Additive involutions and Hamel bases

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Abstract. Our aim is to give other proofs of some slight generalizations of results from Baron (Aequat Math, 2013). We describe larger classes of discontinuous additive involutions $a : X \rightarrow X$ on a topological vector space X such that $a(H) \setminus H \neq \emptyset$ holds for a sufficiently numerous set $H \subset X$ of vectors linearly independent over $\mathbb{Q}$. We also consider the topological vector space $\mathcal{A}_X$ of all additive functions $a : X \rightarrow X$ with the topology induced by the Tychonoff topology of the space X^X . We prove in a simple way that some classes of discontinuous additive involutions are dense in the topological vector space $\mathcal{A}_X$.

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1. Introduction

Following a footnote in [2, p. 325], K. Baron constructed in [1] an example of a class of discontinuous additive involutions $a : X \rightarrow X$ on a real topological vector space X satisfying

$$a(H) \setminus H \neq \emptyset \quad (1)$$

for every at least three-element set $H \subset X$ of vectors linearly independent over $\mathbb{Q}$. The following result was also proved there.

Theorem 1. *Let $X \neq \{0\}$ be a real topological vector space and let us consider the linear topological space X^X of functions from X into X with the usual addition and multiplication by scalars and with the Tychonoff topology. Put*

$$\mathcal{A}_X = \{a \in X^X : a \text{ is additive}\}$$

and consider $\mathcal{A}_X$ with the topology induced by X^X . The sets

- $\{a \in \mathcal{A}_X : a \text{ is a discontinuous involution and (1) holds for every uncountable set } H \subset X \text{ which is linearly independent over } \mathbb{Q}\},$*
- $\{a \in \mathcal{A}_X : a \text{ is a discontinuous involution and } a(H) = H \text{ for a Hamel basis } H \text{ of the space } X \text{ over the field } \mathbb{Q}\}$*

are dense in $\mathcal{A}_X$.

The main purpose of the paper is to present generalizations of the mentioned results which will be equipped with simplest proofs. The new approach is based on the fact that any additive function on a linear space is a $\mathbb{Q}$ -linear operator. We begin with examples of larger classes of discontinuous additive involutions $a : X \rightarrow X$ such that (1) holds for a sufficiently numerous set $H \subset X$ of vectors linearly independent over $\mathbb{Q}$. Further, we give a slight generalization of Theorem 1 with an elementary and simple proof.

2. Additive involutions

By an involution on a nonempty set X we mean every function $f : X \rightarrow X$ such that $f \circ f = \text{id}_X$. It is a trivial observation that if $f, g : X \rightarrow X$ and $f \circ g$ is an injective function, then g is, too. Similarly, if $f \circ g$ is surjective, then so is f . These two simple facts imply that any involution is a bijective mapping.

Assume now that X is a real vector space. By a Hamel basis of the space X we mean an arbitrary basis of the space X over $\mathbb{Q}$. Put

$$\mathcal{A}_X = \{a \in X^X : a \text{ is additive on } X\}.$$

It is known that every additive mapping $a : X \rightarrow X$ is $\mathbb{Q}$ -linear (the proof of this simple fact for $X = \mathbb{R}^n$ can be found in [2] and it is completely the same in the general case for real vector spaces). Hence we may consider a as a linear operator on the space X over the field $\mathbb{Q}$.

Remark 1. Since every $a \in \mathcal{A}_X$ is $\mathbb{Q}$ -linear, it may be properly defined by choosing its values on a Hamel basis of the space X .

Denote

$$\text{Inv}_X = \{a \in \mathcal{A}_X : a \circ a = \text{id}_X\}.$$

We begin with

Fact 1. *For every $a \in \text{Inv}_X$ there is a unique decomposition of the space X over $\mathbb{Q}$ in the form $X = X_a^+ \oplus X_a^-$, where*

$$X_a^+ = \{x \in X : a(x) = x\},$$

$$X_a^- = \{x \in X : a(x) = -x\}$$

are subspaces of X over $\mathbb{Q}$.

Proof. It is obvious that X_a^+ and X_a^- are subspaces of X over $\mathbb{Q}$, $X_a^+ \cap X_a^- = \{0\}$ and

$$X \ni x = \frac{x + a(x)}{2} + \frac{x - a(x)}{2} \in X_a^+ + X_a^-.$$

□

Remark 2. It follows from Fact 1 that every $a \in \text{Inv}_X$ is a symmetry with respect to X_a^+ and along X_a^- .

Fix $a \in \text{Inv}_X$. On account of Fact 1 we may properly define

$$\text{ind}^+(a) = \dim_{\mathbb{Q}} X_a^+,$$

$$\text{ind}^-(a) = \dim_{\mathbb{Q}} X_a^-,$$

where $\dim_{\mathbb{Q}}$ denotes the dimension of the considered subspace over the field $\mathbb{Q}$. Considering the dimensions of X_a^+ and X_a^- (over $\mathbb{Q}$) we will be interested only in distinguishing when the dimensions of the spaces under consideration are finite or not.

Remark 3. In order to construct $a \in \text{Inv}_X \subset \mathcal{A}_X$ satisfying, let us say, $\text{ind}^+(a) = m$, it is enough to take a Hamel basis B of the space X , fix elements $b_1, \dots, b_m \in B$ and define $a : X \rightarrow X$ by

$$a(b_i) = b_i \quad \text{for } i \in \{1, \dots, m\} \quad \text{and} \quad a(b) = -b \quad \text{for } b \in B \setminus \{b_1, \dots, b_m\}.$$

Fix a positive integer m and define classes

$$\text{Ind}_m^+(X) = \{a \in \text{Inv}_X : \text{ind}^+(a) = m\},$$

$$\text{Ind}_m^-(X) = \{a \in \text{Inv}_X : \text{ind}^-(a) = m\}.$$

Theorem 2. *If X is a real topological vector space, $a \in \bigcup_{l=1}^{\infty} (\text{Ind}_l^+(X) \cup \text{Ind}_l^-(X))$ then a is discontinuous.*

Proof. Fix $m \in \mathbb{N}$ and $a \in \text{Ind}_m^+(X)$. Suppose that a is continuous. Then a is $\mathbb{R}$ -linear (for the proof of this fact for $X = \mathbb{R}^n$ we refer the Reader to [2] and its generalization for real topological vector spaces can be proved in the same way) and hence $X_a^+ = \{x \in X : a(x) = x\}$ is a closed and non-trivial subspace of X . This is in contradiction with the fact that $\text{ind}^+(a) = \dim_{\mathbb{Q}} X_a^+$ is finite. Similarly, if we suppose that $a \in \text{Ind}_m^-(X)$ is continuous, and hence $\mathbb{R}$ -linear, then $X_a^- = \{x \in X : a(x) = -x\}$ is a closed and non-trivial subspace of X . This leads to contradiction with the finiteness of $\text{ind}^-(a) = \dim_{\mathbb{Q}} X_a^-$. $\square$

3. Example of classes of involutions satisfying (1)

Let X be a real vector space. We begin with some property of bases in X .

Fact 2. *Let $\mathbf{v} = (v_i)_{i \in I}$ be a Hamel basis of the space X . Fix $k, l \in I$ and define $\mathbf{v}' = (v'_i)_{i \in I}$ by*

$$v'_k = v_k + v_l, \quad v'_l = v_k - v_l \quad \text{and} \quad v'_i = v_i \quad \text{for } i \in I \setminus \{k, l\}.$$

Then $\mathbf{v}'$ is the Hamel basis of X .

We prove

Lemma 1. *Let $a \in \text{Inv}_X$ and let $\mathbf{v} \subset X$ be a set of vectors linearly independent over $\mathbb{Q}$ such that $\text{card } \mathbf{v} = n \in \mathbb{N}$ and $a(\mathbf{v}) = \mathbf{v}$. Then $\text{ind}^+(a) \geq [\frac{n+1}{2}]$, where $[x]$ denotes the integer part of x .*

Proof. Put $\mathbf{v}_1 = \mathbf{v} \cap X_a^+$ and $\mathbf{v}_2 = \mathbf{v} \setminus \mathbf{v}_1$. If $\mathbf{v}_2 = \emptyset$ then $\mathbf{v} \subset X_a^+$, which means that $\text{ind}^+(a) \geq \text{card } \mathbf{v} = n \geq [\frac{n+1}{2}]$.

Assume now that $\mathbf{v}_2 \neq \emptyset$ and let $\text{card } \mathbf{v}_1 = k < n$. Since $a \in \text{Inv}_X$ $\mathbf{v}_2$ consists of $\frac{n-k}{2}$ pairs (v_1^i, v_2^i) such that $a(v_1^i) = v_2^i$ and $a(v_2^i) = v_1^i$ for $i \in \{1, \dots, \frac{n-k}{2}\}$. Replace each pair (v_1^i, v_2^i) by (w_1^i, w_2^i) , where

$$w_1^i = v_1^i + v_2^i \quad \text{and} \quad w_2^i = v_1^i - v_2^i.$$

Then $a(w_1^i) = w_1^i$, which means that $w_1^i \in X_a^+$ for every $i \in \{1, \dots, \frac{n-k}{2}\}$. Put $\mathbf{w} = \{w_1^i : i \in \{1, \dots, \frac{n-k}{2}\}\}$. On account of Fact 2 we get that $\mathbf{v}_1 \cup \mathbf{w}$ is the set of vectors linearly independent over $\mathbb{Q}$. Consequently

$$\text{ind}^+(a) = \dim_{\mathbb{Q}} X_a^+ \geq \text{card } (\mathbf{v}_1 \cup \mathbf{w}) = k + \frac{n-k}{2} = \frac{n+k}{2}.$$

If n is even, then $\frac{n+k}{2} \geq [\frac{n+1}{2}]$ whereas if n is odd then $k \geq 1$ and also $\frac{n+k}{2} \geq [\frac{n+1}{2}]$. This means that $\text{ind}^+(a) \geq [\frac{n+1}{2}]$. $\square$

Now we are in a position to prove our first result.

Theorem 3. *Fix $m \in \mathbb{N}$ and assume that X is a vector space with $\dim_{\mathbb{Q}} X \geq 2m+1$. Let $a \in \text{Ind}_m^+(X)$. Then $a(\mathbf{v}) \setminus \mathbf{v} \neq \emptyset$ holds for every set $\mathbf{v} \subset X$ of vectors linearly independent over $\mathbb{Q}$ with $\text{card } \mathbf{v} \geq 2m+1$.*

Proof. Fix $a \in \text{Ind}_m^+(X)$ and suppose that for some set $\mathbf{v} \subset X$ of vectors linearly independent over $\mathbb{Q}$ with $\text{card } \mathbf{v} \geq 2m+1$ we have $a(\mathbf{v}) \subset \mathbf{v}$. Since each involution is a bijective mapping with at most two-element orbits, we find a finite subset $\mathbf{v}' \subset \mathbf{v}$ of vectors linearly independent over $\mathbb{Q}$ such that $\text{card } \mathbf{v}' \geq 2m+1$ and $a(\mathbf{v}') = \mathbf{v}'$. On account of Lemma 1 we get

$$\text{ind}^+(a) \geq \left\lceil \frac{2m+1+1}{2} \right\rceil = m+1,$$

which leads to contradiction with the assumption $\text{ind}^+(a) = m$. $\square$

A simple consequence of Theorems 2 and 3 is the following

Corollary 1. *Fix $m \in \mathbb{N}$ and assume that X is a non-trivial real topological vector space. Let $a \in \text{Ind}_m^+(X)$. Then a is discontinuous and $a(\mathbf{v}) \setminus \mathbf{v} \neq \emptyset$ for every set $\mathbf{v} \subset X$ of vectors linearly independent over $\mathbb{Q}$ with $\text{card } \mathbf{v} \geq 2m+1$.*

4. Density

Let X be a non-trivial real topological vector space and consider the topological space X^X of all functions from X into X equipped with the Tychonoff topology. We will treat $\mathcal{A}_X$ as a topological space with the topology induced by X^X . Our main result of this section reads as follows

Theorem 4. *Define*

$$\begin{aligned}\mathcal{A}_1 &= \{a \in \text{Inv}_X : a \text{ is discontinuous and } a(H) \setminus H \neq \emptyset \text{ for every infinite} \\ &\quad \text{set } H \subset X \text{ of vectors linearly independent over } \mathbb{Q}\} \\ &\supset \{a \in \text{Inv}_X : \text{ind}^+(a) \in \mathbb{N}\} = \bigcup_{m=1}^{\infty} \text{Ind}_m^+(X) = \mathcal{A}'_1, \\ \mathcal{A}_2 &= \{a \in \text{Inv}_X : a \text{ is discontinuous and } a(H) = H \text{ for some Hamel} \\ &\quad \text{basis } H \subset X\} \\ &\supset \{a \in \text{Inv}_X : \text{ind}^-(a) \in \mathbb{N}\} = \bigcup_{m=1}^{\infty} \text{ind}_m^-(X) = \mathcal{A}'_2.\end{aligned}$$

The sets $\mathcal{A}'_1$ and $\mathcal{A}'_2$, and consequently also $\mathcal{A}_1$ and $\mathcal{A}_2$, are dense in the space $\mathcal{A}_X$.

In the proof of Theorem 4 we use, similarly as it was in the proof of [1, Theorem 2], the following lemma (see [1, Corollary 1]).

Lemma 2. *If for each $m \in \mathbb{N}$ and for every $h_1, \dots, h_m, h_{m+1}, \dots, h_{2m} \in X$ that are linearly independent over $\mathbb{Q}$ there exists $a \in \mathcal{D} \subset \mathcal{A}_X$ such that*

$$a(h_n) = h_{m+n} \quad \text{for every } n \in \{1, \dots, m\},$$

then $\mathcal{D}$ is dense in $\mathcal{A}_X$.

Proof of Theorem 4. Note that Corollary 1 implies $\mathcal{A}'_1 \subset \mathcal{A}_1$. We prove now, in a similar way, that $\mathcal{A}'_2 \subset \mathcal{A}_2$.

Indeed, fix any $a \in \mathcal{A}'_2$. Hence $\text{ind}^-(a) = m$ with some $m \in \mathbb{N}$. On account of Theorem 2 we find that a is discontinuous on X . Since $\text{ind}^-(a) = m$, we find for the function a a Hamel basis $H = (h_i)_{i \in I}$ of X (over $\mathbb{Q}$) and a finite set $\{i_1, \dots, i_m\} \subset I$ such that

$$a(h_{i_k}) = -h_{i_k} \text{ for } k \in \{1, \dots, m\} \quad \text{and} \quad a(h) = h \text{ for } h \in H \setminus \{h_{i_1}, \dots, h_{i_m}\}. \quad (2)$$

Choose arbitrary $\{h_{j_1}, \dots, h_{j_m}\} \subset H \setminus \{h_{i_1}, \dots, h_{i_m}\}$ and define $H' = (h'_i)_{i \in I}$ by

$$\begin{aligned}h'_{i_k} &= h_{j_k} + h_{i_k} \quad \text{and} \quad h'_{j_k} = h_{j_k} - h_{i_k} \quad \text{for } k \in \{1, \dots, m\}, \\ h'_i &= h_i \quad \text{for } i \in I \setminus \{i_1, \dots, i_m, j_1, \dots, j_m\}.\end{aligned}$$

By Fact 2 we know that H' is a Hamel basis of X . Moreover, by (2) we obtain

$$\begin{aligned} a(h'_{i_k}) &= a(h_{j_k} + h_{i_k}) = a(h_{j_k}) + a(h_{i_k}) = h_{j_k} - h_{i_k} = h'_{j_k} \quad \text{for } k \in \{1, \dots, m\}, \\ a(h'_{j_k}) &= a(h_{j_k} - h_{i_k}) = a(h_{j_k}) - a(h_{i_k}) = h_{j_k} + h_{i_k} = h'_{i_k} \quad \text{for } k \in \{1, \dots, m\}, \\ a(h'_i) &= a(h_i) = h_i = h'_i \quad \text{for } i \in I \setminus \{i_1, \dots, i_m, j_1, \dots, j_m\}. \end{aligned}$$

Consequently $a(H') = H'$.

Finally, we prove the density of the sets $\mathcal{A}'_1$ and $\mathcal{A}'_2$ in the space $\mathcal{A}_X$. Let us fix an arbitrary positive integer m and a set $\mathbf{v} = \{h_1, \dots, h_m, h_{m+1}, \dots, h_{2m}\} \subset X$ of vectors linearly independent over $\mathbb{Q}$. Let $H \subset X$ be an extension of $\mathbf{v}$ to a Hamel basis of X . Define $a_1, a_2 \in \mathcal{A}_X$ (on the basis H) by

$$\begin{aligned} a_j(h_i) &= h_{m+i} \quad \text{and} \quad a_j(h_{m+i}) = h_i \quad \text{for } i \in \{1, \dots, m\} \text{ and } j \in \{1, 2\}, \\ a_1(h) &= -h \quad \text{and} \quad a_2(h) = h \quad \text{for } h \in H \setminus \mathbf{v}. \end{aligned} \quad (3)$$

Since a_j are linear on X over $\mathbb{Q}$ and $a_j^2(h) = h$ for each $h \in H$, $a_j \in \text{Inv}_X$ for $j \in \{1, 2\}$.

For the Hamel basis H we define $H' \subset X$ by

$$\begin{aligned} h'_i &= h_i + h_{m+i} \quad \text{and} \quad h'_{m+i} = h_i - h_{m+i} \quad \text{for } i \in \{1, \dots, m\}, \\ h' &= h \quad \text{for } h \in H \setminus \mathbf{v}. \end{aligned}$$

By Fact 2 we find that H' is a Hamel basis of X and from (3) we get

$$\begin{aligned} a_1(h'_i) &= h'_i \quad \text{for } i \in \{1, \dots, m\}, \\ a_1(h'_{m+i}) &= -h'_{m+i} \quad \text{for } i \in \{1, \dots, m\}, \\ a_1(h') &= -h' \quad \text{for } h' \in H' \setminus \mathbf{v}, \end{aligned} \quad (4)$$

as well as

$$\begin{aligned} a_2(h'_i) &= h'_i \quad \text{for } i \in \{1, \dots, m\}, \\ a_2(h'_{m+i}) &= -h'_i \quad \text{for } i \in \{1, \dots, m\}, \\ a_2(h') &= h' \quad \text{for } h' \in H' \setminus \mathbf{v}. \end{aligned} \quad (5)$$

Since H' is the Hamel basis of X (over $\mathbb{Q}$), (4) implies $\text{ind}^+(a_1) = m$, whereas from (5) we derive $\text{ind}^-(a_2) = m$. Hence $a_1 \in \mathcal{A}_1$ and $a_2 \in \mathcal{A}_2$. Applying Lemma 2 we obtain the density of $\mathcal{A}'_1$ and $\mathcal{A}'_2$ in $\mathcal{A}_X$. $\square$

5. Open problems

Baron presented Theorem 2 from [1] during the 15th International Conference on Functional Equations and Inequalities (Ustroń, May 19–25, 2013). Following the question posed by J. Schwaiger after K. Baron's lecture we may ask

Problem 1. (J. Schwaiger) Is it possible to prove similar results (as Theorems 3 and 4) for additive functions satisfying

$$\underbrace{a \circ \dots \circ a}_k = \text{id},$$

where $k \geq 3$ is an integer?

We may also generalize this problem and put the following question.

Problem 2. Is it possible to prove similar results (as Theorems 3 and 4) for additive functions satisfying

$$\alpha_0 \cdot \text{id} + \sum_{j=1}^{k-1} \alpha_j \cdot a^j + a^k = 0,$$

where $\alpha_j \in \mathbb{Q}$ for $j \in \{0, \dots, k-1\}$, $a^j = \underbrace{a \circ \dots \circ a}_j$ and $k \geq 2$ is an integer?

Finally, we saw in Theorem 4 that families $\mathcal{A}'_1, \mathcal{A}'_2$ as well as $\mathcal{A}_1, \mathcal{A}_2$ are dense in $\mathcal{A}_X$. The question is how big are they in the topological sense, i.e. we may ask

Problem 3. Is it true or not that the families $\text{Ind}_m^+(X)$ and $\text{Ind}_m^-(X)$ for $m \in \mathbb{N}$ are of first category in $\mathcal{A}_X$?

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