On Tabor groupoids and stability of some functional equations

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Abstract. Two results are given, which use Tabor groupoids for questions of stability in the sense of Pólya–Szegő–Hyers–Ulam. We also start to study Tabor groupoids in their own right.

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1. Introduction

We use the notations $N = \{1, 2, 3, \dots\}$, $N_0 = N \cup \{0\}$, and R for the space of the reals.

Let S be a groupoid, i.e., S is a set and for all $x, y \in S$ we have a product $xy \in S$. For $x \in S$ and $k \in N_0$ the powers x^{2^k} are recursively defined by

$$x^{2^0} = x^1 = x, \quad x^{2^{k+1}} = x^{2^k} x^{2^k}.$$

Definition. A groupoid S is called a *Tabor groupoid*, if the following condition holds:

(T) For $x, y \in S$ there always exists $k \in N$ such that

$$(xy)^{2^k} = x^{2^k} y^{2^k}. \quad (1.1)$$

This notion had been introduced in [1] on the basis of Józef Tabor’s Remark [9]. Examples are groupoids S with a square-symmetric operation, i.e.

$$(xy)^2 = x^2 y^2 \quad (x, y \in S). \quad (1.2)$$

As a consequence of [11] (cf. also Theorem 1 in [1]) we have the following result:

Theorem 1. *Let S be a Tabor groupoid, and let V be a bounded closed convex subset of a Banach space E . For $f : S \rightarrow E$ we suppose*

$$f(xy) - f(x) - f(y) \in V \quad (x, y \in S). \quad (1.3)$$

Then there exists a (unique) additive function $a : S \rightarrow E$ such that $a(x) - f(x) \in V$ ($x \in S$).

Here the additivity of $a : S \rightarrow E$ means

$$a(xy) = a(x) + a(y) \quad (x, y \in S).$$

In the next Section we shall use Theorem 1 to characterize bounded perturbations of additive functions. In Section 3 we examine semigroups S containing an idempotent element s (i.e., $s^2 = s$). We give sufficient conditions for them to be Tabor groupoids. We also show that the symmetric group S_3 is not a Tabor groupoid.

In Section 4 we study the functional equation

$$\max\{f((xy)y), f(x)\} = f(xy) + f(y) \quad (x, y \in S) \quad (1.4)$$

for functions $f : S \rightarrow R$, where S is a groupoid having a left unit and satisfying the following condition:

($\tilde{T}$) For $x, y \in S$ there always exists $k \in N$ such that

$$(xy)^{2^k} = x^{2^k} y^{2^k}, \quad ((xy)y)^{2^k} = (x^{2^k} y^{2^k}) y^{2^k}. \quad (1.5)$$

($\tilde{T} \Rightarrow T$), hence S is a Tabor groupoid.) We prove the stability of (1.4), which in the special case (1.2) of ($\tilde{T}$) goes back to Attila Gilányi, Kaori Nagatou (長藤 かおり) and Volkmann [4]. In the case of a commutative group $(S, +)$, equation (1.4) is equivalent to

$$\max\{f(x+y), f(x-y)\} = f(x) + f(y) \quad (x, y \in S),$$

the solutions of which are $f(x) = |a(x)|$, $a : (S, +) \rightarrow R$ being additive (Alice Simon and Volkmann [8]; cf. also the comments in [4]).

Finally, let us mention that in [2] we have shown the stability of the functional equation $f(xy) = f(yx)$ for $f : S \rightarrow E$, S being a group and E a normed space.

2. A characterization of bounded perturbations of additive functions

Theorem 2. *Let S be a Tabor groupoid, E a Banach space, $f : S \rightarrow E$, and suppose A to be a bounded closed subset of E . Then the following two conditions are equivalent:*

(P) $f = a + r$, where $a : S \rightarrow E$ is additive and $r(x) \in A$ ($x \in S$).

(Q) *There exist bounded subsets B, C of E such that*

$$f(xy) - f(x) - f(y) \in B \quad (x, y \in S), \quad (2.1)$$

$$2^k f(x) - f(x^{2^k}) \in 2^k A + C \quad (x \in S, k \in \mathbb{N}). \quad (2.2)$$

Proof. (P) $\Rightarrow$ (Q): From (P) we get for $x, y \in S$ that

$$\begin{aligned} f(xy) - f(x) - f(y) &= r(xy) - r(x) - r(y) \in A - A - A, \\ 2^k f(x) - f(x^{2^k}) &= 2^k a(x) + 2^k r(x) - a(x^{2^k}) - r(x^{2^k}) \\ &= 2^k r(x) - r(x^{2^k}) \in 2^k A - A \quad (k \in \mathbb{N}), \end{aligned}$$

and consequently (Q) holds, when taking $B = A - A - A$, $C = -A$.

(Q) $\Rightarrow$ (P): Let V be the closed convex hull of B (which also is bounded). Then (2.1) implies (1.3), and by Theorem 1 we have the existence of an additive $a : S \rightarrow E$ such that $a(x) - f(x) \in V$ ($x \in S$). We get $r(x) := f(x) - a(x) \in (-V)$ ($x \in S$), and it is sufficient to show that

$$r(x) \in A \quad (x \in S). \quad (2.3)$$

Condition (2.2) implies for $x \in S$ that

$$2^k r(x) - r(x^{2^k}) \in 2^k A + C,$$

whence

$$r(x) \in A + \frac{1}{2^k} C - \frac{1}{2^k} V,$$

and $k \rightarrow \infty$ leads to (2.3).

Remark 1. Theorem 2 obviously characterizes perturbations of additive functions, where the perturbing function $r : S \rightarrow E$ has values in a given bounded closed subset A of the Banach space E . In the special case (1.2) and

$$A = \{\xi \mid \xi \in E, \|\xi\| \leq \varepsilon\} \quad (\text{where } \varepsilon > 0), \quad (2.4)$$

Theorem 2 is known from [10].

Remark 2. Let S be a real or complex vector space, consider a Banach space E having the same scalar field, and suppose A to be a bounded closed subset of E . Then, similarly to Theorem 2, functions $f = L + r : S \rightarrow E$ can be characterized, where $L : S \rightarrow E$ is linear and $r(x) \in A$ ($x \in S$); cf. Haifeng Ma (马海凤) and Volkman [7]. In the special case of real spaces S, E , and A as in (2.4), this characterization is known from Roman Ger and Volkman [3].

3. Some Tabor groupoids and the group S_3

Let S^1, S^2 be Tabor groupoids and consider $S^1 \times S^2$ equipped with the product $(x, y)(\bar{x}, \bar{y}) = (x\bar{x}, y\bar{y})$ (for $(x, y), (\bar{x}, \bar{y}) \in S^1 \times S^2$). Then $S^1 \times S^2$ also is a Tabor groupoid, provided that the product in at least one of S^1, S^2 is square-symmetric, i.e., for $S = S^1$ or $S = S^2$ we have (1.2). If in addition $(\tilde{T})$ holds for S^1, S^2 , then this condition also holds for $S^1 \times S^2$.

Now we consider semigroups, that means groupoids, where the multiplication is associative. Every commutative semigroup S obviously is a Tabor groupoid, in fact (1.2) holds. In the next Theorem we require the existence of an idempotent element, consequently groups are admissible.

Theorem 3. *Let S be a semigroup containing an element $s = s^2$. Then S is a Tabor groupoid, if one of the following two conditions holds:*

- I) *For every $x \in S$ there exists $n \in N_0$ such that $x^{2^n} = s$.*
- II) *For every $x \in S$ there exists an odd $m \in N$ such that $x^m = s$.*

Proof. Case I): Let us consider $x, y \in S$. There are $l, m, n \in N_0$ such that

$$x^{2^l} = y^{2^m} = (xy)^{2^n} = s.$$

Then $k := l + m + n + 1 \in N$ and (1.1) holds.

Case II): Let us consider $x, y \in S$. There are odd $m_1, m_2, m_3 \in N$ such that $x^{m_1} = y^{m_2} = (yx)^{m_3} = s$. Then $m = m_1 m_2 m_3$ also is odd and

$$x^m = y^m = (yx)^m = s. \quad (3.1)$$

The number 2 being prime to m , it follows from Fermat's Theorem (cf., e.g., Harriet Griffin [6], p. 88) that m divides $2^{\varphi(m)} - 1$, where φ denotes Euler's function. For $k = \varphi(m) \in N$ we thus have the existence of an $n \in N$ such that

$$2^k - 1 = mn.$$

Now (3.1) implies

$$(yx)^{2^k - 1} = (yx)^{mn} = s = ss = x^{mn} y^{mn} = x^{2^k - 1} y^{2^k - 1}.$$

When multiplying this equation by x from the left and by y from the right, we finally get $(xy)^{2^k} = x^{2^k} y^{2^k}$.

Remark 3. The semigroups treated in Theorem 3 also satisfy condition $(\tilde{T})$.

The group S_3 . We shall show that this group is not a Tabor groupoid; concerning the definition and basic properties of S_3 cf., e.g., Lidiá Ivanovna Golovina [5], pp. 262, 263. Here we describe S_3 as the set of all bijections $\omega : \{1, 2, 3\} \rightarrow \{1, 2, 3\}$, where the product is the composition of functions. Elements ω of S_3 are given in the form $\omega = (\omega(1)\omega(2)\omega(3))$; then $\iota = (1\ 2\ 3)$ is the unit. Let us consider three elements of S_3 :

$$\sigma = (2\ 3\ 1), \quad \tau_2 = (3\ 2\ 1), \quad \tau_3 = (2\ 1\ 3).$$

We have

$$\sigma = \sigma^4 = \sigma^{16} = \sigma^{64} = \dots, \quad \iota \neq \sigma^2 = \sigma^8 = \sigma^{32} = \sigma^{128} = \dots,$$

hence

$$\sigma^{2^n} \neq \iota \quad (n \in N_0). \quad (3.2)$$

Furthermore,

$$\tau_2^2 = \tau_3^2 = \iota, \quad \sigma\tau_3 = \tau_2. \quad (3.3)$$

If S_3 is supposed to be a Tabor groupoid, then there would be a number $k \in N$ such that $(\sigma\tau_3)^{2^k} = \sigma^{2^k}\tau_3^{2^k}$. Taking (3.3) into account, we get $\iota = \sigma^{2^k}$, which is in contradiction with (3.2).

4. A stability result

Lemma. *Let S be a groupoid satisfying $(\tilde{T})$. When $x, y \in S$ are fixed, then (1.5) holds for infinitely many $k \in N$.*

Proof. Let $k \in N$ be such that (1.5) holds. Condition $(\tilde{T})$ applied to x, y replaced by x^{2^k}, y^{2^k} , respectively, yields the existence of $l \in N$ such that

$$(x^{2^k}y^{2^k})^{2^l} = x^{2^{k+l}}y^{2^{k+l}}, \quad ((x^{2^k}y^{2^k})y^{2^k})^{2^l} = (x^{2^{k+l}}y^{2^{k+l}})y^{2^{k+l}}. \quad (4.1)$$

Now it is sufficient to show that (1.5) holds with k replaced by the greater number $k+l \in N$. Indeed, from (1.5) and (4.1) we get

$$(xy)^{2^{k+l}} = ((xy)^{2^k})^{2^l} = (x^{2^k}y^{2^k})^{2^l} = x^{2^{k+l}}y^{2^{k+l}}$$

and

$$((xy)y)^{2^{k+l}} = (((xy)y)^{2^k})^{2^l} = ((x^{2^k}y^{2^k})y^{2^k})^{2^l} = (x^{2^{k+l}}y^{2^{k+l}})y^{2^{k+l}}.$$

Theorem 4. *Let S be a groupoid having a left unit and satisfying condition $(\tilde{T})$. For $f : S \rightarrow R$ suppose that*

$$|\max\{f((xy)y), f(x)\} - f(xy) - f(y)| \leq \varepsilon \quad (x, y \in S). \quad (4.2)$$

Then there exists $g : S \rightarrow R$ such that

$$\max\{g((xy)y), g(x)\} = g(xy) + g(y) \quad (x, y \in S) \quad (4.3)$$

and

$$|f(x) - g(x)| \leq 3\varepsilon \quad (x \in S). \quad (4.4)$$

Proof. As in [4] it can be shown that

$$|f(x^2) - 2f(x)| \leq 3\varepsilon \quad (x \in S). \quad (4.5)$$

For this proof, S can be an arbitrary groupoid with a left unit. From (4.5) we get by [11] (for arbitrary groupoids S) the existence of

$$g(x) = \lim_{n \rightarrow \infty} \frac{1}{2^n} f(x^{2^n}) \quad (x \in S) \quad (4.6)$$

and the validity of (4.4). It is sufficient to verify (4.3): Let us fix $x, y \in S$. When replacing in (4.2) the elements x, y by x^{2^k}, y^{2^k} , so that (1.5) holds, then we have

$$|\max\{f(((xy)y)^{2^k}), f(x^{2^k})\} - f((xy)^{2^k}) - f(y^{2^k})| \leq \varepsilon. \quad (4.7)$$

According to the preceding Lemma this inequality is true for infinitely many $k \in N$, which can be arranged in a sequence

$$k_1 < k_2 < k_3 < \dots \rightarrow \infty.$$

Now we consider (4.7) for all $k = k_m$ ($m \in N$), divide both sides by 2^{k_m} and let m tend to infinity. According to (4.6) we get (4.3).

Remark 4. Under the assumptions of Theorem 4 it is routine to see that there exists only one solution $g : S \rightarrow R$ of the functional equation (4.3) for which the left-hand side of (4.4) is bounded.

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