On locally bounded above solutions of an equation of the Gołab–Schinzel type

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Abstract. We characterize solutions $f, g : \mathbb{R} \rightarrow \mathbb{R}$ of the functional equation $f(x + g(x)y) = f(x)f(y)$ under the assumption that f is locally bounded above at each point $x \in \mathbb{R}$. Our result refers to Gołab and Schinzel (Publ Math Debr 6:113–125, 1959) and Wołodźko (Aequationes Math 2:12–29, 1968).

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1. Introduction

The Cauchy equation of an exponential function

$$f(x + y) = f(x)f(y) \quad (1.1)$$

and the following Pexider equation

$$f(x + y) = g(x)h(y) \quad (1.2)$$

are two of the oldest functional equations. It is very well known that each real solution of (1.1), which is locally bounded above at a point, is continuous. Moreover, if f, g, h are real nonconstant functions satisfying (1.2) such that one of them is locally bounded above at a point, then f, g, h are continuous (see e.g. [9, pp. 155, 343, 355]).

In 1959 Gołab and Schinzel introduced in [4] the following composite-type functional equation

$$f(x + f(x)y) = f(x)f(y) \quad (1.3)$$

in connection with looking for subgroups of the centroaffine group of a field. It turned out that this equation and its generalizations have a lot of applications (an extensive bibliography concerning the Gołab–Schinzel equation, its generalizations and applications can be found in [2] and [7]). It is the main reason,

why so numerous authors considered the equation and its generalizations under various assumptions.

In [6] the author considered the pexiderized Gołąb–Schinzel equation

$$f(x + g(x)y) = h(x)k(y) \quad (1.4)$$

and proved that solutions of this equation can be described by solutions of a partially pexiderized Gołąb–Schinzel equation

$$f(x + g(x)y) = f(x)f(y). \quad (1.5)$$

Therefore Eq. (1.5) plays a very important role among equations of the Gołąb–Schinzel type.

Clearly, Eq. (1.5) is a common generalization of Eqs. (1.1) and (1.3), as well as a partial generalization of the equation

$$f(x + M(f(x))y) = F(f(x), f(y)), \quad (1.6)$$

considered in several papers (a bibliography can be found in the survey paper [7]) and very recently in papers [1] and [10].

For the first time Eq. (1.5) was considered by Chudziak [3]. He determined, among others, solutions $f, g : \mathbb{R} \rightarrow \mathbb{R}$ of (1.5) under the assumption that the function g is continuous. Next, the author proved in [8] that the continuity of f implies the continuity of g , provided $f, g : \mathbb{R} \rightarrow \mathbb{R}$ are nonconstant solutions of (1.5). Moreover, continuous on rays functions satisfying (1.5) were considered in [5].

Here we characterize solutions $f, g : \mathbb{R} \rightarrow \mathbb{R}$ of the functional Eq. (1.5) under the assumption that f is locally bounded above at each point $x \in \mathbb{R}$. Among others, we prove that the local boundedness above of f at each point implies the local boundedness above of g at each point, provided $f(\mathbb{R}) \not\subset \{-1, 0, 1\}$ and $g(\mathbb{R}) \not\subset \{0, 1\}$. Moreover, we give an example of a solution $f : \mathbb{R} \rightarrow \mathbb{R}$ of the Gołąb–Schinzel equation, which is locally bounded at each point of $\mathbb{R}$, but it is not continuous.

In the whole paper we use the following notation:

$$\begin{aligned} A &= f^{-1}(\{1\}), & B &= g^{-1}(\{1\}), & W &= f(\mathbb{R}) \setminus \{0\}, \\ F &= \{x \in \mathbb{R} : f(x) \neq 0\}. \end{aligned}$$

2. Preliminary lemmas

First we prove a lemma.

Lemma 2.1. *If $f : \mathbb{R} \rightarrow \mathbb{R}$ is periodic and locally bounded above at each point of $\mathbb{R}$, then f is bounded above on $\mathbb{R}$.*

Proof. Let $T \in \mathbb{R} \setminus \{0\}$ be a period of f . Take any $a \in \mathbb{R}$. Then f is locally bounded above at each point $x \in [a, a + T]$. The interval $[a, a + T]$ is compact,

whence f is bounded above on it. Consequently, since f is periodic, it has to be bounded above on $\mathbb{R}$, which ends the proof. $\square$

Let us recall some basic properties of functions satisfying (1.5).

Lemma 2.2 ([5, Lemma 1]). *Let X be a real linear space, $f, g : X \rightarrow \mathbb{R}$, $f \neq 1$ and $f \neq 0$. If f and g satisfy Eq. (1.5), then:*

- (i) $f(0) = 1$ and $g(0) \neq 0$;
- (ii) $F = \{x \in X : g(x) \neq 0\}$;
- (iii) $f(g(x)^{-1}(z - x)) = f(x)^{-1}f(z)$ for every $x \in F$ and $z \in X$;
- (iv) $\frac{y-x}{g(x)} \in A$ for every $x, y \in F$ with $f(x) = f(y)$.

Lemma 2.3 ([5, Lemma 2]). *Let X be a real linear space, $f, g : X \rightarrow \mathbb{R}$, $f(X) \setminus \{0, 1\} \neq \emptyset$ and $g(X) \setminus \{0, 1\} \neq \emptyset$. If f and g satisfy (1.5) and A is a linear subspace of X , then there exists $z_0 \in X \setminus A$ such that*

$$f^{-1}(\{f(x)\}) = (g(x) - 1)z_0 + A \quad \text{for each } x \in F.$$

Now we prove further properties of solutions of (1.5), which will be useful in the proof of the main result.

Lemma 2.4. *Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be solutions of (1.5) and f be not periodic. Then:*

- (i) $A \cap B \in \{\emptyset, \{0\}\}$;
- (ii) $A \cap g^{-1}(\{-1\}) \in \{\emptyset, \{x_0\}\}$ with $x_0 \in \mathbb{R}$.

Proof. (i) Suppose that there is $x \in (A \cap B) \setminus \{0\}$. Then, by (1.5), $f(x+y) = f(y)$ for each $y \in \mathbb{R}$. It means that f is periodic, which is a contradiction.
(ii) Suppose that there are $z_1, z_2 \in A \cap g^{-1}(\{-1\})$ with $z_1 \neq z_2$. Then, by (1.5), $f(z_1 - x) = f(x) = f(z_2 - x)$ for $x \in \mathbb{R}$ and thus, setting $x = z_2 - y$ we obtain that $f(z_1 - z_2 + y) = f(y)$ for $y \in \mathbb{R}$. But f can not be periodic, which is a contradiction. $\square$

3. Main results

Let us start with an easy observation.

Proposition 3.1. *Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ satisfy (1.5), $f \neq 0$, $f \neq 1$ and $g(\mathbb{R}) \subset \{0, 1\}$. Then there exist a subgroup G of the group $(\mathbb{R}, +)$ and an additive function $a : G \rightarrow \mathbb{R}$ such that*

$$g(x) = \begin{cases} 1 & \text{for } x \in G; \\ 0 & \text{for } x \in \mathbb{R} \setminus G \end{cases}$$

and

$$f(x) = \begin{cases} e^{a(x)} & \text{for } x \in G; \\ 0 & \text{for } x \in \mathbb{R} \setminus G. \end{cases}$$

Proof. In view of Lemma 2.2(ii) $f(x) = g(x) = 0$ for $x \in \mathbb{R} \setminus F$ and $g(x) = 1$ for $x \in F$. Thus, by (1.5) we have

$$f(x+y) = f(x)f(y) \neq 0 \quad \text{for every } x, y \in F \quad (3.1)$$

and, according to Lemma 2.2(i), (iii),

$$f(-x) = f\left(\frac{0-x}{g(x)}\right) = f(x)^{-1}f(0) \neq 0 \quad \text{for each } x \in F.$$

Hence $F+F \subset F$, $-F \subset F$ and, in view of Lemma 2.2(i), $0 \in F$. It means that F is a subgroup of the group $(\mathbb{R}, +)$. By (3.1) f is an exponential function on F . Hence $f(x) = e^{a(x)}$ for $x \in F$, where $a : F \rightarrow \mathbb{R}$ is an additive function. To end the proof it is enough to put $G = F$. $\square$

Now we prove the main theorem.

Theorem 3.2. *Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ satisfy (1.5), $f(\mathbb{R}) \setminus \{0, 1, -1\} \neq \emptyset$, $g(\mathbb{R}) \setminus \{0, 1\} \neq \emptyset$ and let f be locally bounded above at each point of $\mathbb{R}$. Then there exist $c \in \mathbb{R} \setminus \{0\}$, an infinite subgroup G of the group $(\mathbb{R} \setminus \{0\}, \cdot)$ and a non-constant multiplicative function $\phi : G \rightarrow \mathbb{R}$ such that*

$$g(x) = \begin{cases} cx + 1, & \text{if } cx + 1 \in G; \\ 0, & \text{if } cx + 1 \in \mathbb{R} \setminus G \end{cases} \quad (3.2)$$

and

$$f(x) = \begin{cases} \phi(cx + 1), & \text{if } cx + 1 \in G; \\ 0, & \text{if } cx + 1 \in \mathbb{R} \setminus G. \end{cases} \quad (3.3)$$

Proof. First we prove that $|g(x)| = 1$ for each $x \in A$. To this end we consider two cases.

First suppose that there is $x \in A$ such that $|g(x)| < 1$. According to Lemma 2.2(ii) $g(x) \neq 0$. Then, by (1.5), using induction we have

$$\begin{aligned} f(y) &= f(x)^n f(y) = f(x)^{n-1} f(x + g(x)y) \\ &= f(x)^{n-2} f(x + g(x)(x + g(x)y)) = f(x)^{n-2} f(x(1 + g(x)) + g(x)^2 y) \\ &= \dots = f(x(1 + g(x) + \dots + g(x)^{n-1}) + g(x)^n y) \\ &= f\left(x \frac{1 - g(x)^n}{1 - g(x)} + g(x)^n y\right) \end{aligned}$$

for every $y \in \mathbb{R}$ and $n \in \mathbb{N}$ and

$$\lim_{n \rightarrow \infty} \left(x \frac{1 - g(x)^n}{1 - g(x)} + g(x)^n y \right) = \frac{x}{1 - g(x)} \quad \text{for each } y \in \mathbb{R}.$$

Next suppose that there is $x \in A$ such that $|g(x)| > 1$. Then, by Lemma 2.2(iii), using induction we have

$$\begin{aligned} f(y) &= f(x)^{-n} f(y) = f(x)^{-n+1} f\left(\frac{y-x}{g(x)}\right) \\ &= f(x)^{-n+2} f\left(\frac{\frac{y-x}{g(x)} - x}{g(x)}\right) = f(x)^{-n+2} f\left(\frac{y - x(1+g(x))}{g(x)^2}\right) \\ &= \dots = f\left(\frac{y - x(1+g(x) + \dots + g(x)^{n-1})}{g(x)^n}\right) \\ &= f\left(g(x)^{-n} y - x \frac{1 - g(x)^n}{1 - g(x)} g(x)^{-n}\right) \\ &= f\left(g(x)^{-n} y - x \frac{g(x)^{-n} - 1}{1 - g(x)}\right) \end{aligned}$$

for every $y \in \mathbb{R}$ and $n \in \mathbb{N}$ and

$$\lim_{n \rightarrow \infty} \left(g(x)^{-n} y - x \frac{g(x)^{-n} - 1}{1 - g(x)} \right) = \frac{x}{1 - g(x)} \quad \text{for each } y \in \mathbb{R}.$$

In both cases for each $y \in \mathbb{R}$ there is a sequence $(z_n(y))_{n \in \mathbb{N}} \subset \mathbb{R}$ such that

$$\lim_{n \rightarrow \infty} z_n(y) = \frac{x}{1 - g(x)} \quad \text{and} \quad f(z_n(y)) = f(y). \quad (3.4)$$

Moreover, setting $z = 0$ in Lemma 2.2(iii) and using Lemma 2.2(i), we obtain that W is a subgroup of $(\mathbb{R} \setminus \{0\}, \cdot)$. Since $W \setminus \{-1, 1\} \neq \emptyset$, W is infinite. But f is locally bounded above at the point $\frac{x}{1-g(x)}$; i.e. there exist a real number M_0 and a neighbourhood U of $\frac{x}{1-g(x)}$ such that $f(u) \leq M_0$ for each $u \in U$. Since W is infinite, there is $y_0 \in \mathbb{R}$ with $f(y_0) > M_0$. Consequently, by (3.4), $z_{n_0}(y_0) \in U$ for some $n_0 \in \mathbb{N}$ and

$$M_0 < f(y_0) = f(z_{n_0}(y_0)) \leq M_0,$$

which is a contradiction.

In this way we proved that $g(x) = 1$ or $g(x) = -1$ for each $x \in A$. Moreover, since W is an infinite subgroup of $(\mathbb{R} \setminus \{0\}, \cdot)$, by Lemma 2.1 f is not periodic and thus, in view of Lemma 2.4(i), (ii) $A \cap B = \{\emptyset, \{0\}\}$ and $A \cap g^{-1}(\{-1\}) \in \{\emptyset, \{x_0\}\}$ for $x_0 \in \mathbb{R}$. Hence, according to Lemma 2.2(i) we obtain that either $A = \{0\}$, or $A = \{0, x_0\}$ for $x_0 \neq 0$ with $g(x_0) = -1$.

First consider the case, where $A = \{0, x_0\}$ with $x_0 \neq 0$ such that $g(x_0) = -1$. According to (1.5) $f(x_0 - x) = f(x) \neq 0$ for each $x \in F$. Hence, by Lemma 2.2(iv),

$$\frac{x_0 - 2x}{g(x)} \in A = \{0, x_0\} \quad \text{for } x \in F.$$

Thus, either $x = \frac{x_0}{2}$, or $x \neq \frac{x_0}{2}$ and $\frac{x_0 - 2x}{g(x)} = x_0$ for $x \in F$. Consequently,

$$g(x) = -\frac{2}{x_0}x + 1 \quad \text{for } x \in F \setminus \left\{\frac{x_0}{2}\right\}.$$

Suppose that $\frac{x_0}{2} \in F$. Hence, in view of (1.5), for each $x \in F \setminus \{\frac{x_0}{2}\}$,

$$f\left(\frac{x_0}{2}\right)f(x) = f\left(x + g(x)\frac{x_0}{2}\right) = f\left(x + \left(-\frac{2}{x_0}x + 1\right)\frac{x_0}{2}\right) = f\left(\frac{x_0}{2}\right) \neq 0.$$

Thus $f(x) = 1$ for $x \in F \setminus \{\frac{x_0}{2}\}$ and hence $W = \{1, f(\frac{x_0}{2})\}$. Since W is a subgroup of $(\mathbb{R} \setminus \{0\}, \cdot)$, $f(\frac{x_0}{2}) \in \{-1, 1\}$. But we have assumed that $W \setminus \{-1, 1\} \neq \emptyset$. This contradiction proves that $f(\frac{x_0}{2}) = 0$. Hence

$$g(x) = -\frac{2}{x_0}x + 1 \quad \text{for each } x \in F.$$

Setting $c = -\frac{2}{x_0}$ we obtain (3.5).

Now we consider the second case, where $A = \{0\}$. Then, by Lemma 2.3, there is $z_0 \in \mathbb{R} \setminus \{0\}$ such that $x = (g(x) - 1)z_0$ for each $x \in F$. It means that

$$g(x) = \frac{x}{z_0} + 1 \quad \text{for each } x \in F.$$

Setting $c = \frac{1}{z_0}$ we obtain (3.5).

In this way we proved that g is given by (3.5). Thus, in view of Lemma 2.2(ii), $-\frac{1}{c} \notin F$ and hence $0 \notin cF + 1$. Moreover, by Lemma 2.2(i), $0 \in F$, so $1 \in cF + 1$. Since f and g satisfy (1.5),

$$x + (cx + 1)y = x + g(x)y \in F \quad \text{for } x, y \in F$$

and

$$x + (cx + 1)y = x + g(x)y \in F' \quad \text{for } x \in F, y \in F'.$$

It means that

$$(cx + 1)(cy + 1) = c[x + (cx + 1)y] + 1 \in cF + 1 \quad \text{for } x, y \in F$$

and

$$(cx + 1)(cy + 1) = c[x + (cx + 1)y] + 1 \in cF' + 1 \quad \text{for } x \in F, y \in F'.$$

Consequently we have

$$\begin{cases} (cF + 1)(cF + 1) \subset cF + 1, \\ (cF + 1)(cF' + 1) \subset cF' + 1, \\ 1 \in cF + 1, \\ 0 \notin cF + 1. \end{cases}$$

It is easy to check that $cF' + 1 = (cF + 1)'$. Hence $G = cF + 1$ is a subgroup of $(\mathbb{R} \setminus \{0\}, \cdot)$. We prove that $G \setminus \{-1, 1\} \neq \emptyset$.

On the contrary suppose that $G \subset \{-1, 1\}$. Then $F \subset \{-\frac{2}{c}, 0\}$. Since, by (3.5) and Lemma 2.2(i)

$$\begin{aligned} f\left(-\frac{2}{c}\right)f\left(-\frac{2}{c}\right) &= f\left(-\frac{2}{c} + g\left(-\frac{2}{c}\right)\left(-\frac{2}{c}\right)\right) \\ &= f\left(-\frac{2}{c} - \left(-\frac{2}{c}\right)\right) = f(0) = 1, \end{aligned}$$

we have $f(\mathbb{R}) \subset \{-1, 0, 1\}$, which contradicts the assumption.

Now, define a function $\phi : G \rightarrow \mathbb{R}$ as follows:

$$\phi(x) = f\left(\frac{x-1}{c}\right) \quad \text{for each } x \in G.$$

Then, according to (1.5), for every $x, y \in G$

$$\begin{aligned} \phi(x)\phi(y) &= f\left(\frac{x-1}{c}\right)f\left(\frac{y-1}{c}\right) \\ &= f\left(\frac{x-1}{c} + x\frac{y-1}{c}\right) = f\left(\frac{xy-1}{c}\right) = \phi(xy). \end{aligned}$$

It means that ϕ is multiplicative on the set G . Moreover, ϕ is not constant, because W is infinite. Thus (3.3) holds, which ends the proof. $\square$

Remark 3.3. It is easy to check that functions g, f given by (3.5)–(3.3) with $c \in \mathbb{R} \setminus \{0\}$, a subgroup G of the group $(\mathbb{R} \setminus \{0\}, \cdot)$ and nonconstant multiplicative function $\phi : G \rightarrow \mathbb{R}$, satisfy (1.5).

Remark 3.4. There exist real solutions of (1.5) such that $f(\mathbb{R}) \subset \{-1, 0, 1\}$ and f is not constant; i.e. functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$f(x) = g(x) = \begin{cases} 1 & \text{for } x \in \mathbb{Z}, \\ -1 & \text{for } x \in \{k + \frac{1}{2} : k \in \mathbb{Z}\}, \\ 0 & \text{otherwise.} \end{cases}$$

From Theorem 3.2, we can easily obtain the following result:

Corollary 3.5. *Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be solutions of (1.5) such that f is locally bounded above at each point of $\mathbb{R}$, $g(\mathbb{R}) \setminus \{0, 1\} \neq \emptyset$ and $f(\mathbb{R}) \setminus \{0, 1, -1\} \neq \emptyset$. Then g is also locally bounded above at each point of $\mathbb{R}$.*

Using Theorem 3.2 with $g = M \circ f$, we can obtain the following corollary:

Corollary 3.6. *Let $f, M : \mathbb{R} \rightarrow \mathbb{R}$ satisfy the equation*

$$f(x + M(f(x))) = f(x)f(y),$$

$f(\mathbb{R}) \setminus \{0, 1, -1\} \neq \emptyset$, $(M \circ f)(\mathbb{R}) \setminus \{0, 1\} \neq \emptyset$ and let f be locally bounded above at each point of $\mathbb{R}$. Then there exist $c \in \mathbb{R} \setminus \{0\}$, an infinite subgroup G of the

group $(\mathbb{R} \setminus \{0\}, \cdot)$ and a nonconstant multiplicative function $\phi : G \rightarrow \mathbb{R}$ such that f is given by (3.3) and

$$M(y) = \begin{cases} \phi^{-1}(y) & \text{for } y \in \phi(G); \\ 0 & \text{for } y \in \mathbb{R} \setminus \phi(G). \end{cases} \quad (3.5)$$

Proof. By Theorem 3.2 we obtain that f is given by (3.3) and

$$M(f(x)) = \begin{cases} cx + 1, & \text{if } cx + 1 \in G; \\ 0, & \text{if } cx + 1 \in \mathbb{R} \setminus G \end{cases}$$

for $c \in \mathbb{R} \setminus \{0\}$, an infinite subgroup G of the group $(\mathbb{R} \setminus \{0\}, \cdot)$ and a nonconstant multiplicative function $\phi : G \rightarrow \mathbb{R}$. Thus $M(\phi(y)) = y$ for each $y \in G$. Then ϕ is one-to-one on G and hence

$$M(z) = M(\phi(y)) = y = \phi^{-1}(z)$$

for $z = \phi(y) \in \phi(G)$, which ends the proof. $\square$

Setting $f = g$ in Theorem 3.2, we have

Corollary 3.7. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a solution of (1.3), which is locally bounded above at each point of $\mathbb{R}$. If $f(\mathbb{R}) \setminus \{0, 1, -1\} \neq \emptyset$, then there exist $c \in \mathbb{R} \setminus \{0\}$ and a subgroup G of the group $(\mathbb{R} \setminus \{0\}, \cdot)$ such that*

$$f(x) = \begin{cases} cx + 1, & \text{if } cx + 1 \in G; \\ 0, & \text{if } cx + 1 \in \mathbb{R} \setminus G. \end{cases}$$

The above result refers to a result of Wołodźko [11, Proposition 2]. He obtained the same form of a function f satisfying (1.3) assuming that f is not microperiodic and constant. That is why the following question seems to be interesting:

Does Theorem 3.2 hold, if we replace the assumption that f is locally bounded above at each point of the line by the assumption that f is not periodic?

Example. There is a function satisfying (1.3), which is locally bounded at each point, but it is not continuous. To show it we define a function $f : \mathbb{R} \rightarrow \mathbb{R}$ in the following way:

$$f(x) = \begin{cases} x + 1 & \text{for } x \in \mathbb{Q}; \\ 0 & \text{for } x \in \mathbb{I}\mathbb{Q}. \end{cases}$$

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