0\}$ and $r > 0$ such that f and g have one of the forms:

$$\begin{cases} g(x) = cx + 1 & \text{for } x \in \mathbb{R}, \\ f(x) = |cx + 1|^r & \text{for } x \in \mathbb{R}, \end{cases} \quad (1.3)$$

$$\begin{cases} g(x) = cx + 1 & \text{for } x \in \mathbb{R}, \\ f(x) = |cx + 1|^r \operatorname{sgn}(cx + 1) & \text{for } x \in \mathbb{R}, \end{cases} \quad (1.4)$$

$$\begin{cases} g(x) = \max\{0, cx + 1\} & \text{for } x \in \mathbb{R}, \\ f(x) = (\max\{0, cx + 1\})^r & \text{for } x \in \mathbb{R}. \end{cases} \quad (1.5)$$

Consequently, we obtain that the continuity of each nonconstant the function f implies the continuity of g provided f, g satisfy (1.1).

Here we prove that if f, g satisfy (1.1), then the continuity on rays of the nonconstant function f implies the continuity on rays of g .

For the first time solutions of (1.1) in the class of continuous on rays functions f and g were studied in [6]. J. Chudziak [4] generalized this result, assuming only the continuity on rays of g . We also show, how to obtain forms of solutions of (1.1) using Chudziak's result from [4] under the assumption that f is continuous on rays.

Our paper also refers to the paper [1], where J. Brzdęk considered the equation

$$f(x + M(f(x))y) = H(f(x), f(y))$$

under the assumptions that f is continuous on rays, M is continuous and H is symmetric.

In the whole paper we use the following notation:

$$A := f^{-1}(\{1\}), \quad B := g^{-1}(\{1\}), \quad F := \{x \in X : f(x) \neq 0\}.$$

Moreover, X is a real linear space.

2. Preliminary lemmas

First we recall two lemmas on basic properties of solutions of (1.1), which will be useful in the sequel.

Lemma 2.1. [6, Lemma 1] *Let $f, g : X \rightarrow \mathbb{R}$, $f \neq 1$ and $f \neq 0$. If f and g satisfy equation (1.1), then:*

- (i) $f(0) = 1$;
- (ii) $F = \{x \in X : g(x) \neq 0\}$;
- (iii) $y - x \in g(x)A$ for every $x, y \in F$ with $f(x) = f(y)$;
- (iv) $(g(x)^{-1} - 1)z \in A$ for every $x \in F$ and $z \in B$.

Lemma 2.2. [6, Lemma 2] *Let $f, g : X \rightarrow \mathbb{R}$, $f(X) \setminus \{0, 1\} \neq \emptyset$ and $g(X) \setminus \{0, 1\} \neq \emptyset$. If f and g satisfy (1.1) and A is a linear space, then $A = B$.*

Let us recall that a function $k : X \rightarrow \mathbb{R}$ is continuous on rays if and only if $k_x : \mathbb{R} \rightarrow \mathbb{R}$ given by $k_x(a) := k(ax)$ is continuous for each $x \in X \setminus \{0\}$.

Lemma 2.3. *Let $f, g : X \rightarrow \mathbb{R}$ be nonconstant functions satisfying (1.1) and let f be continuous on rays. Then $g(X) \not\subset \{0, 1\}$.*

Proof. For the proof by contradiction suppose that $g(X) \subset \{0, 1\}$. Then, for each $x \in X \setminus \{0\}$, $f_x, g_x : \mathbb{R} \rightarrow \mathbb{R}$ satisfy (1.1) and f_x is continuous, so f_x, g_x are given by one of conditions (i)–(ii) of Theorem 1.1. In view of Lemma 2.1(i) $f_x(0) = f(0) = 1$ for each $x \in X \setminus \{0\}$, so $f_x \neq 0$. Hence $f_x = 1$, or

$$f_x(\alpha) = e^{c_x \alpha} \quad \text{for } \alpha \in \mathbb{R}$$

for $x \in X \setminus \{0\}$. It means that $0 \notin f(X)$ and, according to Lemma 2.1(ii), $0 \notin g(X)$. Thus $g = 1$, which gives a contradiction. $\square$

3. The main result

Theorem 3.1. *Functions $f, g : X \rightarrow \mathbb{R}$ satisfy (1.1) and f is continuous on rays if and only if one of the following conditions holds:*

- (i) $f = 0$ or $f = 1$;
- (ii) $g = 1$ and there is a linear functional $L : X \rightarrow \mathbb{R}$ such that $f = \exp L$;
- (iii) there are a nontrivial linear functional $L : X \rightarrow \mathbb{R}$ and some $r > 0$ such that f and g have one of the following forms:

$$\begin{cases} g(x) = L(x) + 1 & \text{for } x \in X, \\ f(x) = |L(x) + 1|^r \operatorname{sgn}(L(x) + 1) & \text{for } x \in X, \end{cases} \quad (3.1)$$

$$\begin{cases} g(x) = L(x) + 1 & \text{for } x \in X, \\ f(x) = |L(x) + 1|^r & \text{for } x \in X, \end{cases} \quad (3.2)$$

$$\begin{cases} g(x) = \max\{L(x) + 1, 0\} & \text{for } x \in X, \\ f(x) = (\max\{L(x) + 1, 0\})^r & \text{for } x \in X. \end{cases} \quad (3.3)$$

Proof. Let f and g satisfy (1.1) and f be continuous on rays. Clearly, according to (1.1), if f is constant, then $f = 1$ or $f = 0$ and (i) holds.

So, assume that f is not constant and $g = c$ for some $c \in \mathbb{R}$. Clearly, by (1.1), $c \neq 0$. Suppose that $c \neq 1$. Then, in view of (1.1),

$$f\left(\frac{x}{1-g(x)}\right) = f\left(x + g(x)\frac{x}{1-g(x)}\right) = f(x)f\left(\frac{x}{1-g(x)}\right)$$

for $x \in X$. According to Lemma 2.1(ii) $0 \notin f(X)$, so $f = 1$, which contradicts the assumption. Thus $g = 1$ and, using (1.1), we find that f is a nonconstant exponential function. Hence, according to [10, Theorem 13.1.1], $f = \exp L$, where $L : X \rightarrow \mathbb{R}$ is a continuous on rays nonzero additive function. Since $L_x : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and additive for $x \in X \setminus \{0\}$, there is $c_x \in \mathbb{R}$ such that $L_x(a) = c_x a$ for $a \in \mathbb{R}$ (see e.g. [10, Theorem 13.1.1]). Thus $L(ax) = L_x(a) = c_x a = L_x(1)a = L(x)a$ for $x \in X \setminus \{0\}$ and $a \in \mathbb{R}$. It means that L is a nontrivial linear functional and condition (ii) holds.

Now assume that neither of the functions f, g is constant. Clearly, for each $x \in X \setminus \{0\}$ functions $f_x, g_x : \mathbb{R} \rightarrow \mathbb{R}$ satisfy (1.1) and f_x is continuous. Hence functions f_x, g_x satisfy one of conditions (i)-(iii) of Theorem 1.1 for $x \in X \setminus \{0\}$ and, in view of Lemma 2.1(i), $f_x \neq 0$ for each $x \in X \setminus \{0\}$.

First we prove a property of f, g , which is essential in the whole proof:

(A) for each $x \in A$ either

$$\alpha x \in A \text{ for each } \alpha \in \mathbb{R}, \quad (3.4)$$

or there is $r > 0$ such that

$$\begin{cases} g_x(\alpha) = -2\alpha + 1 & \text{for } \alpha \in \mathbb{R}, \\ f_x(\alpha) = |-2\alpha + 1|^r & \text{for } \alpha \in \mathbb{R}. \end{cases} \quad (3.5)$$

To see it, take any $x \in A$; i.e. $f_x(1) = f(x) = 1$. If $f_x = 1$, then (3.4) holds. By Theorem 1.1, if f_x and g_x satisfy one of conditions (1.2), (1.4), (1.5), then $1 = f_x(1) \in \{e^{c_x}, |c_x + 1|^r \operatorname{sgn}(c_x + 1), (\max\{0, c_x + 1\})^r\}$ with $c_x \neq 0$, which gives a contradiction. Finally, if f_x and g_x satisfy (1.3), then $1 = f_x(1) = |c_x + 1|^r$. Hence $c_x = -2$ and we obtain (3.5).

Next, we will show that g is continuous on rays (then we will be able to apply Chudziak's result from [4]).

Case 1. First consider the case where (3.4) holds for each $x \in A$. It means that $\mathbb{R}A \subset A$. By mathematical induction we prove that

$$\sum_{i=1}^n \mathbb{R}A = \left\{ \sum_{i=1}^n a_i z_i : a_i \in \mathbb{R}, z_i \in A \right\} \subset A \text{ for each } n \in \mathbb{N}. \quad (3.6)$$

So, assume that $\sum_{i=1}^n \mathbb{R}A \subset A$ for some $n \geq 1$. Then, by (1.1) and the first step of induction, for every $z_i \in A$ and $a_i \in \mathbb{R}$, where $i \in \{1, \dots, n+1\}$, we obtain

$$\begin{aligned} f\left(\sum_{i=1}^{n+1} a_i z_i\right) &= f\left(\sum_{i=1}^n a_i z_i + a_{n+1} z_{n+1}\right) \\ &= f\left(a_{n+1} z_{n+1} + g(a_{n+1} z_{n+1}) g(a_{n+1} z_{n+1})^{-1} \sum_{i=1}^n a_i z_i\right) \\ &= f(a_{n+1} z_{n+1}) f\left(\sum_{i=1}^n a_i g(a_{n+1} z_{n+1})^{-1} z_i\right) = 1. \end{aligned}$$

It means that $\sum_{i=1}^{n+1} \mathbb{R}A \subset A$ and hence, by mathematical induction, (3.6) holds and A is a linear subspace of X . Now, from Lemma 2.2 and Lemma 2.3 we have $A = B$. Hence, since (3.4) holds for each $x \in A$, we obtain that $g_x = 1$ for each $x \in A$. Moreover, if $x \in X \setminus A = X \setminus B$, then, in view of Theorem 1.1, g_x is given by one of conditions (1.3)-(1.5). It means that g is continuous on rays.

Case 2. Now, we consider the case where there is $x_0 \in A$ such that g_{x_0}, f_{x_0} are given by (3.5). Then $f(x_0) = f_{x_0}(1) = 1$, $g(x_0) = g_{x_0}(1) = -1$ and, for each $x \in F$,

$$f(x_0 - x) = f(x_0 + g(x_0)x) = f(x_0)f(x) = f(x) \neq 0.$$

Hence, according to Lemma 2.1(iii),

$$\frac{x_0 - 2x}{g(x)} \in A \text{ for } x \in F.$$

Thus, by the property $(\mathcal{A})$, one of conditions (3.4)-(3.5) holds for $x \in F$; i.e. for $x \in F$ either

$$f_{\frac{x_0 - 2x}{g(x)}}(\alpha) = 1 \text{ for } \alpha \in \mathbb{R}, \quad (3.7)$$

or there is $r > 0$ such that

$$\begin{cases} f_{\frac{x_0 - 2x}{g(x)}}(\alpha) = |-2\alpha + 1|^r & \text{for } \alpha \in \mathbb{R}, \\ g_{\frac{x_0 - 2x}{g(x)}}(\alpha) = -2\alpha + 1 & \text{for } \alpha \in \mathbb{R}. \end{cases} \quad (3.8)$$

First we show that (3.8) holds for each $x \in F$. Suppose for contradiction that there is $x \in F$ such that (3.7) holds. Then

$$\begin{aligned}
1 &= f(\beta(x_0 - 2x)) = f(\beta x_0 - 2\beta x) = f\left(\beta x_0 + g(\beta x_0) \frac{-2\beta x}{g(\beta x_0)}\right) \\
&= f_{x_0}(\beta) f\left(\frac{-2\beta x}{g_{x_0}(\beta)}\right) = |-2\beta + 1|^r f_x\left(\frac{-2\beta}{-2\beta + 1}\right) = |-2\beta + 1|^r f_x\left(\frac{1}{1 - \frac{1}{2\beta}}\right)
\end{aligned}$$

for each $\beta \in \mathbb{R} \setminus \{0, \frac{1}{2}\}$. Hence

$$f_x\left(\frac{1}{1 - \frac{1}{2\beta}}\right) = \left|\frac{1}{-2\beta + 1}\right|^r \quad \text{for } \beta \in \mathbb{R} \setminus \left\{0, \frac{1}{2}\right\}.$$

Take any sequence $(\beta_n)_{n \in \mathbb{N}} \subset \mathbb{R} \setminus \{0, \frac{1}{2}\}$ which tends to ∞ . Then, the continuity of f_x implies the following equality:

$$f(x) = f_x(1) = \lim_{n \rightarrow \infty} f_x\left(\frac{1}{1 - \frac{1}{2\beta_n}}\right) = \lim_{n \rightarrow \infty} \left|\frac{1}{-2\beta_n + 1}\right|^r = 0,$$

which contradicts $x \in F$.

Now, since (3.8) holds for each $x \in F$, we get

$$f\left(\frac{x_0}{2} - x\right) = f\left(\frac{x_0 - 2x}{2g(x)} g(x)\right) = f_{\frac{x_0 - 2x}{g(x)}}\left(\frac{g(x)}{2}\right) = |-g(x) + 1|^r$$

and, consequently, in view of Lemma 2.1(ii),

$$\begin{aligned}
|1 - g(x)|^r &= f\left(\frac{x_0}{2} - x\right) = f\left(x_0 - \left(x + \frac{x_0}{2}\right)\right) \\
&= f\left(x_0 + g(x_0)\left(x + \frac{x_0}{2}\right)\right) = f(x_0) f\left(x + \frac{x_0}{2}\right) \\
&= f\left(x + g(x) \frac{x_0}{2g(x)}\right) = f(x) f_{x_0}\left(\frac{1}{2g(x)}\right) \\
&= f(x) \left|-2 \frac{1}{2g(x)} + 1\right|^r = f(x) \left|\frac{1 - g(x)}{g(x)}\right|^r
\end{aligned}$$

for $x \in F$. Thus $f(x) = |g(x)|^r$ for each $x \in F \setminus B$. By (3.8) (for $\alpha = \frac{1}{4}$) $\frac{1}{2} \in g(X)$ and whence, according to Lemma 2.1(iv), $B \subset A$. In this way we obtain that $f(x) = |g(x)|^r$ for $x \in F$ and thus, by Lemma 2.1(ii),

$$f(x) = |g(x)|^r \quad \text{for each } x \in X. \quad (3.9)$$

Having (3.9), using Theorem 1.1, it is easy to check that for each $x \in X \setminus \{0\}$ functions f_x and g_x have one of the following forms:

$$f_x = 1 \quad \text{and} \quad g_x(\mathbb{R}) \subset \{-1, 1\}, \quad (3.10)$$

$$\begin{cases} f_x(\alpha) = |c_x \alpha + 1|^r & \text{for } \alpha \in \mathbb{R}, \\ g_x(\alpha) = c_x \alpha + 1 & \text{for } \alpha \in \mathbb{R}, \end{cases} \quad (3.11)$$

$$\begin{cases} f_x(\alpha) = (\max\{0, c_x \alpha + 1\})^r & \text{for } \alpha \in \mathbb{R}, \\ g_x(\alpha) = \max\{0, c_x \alpha + 1\} & \text{for } \alpha \in \mathbb{R}, \end{cases} \quad (3.12)$$

where $r > 0$ (conditions (1.2), (1.4) do not hold, because $c_x \neq 0$).

Now, we prove that if f_y, g_y fulfill (3.10) for some $y \in X \setminus \{0\}$, then $g_y = 1$. So, let $f_y = 1$ and $g_y(\mathbb{R}) \subset \{-1, 1\}$ for $y \in X \setminus \{0\}$ and suppose for contradiction that there is $\alpha \in \mathbb{R}$ with $g_y(\alpha) = -1$. Hence, by Lemma 2.1(ii), for each $x \in F$ we have

$$\begin{aligned} f_x(1) &= f(x)f_y\left(\frac{\alpha}{g(x)}\right) = f\left(x + g(x)\frac{\alpha y}{g(x)}\right) = f(x + \alpha y) \\ &= f(\alpha y + g(\alpha y)(-x)) = f(\alpha y)f(-x) = f_x(-1). \end{aligned}$$

But f_x is given by (3.11) or (3.12) for $x \in F \setminus A$, so

$$|c_x + 1|^r = f_x(1) = f_x(-1) = |-c_x + 1|^r$$

and whence $c_x = 0$, which is a contradiction.

In this way we obtain that either $f_x = 1$ and $g_x = 1$, or g_x, f_x satisfy one of conditions (3.11), (3.12) for $x \in X \setminus \{0\}$. It means that g is continuous on rays.

In both cases the continuity on rays of g was proved. Now, we can apply Chudziak's result; by [4, Proposition 3] f and g are given by one of the following two forms:

$$\begin{cases} g(x) = L(x) + 1 & \text{for } x \in X, \\ f(x) = \psi(L(x) + 1) & \text{for } x \in X, \end{cases} \quad (3.13)$$

where $L : X \rightarrow \mathbb{R}$ is a nontrivial linear functional and $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is a nonconstant multiplicative function, or

$$\begin{cases} g(x) = \max\{0, L(x) + 1\} & \text{for } x \in X, \\ f(x) = \begin{cases} \psi(L(x) + 1) & \text{for } x \in X \text{ with } L(x) + 1 \geq 0; \\ 0 & \text{otherwise} \end{cases} \end{cases} \quad (3.14)$$

with a nontrivial linear functional $L : X \rightarrow \mathbb{R}$ and a nonconstant multiplicative function $\psi : [0, \infty) \rightarrow [0, \infty)$.

First assume that f, g satisfy (3.13). Then there is $x \in X \setminus \{0\}$ such that $L(x) \neq 0$. Moreover,

$$f_x(\alpha) = \psi(L(\alpha x) + 1) = \psi(L(x)\alpha + 1) \quad \text{for } \alpha \in \mathbb{R}. \quad (3.15)$$

For each $u \in \mathbb{R}$ put $\alpha = \frac{u-1}{L(x)}$. Then $u = L(x)\alpha + 1$ and, by (3.15),

$$\psi(u) = f_x\left(\frac{u-1}{L(x)}\right) \quad (3.16)$$

for $u \in \mathbb{R}$. Consequently, the continuity of f_x implies the continuity of ψ . But ψ is a nonconstant multiplicative function, so there is $r > 0$ such that one of conditions (3.1), (3.2) holds (see e.g. [10, Theorem 13.1.6]).

Finally, let f, g satisfy (3.14). Then there is $x \in X \setminus \{0\}$ such that $L(x) > 0$. For each $u \in [0, \infty)$ put $\alpha = \frac{u-1}{L(x)}$. Then $u = L(x)\alpha + 1$ and (3.15), (3.16) hold for $u \in [0, \infty)$. Hence ψ is a nonconstant multiplicative continuous function, so there is $r > 0$ such that (3.3) holds (see e.g. [10, Theorem 13.1.6]).

It is easy to check that functions f and g given by one of conditions (i)-(iii) satisfy (1.1) and f is continuous on rays. This ends the proof. $\square$

Open Access. This article is distributed under the terms of the Creative Commons Attribution License which permits any use, distribution, and reproduction in any medium, provided the original author(s) and the source are credited.

References

- [1] Brzdęk, J.: A generalization of addition formulae. *Acta Math. Hungar.* **101**, 281–291 (2003)
- [2] Brzdęk, J.: The Gołąb–Schinzel equation and its generalizations. *Aequat. Math.* **70**, 14–24 (2005)
- [3] Chudziak, J.: Semigroup-valued solutions of the Gołąb–Schinzel functional equation. *Abh. Math. Sem. Univ. Hamburg* **76**, 91–98 (2006)
- [4] Chudziak, J.: Semigroup-valued solutions of some composite equations. *Aequat. Math.* (2013). doi:[10.1007/s00010-013-0228-4](https://doi.org/10.1007/s00010-013-0228-4) (in press)
- [5] Gołąb, S., Schinzel, A.: Sur l'équation fonctionnelle $f(x + f(x)y) = f(x)f(y)$. *Publ. Math. Debrecen* **6**, 113–125 (1959)
- [6] Jabłońska, E.: Continuous on rays solutions of an equation of the Gołąb–Schinzel type. *J. Math. Anal. Appl.* **375**, 223–229 (2011)
- [7] Jabłońska, E.: Remarks concerning the pexiderized Gołąb–Schinzel functional equation. *J. Math. Appl.* **35**, 33–38 (2012)
- [8] Jabłońska, E.: On solutions of some generalizations of the Gołąb–Schinzel equation. In: Brzdęk, J., Rassias, Th.M. (eds.) *Functional Equations in Mathematical Analysis*, pp. 509–521. Springer Optimization and its Applications, vol. 52. Springer, New York (2012)
- [9] Jabłońska, E.: On continuous solutions of an equation of the Gołąb–Schinzel type. *Bull. Aust. Math. Soc.* **87**, 10–17 (2013)
- [10] Kuczma, M.: An introduction to the theory of functional equations and inequalities. In: Gilányi, A. (ed.) *Cauchy's Equation and Jensen's Inequality*, 2nd edn. Birkhäuser Verlag, Basel (2009)

Eliza Jabłońska
Department of Mathematics
Rzeszów University of Technology
Powstańców Warszawy 12
35-959 Rzeszów
Poland
e-mail: elizapie@prz.edu.pl

Received: September 21, 2013

Revised: October 29, 2013