Remark on hyperstability of the general linear equation

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Abstract. We present a result concerning the hyperstability of the general linear equation. Namely, we show that a function satisfying the equation approximately must be actually a solution to it.

Mathematics Subject Classification (2010). Primary 39B82, 39B62; Secondary 47H14, 47H10.

Keywords. Hyperstability, general linear equation.

Throughout the paper $\mathbb{F}$, $\mathbb{K}$ denote the fields of real or complex numbers. Let X , Y be normed spaces over $\mathbb{F}$, $\mathbb{K}$, respectively. The aim of the paper is to give a hyperstability result for the general linear equation

$$g(ax + by) = Ag(x) + Bg(y), \quad (1)$$

where $g: X \rightarrow Y$ and $a, b \in \mathbb{F} \setminus \{0\}$, $A, B \in \mathbb{K}$. We see that for $a = b = A = B = 1$ in (1) we get the Cauchy equation while the Jensen equation corresponds to $a = b = A = B = \frac{1}{2}$. The general linear equation has been studied by many authors, see for example [6, 8, 10–12, 14].

Probably the first hyperstability result was published in [3] and concerned ring homomorphisms. However the term hyperstability was used for the first time in [13].

We recall the most classical results concerning the Hyers-Ulam stability of the Cauchy equation

$$f(x + y) = f(x) + f(y), \quad x, y \in X, \quad (2)$$

which were the motivation to write the paper.

If X is a normed space, Y is a Banach space, $c \geq 0$, p is a real number such that $0 < p < 1$ and $f: X \rightarrow Y$ satisfies the inequality

$$\|f(x + y) - f(x) - f(y)\| \leq c(\|x\|^p + \|y\|^p), \quad x, y \in X, \quad (3)$$

then there exists a unique solution $T: X \rightarrow Y$ of (2) with

$$\|f(x) - T(x)\| \leq \frac{c\|x\|^p}{|1 - 2^{p-1}|}, \quad x \in X.$$

It is due to Aoki [1] (see also [15]). Later Gajda [7] obtained this result for $p > 1$ and Rassias [16] for $p < 0$. Moreover, in [7] we can find an example showing that an analogous result for $p = 1$ is not true. For $p = 0$ it is the first result of stability proved by Hyers [9]. Now, it is known that for $p < 0$ we have the hyperstability result, that is f satisfying (3) must be additive for $x, y \in X \setminus \{0\}$ (see [4]).

The method of the proof used in the main theorem is based on a fixed point result that can be derived from [5] (Theorem 1). To present it we need the following three hypothesis:

- (H1) U is a nonempty set, Y is a Banach space, $f_1, \dots, f_k: U \rightarrow U$ and $L_1, \dots, L_k: U \rightarrow \mathbb{R}_+$ are given.
 (H2) $\mathcal{T}: Y^U \rightarrow Y^U$ is an operator satisfying the inequality

$$\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\| \leq \sum_{i=1}^k L_i(x) \|\xi(f_i(x)) - \mu(f_i(x))\|, \\ \xi, \mu \in Y^U, x \in U.$$

- (H3) $\Lambda: \mathbb{R}_+^U \rightarrow \mathbb{R}_+^U$ is defined by

$$\Lambda\delta(x) := \sum_{i=1}^k L_i(x) \delta(f_i(x)), \quad \delta \in \mathbb{R}_+^U, x \in U.$$

Now we are in a position to present the above mentioned fixed point theorem.

Theorem 1. *Let hypotheses (H1)–(H3) be valid and functions $\varepsilon: U \rightarrow \mathbb{R}_+$ and $\varphi: U \rightarrow Y$ fulfil the following two conditions*

$$\|\mathcal{T}\varphi(x) - \varphi(x)\| \leq \varepsilon(x), \quad x \in U, \\ \varepsilon^*(x) := \sum_{n=0}^{\infty} \Lambda^n \varepsilon(x) < \infty, \quad x \in U.$$

Then there exists a unique fixed point ψ of $\mathcal{T}$ with

$$\|\varphi(x) - \psi(x)\| \leq \varepsilon^*(x), \quad x \in U.$$

Moreover

$$\psi(x) := \lim_{n \rightarrow \infty} \mathcal{T}^n \varphi(x), \quad x \in U.$$

Theorem 2. *Let X be a normed space over a field $\mathbb{F}$, Y be a Banach space over $\mathbb{K}$, $a, b \in \mathbb{F} \setminus \{0\}$, $A, B \in \mathbb{K}$, $c \geq 0$, $p < 0$ and $g: X \rightarrow Y$ satisfy*

$$\|g(ax + by) - Ag(x) - Bg(y)\| \leq c(\|x\|^p + \|y\|^p), \quad x, y \in X \setminus \{0\}. \quad (4)$$

Then g satisfies the equation

$$g(ax + by) = Ag(x) + Bg(y), \quad x, y \in X \setminus \{0\}. \quad (5)$$

Proof. Replacing x by $\frac{1}{a}(m+1)x$ and y by $-\frac{1}{b}mx$ for $m \in \mathbb{N}$ in (4) we get

$$\begin{aligned} & \left\| g(x) - Ag\left(\frac{1}{a}(m+1)x\right) - Bg\left(-\frac{1}{b}mx\right) \right\| \\ & \leq c \left(\left| \frac{1}{a}(m+1) \right|^p + \left| \frac{1}{b}m \right|^p \right) \|x\|^p, \quad x \in X \setminus \{0\}. \end{aligned} \quad (6)$$

Further put

$$\begin{aligned} \mathcal{T}_m \xi(x) &:= A\xi\left(\frac{1}{a}(m+1)x\right) + B\xi\left(-\frac{1}{b}mx\right), \quad x \in X \setminus \{0\}, \xi \in Y^{X \setminus \{0\}}, \\ \varepsilon_m(x) &:= c \left(\left| \frac{1}{a}(m+1) \right|^p + \left| \frac{1}{b}m \right|^p \right) \|x\|^p, \quad x \in X \setminus \{0\}. \end{aligned}$$

Then the inequality (6) takes the form

$$\|\mathcal{T}_m g(x) - g(x)\| \leq \varepsilon_m(x), \quad x \in X \setminus \{0\}.$$

The operator

$$\Lambda_m \eta(x) := |A|\eta\left(\frac{1}{a}(m+1)x\right) + |B|\eta\left(-\frac{1}{b}mx\right), \quad \eta \in \mathbb{R}_+^{X \setminus \{0\}}, x \in X \setminus \{0\}$$

has the form described in (H3) with $k = 2$ and $f_1(x) = \frac{1}{a}(m+1)x$, $f_2(x) = -\frac{1}{b}mx$, $L_1(x) = |A|$, $L_2(x) = |B|$ for $x \in X$. Moreover, for every $\xi, \mu \in Y^{X \setminus \{0\}}, x \in X \setminus \{0\}$

$$\begin{aligned} & \|\mathcal{T}_m \xi(x) - \mathcal{T}_m \mu(x)\| \\ &= \left\| A\xi\left(\frac{1}{a}(m+1)x\right) + B\xi\left(-\frac{1}{b}mx\right) - A\mu\left(\frac{1}{a}(m+1)x\right) - B\mu\left(-\frac{1}{b}mx\right) \right\| \\ &\leq |A| \left\| \xi\left(\frac{1}{a}(m+1)x\right) - \mu\left(\frac{1}{a}(m+1)x\right) \right\| + |B| \left\| \xi\left(-\frac{1}{b}mx\right) - \mu\left(-\frac{1}{b}mx\right) \right\| \\ &= \sum_{i=1}^2 L_i(x) \|\xi(f_i(x)) - \mu(f_i(x))\|. \end{aligned}$$

We see that there exists $m_0 \in \mathbb{N}$ such that

$$|A| \left| \frac{1}{a}(m+1) \right|^p + |B| \left| \frac{1}{b}m \right|^p < 1 \quad \text{for } m \geq m_0.$$

Therefore

$$\begin{aligned}
 \varepsilon^*(x) &:= \sum_{n=0}^{\infty} \Lambda_m^n \varepsilon_m(x) \\
 &= \sum_{n=0}^{\infty} c \left(\left| \frac{1}{a}(m+1) \right|^p + \left| \frac{1}{b}m \right|^p \right) \left(|A| \left| \frac{1}{a}(m+1) \right|^p + |B| \left| \frac{1}{b}m \right|^p \right)^n \|x\|^p \\
 &= \frac{c \left(\left| \frac{1}{a}(m+1) \right|^p + \left| \frac{1}{b}m \right|^p \right) \|x\|^p}{1 - |A| \left| \frac{1}{a}(m+1) \right|^p - |B| \left| \frac{1}{b}m \right|^p}, \quad x \in X \setminus \{0\}, m \geq m_0.
 \end{aligned}$$

Thus, according to Theorem 1, for each $m \geq m_0$ there exists a unique solution $G_m: X \setminus \{0\} \rightarrow Y$ of the equation

$$G_m(x) = AG_m \left(\frac{1}{a}(m+1)x \right) + BG_m \left(-\frac{1}{b}mx \right)$$

such that

$$\|g(x) - G_m(x)\| \leq \frac{c \left(\left| \frac{1}{a}(m+1) \right|^p + \left| \frac{1}{b}m \right|^p \right) \|x\|^p}{1 - |A| \left| \frac{1}{a}(m+1) \right|^p - |B| \left| \frac{1}{b}m \right|^p}, \quad x \in X \setminus \{0\}.$$

We show that

$$\begin{aligned}
 &\|T_m^n g(ax + by) - AT_m^n g(x) - BT_m^n g(y)\| \\
 &\leq c \left(\left| A \left| \frac{1}{a}(m+1) \right|^p + |B| \left| \frac{1}{b}m \right|^p \right)^n (\|x\|^p + \|y\|^p) \right) \quad (7)
 \end{aligned}$$

for every $x, y \in X \setminus \{0\}$, $n \in \mathbb{N}_0$.

If $n = 0$, then (7) is simply (4). So, take $r \in \mathbb{N}_0$ and suppose that (7) holds for $n = r$ and $x, y \in X \setminus \{0\}$. Then

$$\begin{aligned}
 &\|T_m^{r+1} g(ax + by) - AT_m^{r+1} g(x) - BT_m^{r+1} g(y)\| \\
 &= \left\| AT_m^r g \left(\frac{1}{a}(m+1)(ax + by) \right) + BT_m^r g \left(-\frac{1}{b}m(ax + by) \right) \right. \\
 &\quad - A^2 T_m^r g \left(\frac{1}{a}(m+1)x \right) - AB T_m^r g \left(-\frac{1}{b}mx \right) \\
 &\quad \left. - BAT_m^r g \left(\frac{1}{a}(m+1)y \right) - B^2 T_m^r g \left(-\frac{1}{b}my \right) \right\| \\
 &\leq |A| \left\| T_m^r g \left(\frac{1}{a}(m+1)(ax + by) \right) - AT_m^r g \left(\frac{1}{a}(m+1)x \right) \right. \\
 &\quad \left. - BT_m^r g \left(\frac{1}{a}(m+1)y \right) \right\| \\
 &\quad + |B| \left\| T_m^r g \left(-\frac{1}{b}m(ax + by) \right) - AT_m^r g \left(-\frac{1}{b}mx \right) - BT_m^r g \left(-\frac{1}{b}my \right) \right\| \\
 &\leq |A|c \left(\left| A \left| \frac{1}{a}(m+1) \right|^p + |B| \left| \frac{1}{b}m \right|^p \right)^r \left(\left\| \frac{1}{a}(m+1)x \right\|^p + \left\| \frac{1}{a}(m+1)y \right\|^p \right) \right)
 \end{aligned}$$

$$\begin{aligned}
& + |B|c \left(|A| \left| \frac{1}{a}(m+1) \right|^p + |B| \left| \frac{1}{b}m \right|^p \right)^r \left(\left\| -\frac{1}{b}mx \right\|^p + \left\| -\frac{1}{b}my \right\|^p \right) \\
& \leq c \left(|A| \left| \frac{1}{a}(m+1) \right|^p + |B| \left| \frac{1}{b}m \right|^p \right)^{r+1} (\|x\|^p + \|y\|^p).
\end{aligned}$$

Letting $n \rightarrow \infty$ in (7), we obtain that

$$G_m(ax + by) = AG_m(x) + BG_m(y), \quad x, y \in X \setminus \{0\}.$$

So, we have a sequence $(G_m)_{m \geq m_0}$ of functions satisfying equation (5) such that

$$\|g(x) - G_m(x)\| \leq \frac{c \left(\left| \frac{1}{a}(m+1) \right|^p + \left| \frac{1}{b}m \right|^p \right) \|x\|^p}{1 - |A| \left| \frac{1}{a}(m+1) \right|^p - |B| \left| \frac{1}{b}m \right|^p}, \quad x \in X \setminus \{0\}.$$

It follows, with $m \rightarrow \infty$, that g also satisfies (5) for $x, y \in X \setminus \{0\}$. $\square$

Corollary 3. Let X be a normed space over a field $\mathbb{F}$, Y be a Banach space over $\mathbb{K}$, $a, b \in \mathbb{F} \setminus \{0\}$, $A, B \in \mathbb{K}$, $A + B \neq 1$, $C \in Y$, $c \geq 0$, $p < 0$ and $f: X \rightarrow Y$ satisfy

$$\|f(ax + by) - Af(x) - Bf(y) - C\| \leq c(\|x\|^p + \|y\|^p), \quad x, y \in X \setminus \{0\}.$$

Then f satisfies the equation

$$f(ax + by) = Af(x) + Bf(y) + C, \quad x, y \in X \setminus \{0\}.$$

Proof. Considering $g(x) = f(x) + \frac{C}{A+B-1}$, $x \in X$ we see that

$$\|g(ax + by) - Ag(x) - Bg(y)\| \leq c(\|x\|^p + \|y\|^p), \quad x, y \in X \setminus \{0\}.$$

So, by Theorem 2 we have

$$g(ax + by) = Ag(x) + Bg(y), \quad x, y \in X \setminus \{0\},$$

thus

$$f(ax + by) = Af(x) + Bf(y) + C, \quad x, y \in X \setminus \{0\}.$$

$\square$

Remark 4. If g satisfies (4) with $A = B = 0$, then, by (6) with $m \rightarrow \infty$, it follows that $g(x) = 0$ for $x \in X \setminus \{0\}$.

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Received: March 1, 2013

Revised: April 18, 2013