Recent results on iteration theory: iteration groups and semigroups in the real case

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Abstract. In this survey paper we present some recent results in iteration theory. Mainly, we focus on the problems concerning real iteration groups (flows) and semigroups (semiflows) such as existence, regularity and embeddability. We also discuss some issues associated to the problem of embeddability, i.e. iterative roots and approximate iterability. The topics of this paper are: (1) measurable iteration semigroups; (2) embedding of diffeomorphisms in regular iteration semigroups in the space $\mathbb{R}^n$; (3) iteration groups of fixed point free homeomorphisms on the plane; (4) embedding of interval homeomorphisms with two fixed points in a regular iteration group; (5) commuting functions and embeddability; (6) iterative roots; (7) the structure of iteration groups on an interval; (8) iteration groups of homeomorphisms of the circle; (9) approximately iterable functions; (10) set-valued iteration semigroups; (11) iterations of mean-type mappings; (12) Hayers–Ulam stability of the translation equation. Most of the results presented here were obtained by means of functional equations. We indicate the relations between iteration theory and functional equations.

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1. Introduction

Iteration groups and semigroups are the main objects of study in iteration theory. They are also called flows and semiflows in the theory of dynamical systems. In this survey paper we present some selected achievements concerning iteration groups and semigroups that have been made during the last years. Let us recall the general definition.

Definition 1. Let G be an additive subgroup or subsemigroup of $\mathbb{R}$ or $\mathbb{C}$ and $X \neq \emptyset$ be a given set. A family of mappings $\{f^t: X \rightarrow X, t \in G\}$ is said to be an iteration group or iteration semigroup on X over G respectively if

$$f^t(f^s(x)) = f^{t+s}(x), \quad s, t \in G, \quad x \in X. \quad (1)$$

Usually we assume that $G = \mathbb{R}$ or $G = \mathbb{R}^+$ and then we call $\{f^t, t \in G\}$ an iteration group (flow) or semigroup (semiflow), respectively. Extending the domain of the iterative index t on an arbitrary algebraic structure G and introducing the notation $F(x, t) := f^t(x)$ we write (1) as the translation equation

$$F(F(x, t), s) = F(x, t + s), \quad s, t \in G, \quad x \in X. \quad (2)$$

The translation equation on algebraic structures was studied by Moszner and his collaborators. More information on this topic may be found in the survey article [102].

General enough but still a very useful condition is that X is a Banach space. Under suitable differentiability conditions the translation equation can be transformed into differential equations. Introducing

$$g(x) := \left. \frac{\partial F(x, t)}{\partial t} \right|_{t=0}$$

we obtain three equations

$$\frac{\partial F}{\partial t} = g \frac{\partial F}{\partial x}, \quad \frac{\partial F}{\partial t} = g \circ F, \quad g \frac{\partial F}{\partial x} = g \circ F$$

called Jabotinsky equations. The exact relations between the translation equation and Jabotinsky equations have been described by Aczél and Gronau [2]. For detailed results we refer the reader to the articles [35–37].

One origin of the notion of iteration groups is a natural extension of the domain of indices of discrete iterations f^n of a given mapping $f: X \rightarrow X$ to the real space. Another is a description of deterministic processes originated in the theory of ordinary differential equations. If we interpret an iterative index as time, then iteration semigroups and groups are the models of deterministic processes. A deterministic process, roughly speaking, describes the change of the position in time of the points in a given space X in such a way that after the given time they are uniquely determined. Usually one assumes additionally that the positions of the points are uniquely determined also in the past. In other words, the state of each object in the future and in the past is uniquely determined by the state of the object in the present moment independently of its itinerary. The points from the space X are called *objects* and their positions after time t are called the *states at the moment t* . Denote by $f^s(x)$ the state of x after time s . Then $f^t(f^s(x))$ means the state of $f^s(x)$ after time t . Hence it is the state of x after time $s + t$. On the other hand, $f^{t+s}(x)$ is also the state of x after time $t + s$ thus, by the uniqueness, we get the equality $f^t(f^s(x)) = f^{t+s}(x)$. This gives equation (1) for $G = \mathbb{R}$ or $G = \mathbb{R}^+$.

In iteration theory we generally assume that X is a metric space and the mappings f^t are of a suitable regularity. One can say that iteration theory is

a part of the translation equation theory but with suitable regularities. Iteration groups are strictly connected to dynamical systems. Their significance lies in the fact that they describe deterministic processes. For more information on iteration theory see [6, 64, 66, 152–154, 180]. This note is a continuation of the above papers. In the present survey we concentrate on selected topics connected to real iteration groups. Complex iteration groups have their own specificity which follows from the properties of holomorphic functions and the rings of formal power series. This is a very large domain and the subject is beyond the scope of this paper. A lot of information and references of this topic can be found in the survey articles [1, 120, 121].

We focus here on the selected problems of iteration theory strictly related to functional equations. We point out the role of functional equations as the basic research tools in the theory of iteration groups. We consider the following topics.

1. Measurable iteration semigroups.
2. Embedding of diffeomorphisms in regular iteration semigroups in the $\mathbb{R}^n$ space.
3. Iteration groups of fixed point free homeomorphisms on the plane.
4. Embedding of interval homeomorphisms with two fixed points in a regular iteration group.
5. Commuting functions and embeddability.
6. Iterative roots.
7. The structure of iteration groups of homeomorphisms on an interval.
8. Iteration groups of homeomorphisms of the circle.
9. Approximately iterable functions.
10. Set-valued iteration semigroups.
11. Iterations of mean-type mappings.
12. Hayers–Ulam stability of the translation equation.

2. Measurable iteration semigroups

First we discuss the problem of continuity of measurable iteration semigroups. Let X be a metric space. We begin with the following.

Definition 2. An iteration semigroup $\{f^t: X \rightarrow X, t > 0\}$ is said to be continuous if all functions f^t are continuous and for every $x \in X$ the mapping $t \mapsto f^t(x)$ is continuous.

Definition 3. An iteration semigroup $\{f^t: X \rightarrow X, t > 0\}$ is said to be measurable if all functions f^t are continuous and for every $x \in X$ the mapping $t \mapsto f^t(x)$ is Lebesgue measurable.

The question is, when a measurable iteration semigroup is continuous. For an arbitrary metric space the problem is still unsolved, but the answer is positive for particular wide classes of metric spaces. In 1979 in [159] the fact that

a measurable iteration semigroup is continuous was proved for closed bounded intervals. The first such result was extended for compact metric spaces (see [163]). Next Baron and Jarczyk in [8] generalized this result for locally compact metric spaces. The same authors jointly with Chojnacki in paper [9] proved this fact for separable metric spaces. Their results are stated as follows.

Theorem 1 (see [9]). *Let $\{f^t: X \rightarrow X, t > 0\}$ be an iteration semigroup such that every mapping f^t is continuous. Let $t_0 > 0$ and X be a separable metric space. If there exists a Lebesgue measurable set $M \subset (0, t_0)$ with positive Lebesgue measure such that for every $x \in X$ the mappings $t \mapsto f^t(x)$ restricted to M are Lebesgue measurable, then the restricted iteration group $\{f^t: X \rightarrow X, t > t_0\}$ is continuous with respect to both variables.*

This theorem also generalizes the previous result of Baron and Jarczyk from their paper [7], where X was a compact metric space and measurability was considered on a subset $M \subset (0, \infty)$. The open problem is if it is possible to omit the assumption of separability. The same authors also obtained the following result.

Theorem 2 (see [9]). *If X is a metric space and $\mathcal{F} = \{f^t: X \rightarrow X, t \in \mathbb{R}\}$ is an iteration group such that the mapping $(x, t) \mapsto f^t(x)$ is continuous separately with respect to t and x , then this mapping is continuous with respect to both variables.*

It is interesting that the assumption that X is a metric space cannot be replaced by the requirement that X is a topological space, even if it is separable (see [9]).

In the case of a real interval we have a stronger result.

Theorem 3 (see [159]). *If I is an interval and $\mathcal{F} = \{f^t: I \rightarrow I, t > 0\}$ is an iteration semigroup of continuous functions such that at least one $f^{\bar{t}}$ is injective but not surjective and $f^{\bar{t}} < \text{id}$ or $f^{\bar{t}} > \text{id}$, then $\mathcal{F}$ is continuous.*

Let X be a compact metric space and $cc(X)$ be a space of all non-empty compact subsets of X endowed with the Hausdorff metric. Let $\{f^t: X \rightarrow cc(X), t > 0\}$ be an iteration semigroup of continuous set-valued functions. Smajdor in paper [125] showed that if such an iteration semigroup is measurable, then it is continuous. In the same paper he also gave several generalizations of this theorem for a separable locally compact space.

In the further part we shall concentrate only on iteration groups defined on the N -dimensional Euclidean space as well as on intervals and the unit circle.

3. Embedding of diffeomorphisms in regular iteration semigroups in the $\mathbb{R}^N$ space

Definition 4. We say that a function $f: X \rightarrow X$ is embeddable in an iteration semigroup $\mathcal{F} = \{f^t: X \rightarrow X, t \in \mathbb{R}^+\}$ if $f^1 = f$. Then $\mathcal{F}$ is said to be an iteration semigroup of f .

If we consider the deterministic process that evolve in continuous time and the transition rule describing the iteration semigroup $\{f^t, t \geq 0\}$, then embeddability might be interpreted as follows: if the mapping f describes the change of all states in the phase space X after time 1, then the iterate f^t describes the change of the states after time t .

Koenigs (see [56]) was the first who considered the problem of embeddability for analytic functions. Next Szekeres (see [147]) in 1958 created the whole theory of embeddability for regular functions on an interval with one fixed point. This theory was further developed among others by L. Berg, R. Coifman, M. Kuczma, A. Lundberg, H. Michel, A. Smajdor (see e.g. [64] Chapter 9 and [66] Chapter 11). The embeddability of a given continuous mapping in continuous iteration semigroups on intervals and on the circle is described in details in the papers [39, 61, 158–160, 172]. Here we consider the problem of the embeddability of given diffeomorphisms defined on a subset of the $\mathbb{R}^N$ space in regular iteration semigroups. The problem is solved only for diffeomorphisms satisfying relatively strong assumptions.

Let U be subset of $\mathbb{R}^N$ with a non-empty interior.

Definition 5. A continuous iteration semigroup $\{F^t: U \rightarrow U, t \geq 0\}$ is said to be of class C^r if all mappings F^t for $t > 0$ are of class C^r .

We start with the iteration groups of linear operators in the $\mathbb{R}^N$ space. For this purpose we introduce the following.

Definition 6. We say that a square matrix S has a real logarithm if there exists a matrix A such that $S = \exp A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$.

Let S have a real logarithm A , that is, $S = e^A$, then putting $S^t := e^{tA}$, $t \in \mathbb{R}$ we have

$$S^t S^s = S^{t+s}, \quad t, s \in \mathbb{R}$$

and $S^1 = S$. This means that the family of matrices S^t for $t \in \mathbb{R}$ form a regular iteration group of S . Obviously we can treat a matrix as a linear operator in the $\mathbb{R}^N$ space. Conversely, if $\{S^t, t \in \mathbb{R}\}$ is a continuous iteration group of linear mappings, then there exists a matrix A such that $S^t = e^{tA}$ (see e.g. [43]).

A real matrix S has a real logarithm if and only if S is nonsingular and the geometric multiplicity of each real negative eigenvalue λ of S is even, that is the dimension of the subspace of eigenvectors $\{x: Sx = \lambda x\}$ is even. This is

equivalent to the property that each Jordan block of S belonging to a negative eigenvalue occurs an even number of times (see [29, 143]).

A real logarithm is usually not uniquely determined. A matrix S has a unique real logarithm if and only if all eigenvalues are real and positive and their geometric multiplicities are equal to one, in other words, no Jordan blocks of S belonging to the same eigenvalue appear more than once (see [29, 143]).

It is also known (see [34]) that S has a real logarithm if and only if S has a square root, that means there exists a real matrix T such that $S = T^2$. Moreover, if the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_N$ of S satisfy the inequalities $|\lambda_k - 1| < 1, k = 1, 2, \dots, N$, then $\log S := \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n!} (S - I)^n$ is a real logarithm of S (see [34]).

Denote by $C_\delta^r(U)$ the set of all functions $F: U \rightarrow \mathbb{R}^N$ of class C^r such that $F(0) = 0$ and

$$d^{(r)}F(x) = d^{(r)}F(0) + O(\|x\|^\delta), \quad \|x\| \rightarrow 0, \quad 0 \leq \delta \leq 1.$$

Let us consider the following general assumptions.

Hypothesis (i). $U \subset \mathbb{R}^N$ is a neighbourhood of the origin, $F: U \rightarrow U$ is a diffeomorphism, $F(0) = 0$ and $|\lambda_1| \leq |\lambda_2| \leq \dots \leq |\lambda_N| < 1$, where $\lambda_1, \lambda_2, \dots, \lambda_N$ are the eigenvalues of $S := dF(0)$.

Let r be an integer and $\delta \in [0, 1]$ be such that

$$|\lambda_N|^{r+\delta} < |\lambda_1|. \quad (3)$$

If $r \geq 3$ then we assume that F has no resonance up to r -th order that is $(H_r): \lambda_1^{q_1} \dots \lambda_N^{q_N} \neq \lambda_i$ for $i = 1, 2, \dots, N, q_1, \dots, q_N \in \mathbb{N}$ and $2 \leq \sum_{j=1}^N q_j \leq r$.

It is easy to see that, in the two dimensional case condition (H_r) is satisfied if and only if $\lambda_2^q \neq \lambda_1$ for $2 \leq q \leq r$.

A polynomial function $\eta_r: \mathbb{R}^N \rightarrow \mathbb{R}^N$ such that $\eta_r(x) = x + \sum_{k=2}^r L_k(x)$, where L_k are homogeneous polynomials of degree k satisfying the system

$$d^k(\eta_r \circ F - S\eta_r)(0) = 0, \quad k = 2, 3, \dots, r, \quad (4)$$

is said to be a *formal solution* of the Schröder equation

$$\varphi(F(x)) = S\varphi(x), \quad x \in U \quad (5)$$

of r -th order.

If the characteristic roots of S satisfy condition (H_r) , then there exists a unique polynomial function $\eta_r: \mathbb{R}^N \rightarrow \mathbb{R}^N$ of order r satisfying system (4) (see e.g. [133, 145]). If (5) has a C^r solution such that $\varphi(0) = 0$ and $d\varphi(0) = E$, then $d^{(k)}\varphi(0) = d^{(k)}\eta_r(0)$.

Let us recall the following generalization of Sternberg's and Kuczma's theorems on linearization (see [41, 65, 145]).

Theorem 4. *Let F satisfy (i), $r \geq 1, F \in C_\delta^r(U)$ with r and δ satisfying (3). If $r = 1$ and $0 \leq \delta \leq 1$ or $r = 2$ with $\delta = 0$ then in a neighbourhood V of the origin there exists the limit*

$$\varphi(x) := \lim_{n \rightarrow \infty} S^{-n} F^n(x). \quad (6)$$

If $r \geq 2, \delta > 0$ and there exist polynomials η_r satisfying system (4), then in a neighbourhood V of the origin, there exists the limit

$$\varphi(x) := \lim_{n \rightarrow \infty} \left(S^{-n} F^n(x) + \sum_{k=2}^r S^{-n} L_k(F^n(x)) \right), \quad (7)$$

where $L_k(x) = \frac{d^{(k)} \eta_r(0)}{k!}(x, \dots, x)$. The convergence is uniform in V . Moreover, the function φ given by (7) or (6) is of class $C_\delta^r(V)$ and φ satisfies Eq. (5).

Theorem 4 implies the following theorems.

Theorem 5. Let $F \in C_\delta^r(U)$ satisfy (i), condition (3) hold, 0 be a globally attractive fixed point of F and let $dF(0)$ have a real logarithm A . Then in a neighbourhood V of the origin there exists a unique C^r embedding $\{F^t: V \rightarrow V, t \geq 0\}$ of F such that $dF^t(0) = e^{tA}$, $t \geq 0$. Moreover,

$$F^t(x) = \varphi^{-1}(e^{tA} \varphi(x)), \quad t \geq 0, x \in V, \quad (8)$$

where φ is given by formula (6) if $r = 1$ and $0 \leq \delta \leq 1$ or $r = 2$ with $\delta = 0$ and by (7) if $r \geq 3$ and (H_r) holds.

If $F(U) = U$ then $V = U$ and formula (8) holds for every $t \in \mathbb{R}$.

It follows by the Grobman- Hartman theorem (see e.g. [123, 148]) that formula (8) occurs for C^1 iteration semigroups but the function φ although it satisfies Eq. (5) is not uniquely determined.

Theorem 6. Let $F: U \rightarrow U$ be a bijection of class C^2 satisfying (i), $|\lambda_N|^2 < |\lambda_1|$, 0 be a globally attractive fixed point of F and $dF(0)$ have a real logarithm. Then for every real matrix A such that $e^A = dF(0)$ there exists the limit

$$\lim_{n \rightarrow \infty} F^{-n}(e^{tA} F^n(x)) =: F^t(x), \quad t \geq 0 \quad (9)$$

for $x \in U$. The family $\{F^t, t \geq 0\}$ forms a C^2 embedding of F in U such that $dF^t(0) = e^{tA}$.

A different approach to the problem of embeddability was presented by Arango and Gómez [4]. They showed that for a mapping $F \in C_0^3(U)$ satisfying (i) and (3) with $r = 2, \delta = 0$ and $dF(0) = S$ possessing a real logarithm there exists the limit

$$G(x) := \lim_{n \rightarrow \infty} (dF^n(x))^{-1} \log S F^n(x).$$

This vector field is of class C^2 and generates the iteration group of diffeomorphisms $\{F^t, t \in \mathbb{R}\}$ of F such that $\frac{\partial F^t(x)}{\partial t}|_{t=0} = G(x)$.

In the case of one dimensional space formula (9) for a homeomorphism $F: [0, \infty) \rightarrow [0, \infty)$ without fixed points in $(0, \infty)$, differentiable at 0 such that $F'(0) =: s > 1$ was extensively investigated by Domsta in [30]. He showed that:

1. Every iteration semigroup $\{F^t, t > 0\}$ of F such that all F^t are differentiable at 0 is given by formula (9).
2. If F is continuous and differentiable at zero with $F'(0) > 1$, but is not sufficiently regular, then the following singularities may happen:
 - (a) limit (9) does not need to exist,
 - (b) limit (9) exists, but (9) does not define a semigroup,
 - (c) limit (9) exists and define a semigroup, but $F \neq F^1$.

For a mapping defined on the plane the form of C^2 iteration semigroups simplifies. The description of iteration semigroups of F depends on the eigenvalues of $dF(0) = S$. We obtain the following four different forms.

Let matrix P be such that $P^{-1}SP$ is a real Jordan form of S .

Theorem 7 ([142]). *Let $0 \in \text{Int } U \subset \mathbb{R}^2, F: U \rightarrow U$ be of class C^2 and $F(0)=0$.*

- (i) *If $dF(0) =: S$ has two different real eigenvalues $\lambda_1 < \lambda_2 < 1$ and $|\lambda_2|^2 < |\lambda_1|$, then F is C^2 -embeddable if and only if $\lambda_1, \lambda_2 > 0$. Moreover, F has a unique C^2 embedding and this embedding is given by*

$$F^t(x) = \varphi^{-1}(\lambda_1^t \varphi_1(x), \lambda_2^t \varphi_2(x)), \quad t \geq 0, x \in V.$$

- (ii) *If $dF(0)$ has one double eigenvalue $|\lambda| < 1$ and $dF(0) \neq \lambda E$, then F is C^2 -embeddable if and only if $\lambda > 0$. Moreover, F has a unique C^r -embedding and this embedding is given by*

$$F^t(x) = \varphi^{-1} \left(\lambda^t \left(\varphi_1(x) + \frac{t}{\lambda} \varphi_2(x) \right), \lambda^t \varphi_2(x) \right), \quad t \geq 0, x \in V.$$

- (iii) *If $dF(0)$ has a pure complex eigenvalue $|\lambda| < 1$, then F has countably many C^2 -embeddings. These embeddings are given by the formula*

$$F^t(x) = \varphi^{-1} \left(|\lambda|^t \begin{bmatrix} \cos t(\text{Arg } \lambda + 2k\pi) & -\sin t(\text{Arg } \lambda + 2k\pi) \\ \sin t(\text{Arg } \lambda + 2k\pi) & \cos t(\text{Arg } \lambda + 2k\pi) \end{bmatrix} \varphi(x) \right)$$

for $t \geq 0, x \in V$.

- (iv) *If $dF(0) = \lambda E$ and $|\lambda| < 1$ then F has uncountably many C^2 -embeddings. These embeddings are given by the formula*

$$F^t(x) = \varphi^{-1} \left(\text{sgn } \lambda |\lambda|^t Q \begin{bmatrix} \cos 2tk\pi & -\sin 2tk\pi \\ \sin 2tk\pi & \cos 2tk\pi \end{bmatrix} Q^{-1} \varphi(x) \right)$$

for $t \geq 0, x \in V$, where Q is an arbitrary nonsingular matrix.

In all the above cases $V \subset U$ is a neighbourhood of the origin and $(\varphi_1, \varphi_2) = \varphi$ is given by

$$\varphi(x) = P \lim_{n \rightarrow \infty} S^{-n} F^n(x), \quad x \in V.$$

4. Iteration groups of fixed point free homeomorphisms on the plane

In this section we describe the continuous iteration groups of mappings without fixed points defined on the plane.

Definition 7. A Brouwer homeomorphism (free mapping) is a homeomorphism of the plane onto itself without fixed points which preserves the orientation.

Using the Abel functional equation one can obtain a description of a Brouwer homeomorphism. A connection between free mappings and a translation was discovered by Brouwer. The modified version of his theorem is the following.

Theorem 8 (Brouwer Translationssatz, [15, 38, 124]). *Let f be a free mapping. For every point $p \in \mathbb{R}^2$ there exist a simply connected region U_p such that $p \in U_p$, $f[U_p] = U_p$ and a homeomorphism $\varphi: U_p \rightarrow \mathbb{R}^2$ satisfying the Abel functional equation*

$$\varphi(f(x)) = \varphi(x) + e_1, \quad x \in U_p, \quad e_1 = (1, 0) \quad (10)$$

so that for every $t \in \mathbb{R}$ the preimage $\varphi^{-1}[\{t\} \times \mathbb{R}]$ is closed on the plane.

Definition 8. Let $D \subset \mathbb{R}^2$ be a simply connected region. A homeomorphism f of D onto itself such that every Jordan domain $B \subset D$ meets at most a finite number of its images $f^n[B]$, $n \in \mathbb{Z}$, is said to be a *Sperner homeomorphism*.

Theorem 9 ([3, 146]). *Let D be a simply connected region. A mapping $f: D \rightarrow D$ is a Sperner homeomorphism which preserves the orientation if and only if it is conjugated to a translation, that is there exists a homeomorphic solution of the Abel equation (10) in D .*

The iteration groups of a Sperner homeomorphism have a very simple form.

Theorem 10 ([80]). *Let D be a simply connected region. If $\{f^t: D \rightarrow D, t \in \mathbb{R}\}$ is an iteration group of a Sperner homeomorphism, then it is conjugated to the group of translations, that is*

$$f^t(x) = \varphi^{-1}(\varphi(x) + te_1), \quad x \in D, \quad t \in \mathbb{R}$$

for a homeomorphism $\varphi: D \rightarrow \mathbb{R}^2$.

Let f be a free homeomorphism. Define in $\mathbb{R}^2$ the following co-divergence relation $p \sim q$ if $p = q$ or p and q are endpoints of some arc K for which

$$f^n[K] \rightarrow \infty \quad \text{as } n \rightarrow \pm\infty.$$

This is an equivalence relation. Each equivalence class G of this relation is simply connected and $f|_G$ is conjugated to a translation. The above relation was introduced by Andrea [3], he also investigated the equivalence classes of “ $\sim$ ”. His ideas were modified and developed by Brown [16] and Leśniak in papers [69, 71–74]. Moreover, in papers [75–77] Leśniak used this notion to describe the behaviour of the iteration groups of free mappings. The following results

are the starting point for the general construction of the iteration groups of homeomorphisms.

Theorem 11 ([73]). *If $\{f^t, t \in \mathbb{R}\}$ is a continuous iteration group of Brouwer homeomorphisms, then in each equivalence class G of $f = f^1$, the iteration group $\{f^t|_{\text{Int } G}, t \in \mathbb{R}\}$ is conjugated to a group of translations, i.e. this is a group of Sperner homeomorphisms.*

Theorem 12 ([73]). *For every equivalence class G there exists a maximal simply connected region H invariant under $\{f^t, t \in \mathbb{R}\}$ such that $G \subset H$ and $\{f^t|_H, t \in \mathbb{R}\}$ is conjugated to a group of translations.*

Such a region is said to be parallelizable with respect to the iteration group $\{f^t, t \in \mathbb{R}\}$.

Theorem 13 ([72]). *A maximal parallelizable region H of $\{f^t, t \in \mathbb{R}\}$ is a union of equivalence classes with respect to f^1 .*

Applying the above theorems Leśniak gave a complete description of all iteration groups of Brouwer homeomorphisms. The detailed construction is described in paper [78].

The idea of the general construction of the above iteration groups is to cover the plane by a sum of parallelizable regions and use the foliation induced by the flow (see [69, 156]). These regions need not be pairwise disjoint. The problem is that there is a lot of combinations of the coincidence conditions of generating homeomorphisms. For describing these coincidence relations Leśniak built a special diagram consisting of finite sequences with a clever partial order which describes the appropriate family of the maximal parallelizable regions with its generating homeomorphisms. Its construction is based on the conception of the Kaplan diagram (see [11, 52, 53]). This ordered space and a system of specially associated generating homeomorphisms make a skeleton for the construction of the global iteration group of fixed point free homeomorphisms.

5. Embedding of interval homeomorphisms with two fixed points in a regular iteration group

The embeddability of diffeomorphisms with two hyperbolic fixed points in a C^1 iteration group is very exceptional. If $f: I \rightarrow I$, where I is an interval, is a C^r diffeomorphism with no fixed points, then it is embeddable in infinitely many C^r iteration groups. If a C^r diffeomorphism has only one hyperbolic fixed point and $r \geq 2$, then f is also C^r embeddable, but has a unique C^r embedding in this case. If f has more than one hyperbolic fixed point, then even strong regularity of f does not imply regular embeddability. The problem of the criterion for the function to have a C^1 embedding is still open. There are known only noneffective criterions of C^1 embeddability (see [12, 40, 60, 67]). The possibility of the existence of such an embedding was also studied in [182].

The results in [182] imply that the set of C^1 embeddable functions is of the first category in the space of C^1 diffeomorphisms with two hyperbolic fixed points which have the second derivative at fixed points with the topology induced by C^1 -norm.

Karlin and McGregor in papers [54, 55] investigated this problem for probability generating functions. Let f be a probability generating function and $f(0) = 0$, then $f(1) = 1$ and $f(x) < x$ for $x \in (0, 1)$. They proved that if f is single valued and the set of singularities of f in the extended complex plane is at most countable, then f is embeddable in an iteration group of analytic functions if and only if f is a linear fractional function. Moreover, under slightly weaker assumptions the only functions embeddable in the probability generating iteration groups are mappings of the form

$$f(x) = \frac{x(1-p)^{\frac{1}{k}}}{(1-px^k)^{\frac{1}{k}}}, \quad k \in \mathbb{N}, \quad 0 < p < 1$$

and the only embeddable meromorphic single valued function in the extended complex plane in an iteration group of analytic functions are the linear fractional mappings.

For C^1 iteration groups the problem was solved only for the mappings which are locally linear in a neighbourhood of a fixed point.

Theorem 14 ([178]). *Let $0 < a < b < 1$ and the function $g: [0, a] \cup [b, 1] \rightarrow I$ satisfy conditions $g(0) = 0, g(1) = 1$ and let both $g|_{[0,a]}$ and $g|_{[b,1]}$ be linear.*

- (i) *If $g(b) < a$, then any extension of g does not have a C^1 embedding.*
- (ii) *If $g(b) = a$ and $g'(1)^a \neq g'(0)^{a-1}$, then any extension of g does not have a C^1 embedding.*
- (iii) *If $g(b) = a$ and $g'(1)^a = g'(0)^{a-1}$, then there exists a unique extension of g which has a C^1 embedding.*
- (iv) *If $g(b) > a$, then g has infinitely many C^1 extensions which have C^1 embeddings.*

It turns out that if we modify the C^2 function $f: [0, 1] \rightarrow [0, 1]$ in some small subinterval $J \subset (0, 1)$, then we get a C^1 embeddable function. To precisely describe this property let us consider the family of functions satisfying the following assumption

(P) $f \in \text{Diff}^1[0, 1]$, $0 < f(x) < x, x \in (0, 1)$, $f'(0) \neq 1$, $f'(1) \neq 1$ and there exists $\delta > 0$ such that

$$f'(x) = f'(0) + O(x^\delta), \quad x \rightarrow 0 \quad \text{and} \quad f'(x) = f'(1) + O((x-1)^\delta), \quad x \rightarrow 1.$$

If a mapping f satisfies (P), then there exist the limits

$$\Phi_0(x) := \log f'(0) \lim_{n \rightarrow \infty} \frac{f^n(x)}{(f^n)'(x)}, \quad \Phi_1(x) := -\log f'(1) \lim_{n \rightarrow \infty} \frac{f^{-n}(x) - 1}{(f^{-n}(x))'(x)}$$

for $x \in [0, 1]$.

Theorem 15 ([177]). *Let $b \in (0, 1)$, $a \in (f(b), b)$, $h: [a, b] \rightarrow [f(a), f(b)]$ be an injection of class C^1 , $h' > 0$, $h(x) \neq x$, $x \in (a, b)$, $h'(a) = f'(a)$ and $h'(b) = f'(b) \frac{\Phi_0(b)}{\Phi_1(b)}$. Then there exists a unique C^1 embeddable function $\tilde{f}$ defined on $[0, 1]$ such that*

$$\tilde{f}|_{[a,b]} = h \quad \text{and} \quad \tilde{f}(x) = f(x) \quad \text{for } x \in [0, a] \cup [f^{-1}(b), 1].$$

Moreover, the C^1 iteration group $\{\tilde{f}^t, t \in \mathbb{R}\}$ of $\tilde{f}$ coincides locally with C^1 iteration groups of $f|_{[0,1]}$ and $f|_{(0,1]}$.

If $\Phi_0(b) = \Phi_1(b)$ then there exists a unique C^1 embeddable function $\hat{f}$ such that

$$\hat{f}(x) = f(x), \quad x \in [0, 1] \setminus (b, f^{-1}(b)).$$

Now we discuss the problem of normal forms for C^1 iteration groups with two hyperbolic fixed points. Let $0 < s < 1 < M$. Consider the following family of functions

$$p^t(x) := \frac{s^t x}{(1 + (s^{tk} - 1)x^k)^{\frac{1}{k}}}, \quad x \in [0, 1], \quad t \in \mathbb{R},$$

where $k := -\frac{\log M}{\log s}$. This family is a C^∞ iteration group such that $(p^t)'(0) = s^t$ and $(p^t)'(1) = M^t$. We have the following conjugacy property.

Theorem 16 ([60]). *Let f satisfy (P). If $\{f^t: [0, 1] \rightarrow [0, 1], t \in \mathbb{R}\}$ is a C^1 iteration group of f , then*

$$f^t = \psi^{-1} \circ p^t \circ \psi, \quad t \in \mathbb{R},$$

where $\psi: [0, 1] \rightarrow [0, 1]$ is a diffeomorphism given by

$$\psi(x) = \lim_{n \rightarrow \infty} \frac{f^n(x)}{[(f^n(x_0))^k + (f^n(x))^k]^{1/k}}, \quad x \in [0, 1]$$

and $x_0 \in (0, 1)$ is arbitrarily chosen.

This theorem generalizes the result presented in [12] where it was proved that for $r \geq 2$ all C^r flows with exactly two hyperbolic fixed points are C^r conjugated.

Such a property is not true for C^1 diffeomorphisms.

Theorem 17 ([60]). *There exist C^1 iteration groups of a function f satisfying (P) such that $(f^t)'(0) \neq 1$, $(f^t)'(1) \neq 1$ for $t \neq 0$ which are not diffeomorphically conjugated to any group $\{p^t: [0, 1] \rightarrow [0, 1], t \in \mathbb{R}\}$.*

6. Commuting functions and embeddability

Let f and g satisfy the following assumption

- (C) I is an open interval, $f, g: I \rightarrow I$ are continuous and strictly increasing, $f(x) < x, g(x) < x$ for $x \in I$ and

$$f \circ g = g \circ f.$$

In paper [63] Kuczma proved that if $f \in C^{1+\delta}[[a, b]]$ and $g \in C^1[[a, b]]$ satisfy (C) on (a, b) , then f and g are embeddable in a C^1 iteration semigroup and this semigroup is uniquely determined. We consider a similar problem for functions defined on an open interval, without fixed points where even the asymptotic regularity at the ends of the interval does not have any influence on embeddability.

For every $x \in I$ there exists a unique sequence $\{m_k(x) : k \in \mathbb{N}\}$ of positive integers such that

$$f^{m_k(x)+1}(x) \leq g^k(x) \leq f^{m_k(x)}(x). \quad (11)$$

Moreover, there exists the finite limit

$$\lim_{k \rightarrow \infty} \frac{m_k(x)}{k} =: s(f, g) \quad (12)$$

and it does not depend on x . Notice that $s(f, g) \notin \mathbb{Q}$ if and only if f and g are iteratively incommensurable, i.e. for every $x \in I$ and every $n, m \in \mathbb{N}$ $f^n(x) \neq g^m(x)$ (see [101, 166]).

We have the following result.

Theorem 18 ([59]). *If f and g satisfy (C), $s(f, g) \notin \mathbb{Q}$, $f, g \in \text{Diff}^1(I)$ and f', g' are of bounded variation in I , then there exists a unique iteration semigroup $\{h^t, t \geq 0\}$ such that $h^1 = f$ and $g \in \{h^t, t \geq 0\}$. Then $g = f^{s(f, g)}$. Moreover,*

$$h^t(x) = \lim_{k \rightarrow \infty} g^{-m_k} \circ f^{n_k}(x), \quad x \in I, \quad t \geq 0$$

for every sequence $\{(n_k, m_k)\}$ such that $\lim_{k \rightarrow \infty} (n_k - s(f, g)m_k) = t$.

In the particular case if f is convex and g is concave, then h^t are affine functions.

The last statement is related to the results of Matkowski in [94] for M -convex functions. He dealt with continuous iteration groups such that

$$f^t(M(x, y)) \leq M(f^t(x), f^t(y)) \quad \text{or} \quad f^t(M(x, y)) \geq M(f^t(x), f^t(y)),$$

where $M: (0, \infty) \times (0, \infty) \rightarrow (0, \infty)$ is a continuous function and proved that if f^a and f^b are M -convex and $a < 0 < b$, then the group $\{f^t, t \in \mathbb{R}\}$ is M -affine, that is $f^t(M(x, y)) = M(f^t(x), f^t(y))$ for $t \in \mathbb{R}$.

The case where f^a and f^b are subadditive was considered in [93]. These results were generalized by Krassowska [57] for (M, N) convex iteration groups.

Let $\mathcal{F} = \{f^t: I \rightarrow I, t \in \mathbb{R}\}$ be a continuous iteration group of a fixed point free homeomorphisms. Matkowski (see [92]) proved that if a function $g: I \rightarrow I$ is continuous at least at one point and commutes with two mappings f^a, f^b and $\frac{a}{b}$ is irrational, then $g \in \mathcal{F}$. In [97] he also showed that if, moreover, $a < 0 < b$, then the commutativity of g with f^a and f^b can be replaced by the inequalities

$$g \circ f^a \leq f^a \circ g, \quad g \circ f^b \leq f^b \circ g. \quad (13)$$

Now let V be at least a two-dimensional subspace of the vector space $(\mathbb{R}, \mathbb{Q}, +, \cdot)$. Let a family of homeomorphisms $\mathcal{F} = \{f^v: I \rightarrow I, v \in V\}$ be an iteration group. If $\mathcal{F}$ is a dense iteration group, then there exist a homeomorphism $\varphi: I \rightarrow \mathbb{R}$ and an additive function $c: V \rightarrow \mathbb{R}$ such that $f^v(x) = \varphi^{-1}(\varphi(x) + c(v)), x \in I, v \in V$ (see Tabor [149]). Ciepliński gave the following generalization of Matkowski's results.

Theorem 19 ([26]). *If $\mathcal{F}$ is a dense iteration group and a function $g: I \rightarrow I$ is continuous at least at one point and commutes with two mappings $f^a, f^b \in \mathcal{F}$ such that $\frac{b}{a}$ is irrational, then g is topologically conjugated to a translation. If the additive function c is surjective, then $g \in \mathcal{F}$. If, moreover, $f^a < \text{id} < f^b$, then the commutativity of g with f^a and f^b can be replaced by inequalities (13).*

For further discussion on the regularity of iteration groups we need to introduce the following set. Write $x \in \mathbb{R} \setminus \mathbb{Q}$ in the form of an infinite continuous fraction

$$x = [x] + \frac{1}{a_1(x) + \frac{1}{a_2(x) + \frac{1}{a_3(x) + \ddots}}},$$

where $a_i(x), i = 1, 2, \dots$ are positive integers.

Denote by A the set of all $x \in \mathbb{R} \setminus \mathbb{Q}$ such that

$$\lim_{B \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{\sum_{i \leq n, a_i(x) \geq B} \log(a_i(x) + 1)}{\sum_{i \leq n} \log(a_i(x) + 1)} = 0.$$

The set A is of full Lebesgue measure in $\mathbb{R}$ (see [42], Ch.V.9).

Theorem 20 ([162]). *If $f, g \in \text{Diff}^r[I], 3 \leq r \leq \infty$ and $s(f, g) \in A$, then there exists a unique C^{r-2} iteration semigroup such that $f^1 = f, f^s = g$ for an $s \in \mathbb{R}$.*

If f and g are surjections, then putting $h^{-t} = (h^t)^{-1}$ for $t > 0$ we can extend the above iteration semigroups to groups.

Now we drop the assumption of regularity of f and g but assume that they are surjective. To give a necessary and sufficient condition for the embeddability of commuting homeomorphisms in an irregular iteration group we need to introduce two sets $L_{f,g}$ and $K_{f,g}$. The first one is defined as follows

$$L_{f,g} := \{f^m \circ g^n(x), n, m \in \mathbb{Z}\}^d.$$

This set does not depend on x and either $L_{f,g}$ is a Cantor set or $L_{f,g} = I$ (see [58, 161]). If $s(f, g) \notin \mathbb{Q}$, then the system of Abel equations

$$\begin{aligned}\varphi(f(x)) &= \varphi(x) + 1, \\ \varphi(g(x)) &= \varphi(x) + s(f, g)\end{aligned}$$

has a unique monotonic continuous solution up to an additive constant. This solution is invertible if and only if $L_{f,g} = cl I$. If $L_{f,g}$ is a Cantor set then the solution is constant on every component of the set $I \setminus L_{f,g}$ (see [32, 164]).

Theorem 21 ([165]). *Let $s(f, g) \notin \mathbb{Q}$ and put $K_{f,g} := \varphi[I \setminus L_{f,g}]$. The functions f and g are embeddable in an iteration group of continuous functions if and only if $K_{f,g} = \emptyset$ or $K_{f,g} = \mathbb{Q} + s(f, g)\mathbb{Q} + T$, where a set T is at most countable.*

If $K_{f,g} = \emptyset$ then φ is invertible and

$$f^t(x) = \varphi^{-1}(\varphi(x) + a(t)),$$

where a is an additive function.

The problem of the embeddability of commuting homeomorphisms in an abelian group was also considered by Winkler [157]. The author completely classified the maximal abelian subgroups of $\text{Aut}(\mathbb{R}, \leq)$, the automorphism group of the real line.

Further we consider the case where f and g are not embeddable. For such functions we consider the sets of “phantom iterates” which extend the set of the original objects of iteration. The phantoms here are set-valued mappings. We construct a special iteration group of set-valued functions in which f and g can be embedded as some selectors. The idea of generalized embeddings appeared already in papers of Reich and Schweiger [119, 122] and was used to find the necessary and sufficient conditions for the embeddability of formally biholomorphic mappings. This idea was popularized by Gy.Targonski (see [151, 153, 154]) but he defined phantom iterates using the Koopman operator and semigroups of linear operators on suitable commutative algebras.

Let f and g be homeomorphisms satisfying (C). Define the following two families of functions

$$\begin{aligned}f_-^t(x) &:= \sup\{f^n \circ g^{-m}(x) : n - sm > t, n, m \in \mathbb{N}\}, \\ f_+^t(x) &:= \inf\{f^n \circ g^{-m}(x) : n - sm < t, n, m \in \mathbb{N}\}\end{aligned}$$

for $x \in I, t \in \mathbb{R}$, where $s = s(f, g) \notin \mathbb{Q}$. The family of functions $\{f_+^t, t \in \mathbb{R}\}$ and $\{f_-^t, t \in \mathbb{R}\}$ are iteration groups of non-decreasing functions and $t \mapsto f_+^t(x)$ and $t \mapsto f_-^t(x)$ are non-increasing. If $L_{f,g} \neq cl I$ then the above iteration groups are discontinuous with respect to both variables (see [176]).

Denote $cc[I] := \{[c, d] \subset I\}$ and put

$$F^t(x) := [f_-^t(x), f_+^t(x)]. \quad (14)$$

Theorem 22 ([176]). *The family $\{F^t: I \rightarrow cc[I], t \in \mathbb{R}\}$ is an iteration group in the sense of set-valued functions, i.e.*

$$F^u \circ F^v(x) = F^{u+v}(x), \quad u, v \in \mathbb{R},$$

where

$$F^u \circ F^v(x) := \bigcup_{y \in F^v(x)} F^u(y), \quad (15)$$

such that $f(x) \in F^1(x)$ and $g(x) \in F^{s(f,g)}(x)$. Moreover, for every $t \in \mathbb{R}, x \mapsto F^t(x)$ is increasing and for every $x \in I, t \mapsto F^t(x)$ is strictly decreasing.

Here we understand the monotonicity of set-valued functions in the following sense:

if $x < y$ then either $F^t(x) = F^t(y)$ or for every $u \in F^t(x)$ and every $v \in F^t(y)$ we have $u < v$ and if $t < s$ then for every $u \in F^t(x)$ and every $v \in F^s(x)$ we have $u < v$.

Theorem 23 ([176]). *If an iteration group $\{f^t, t \in \mathbb{R}\}$ is such that $f^1 = f$ and $f^s = g$ for an $s \in \mathbb{R}$, then there exists an additive function γ such that $\gamma(1) = 1, \gamma(s) = s(f, g)$ and*

$$f^t(x) \in F^{\gamma(t)}(x).$$

If $\text{Int } L_{f,g} \neq \emptyset$ then for every $t \in \mathbb{R}, F^t(x)$ is a singleton. If $\text{Int } L_{f,g} = \emptyset$ then for every $t \in \mathbb{R}$ and every $x \in I \setminus L_{f,g}, F^{\gamma(t)}(x)$ is an interval.

7. Iterative roots

Iterative roots are strongly associated to the problem of embeddability. Let f be a self-mapping. Every solution φ of the functional equation $\varphi^n = f$ is called an *iterative root of n -th order of f* . If a given self-mapping f is embeddable in an iteration semigroup $\{f^t, t > 0\}$, then the function $\varphi := f^{\frac{1}{n}}$ is an iterative root of f of n -th order. The converse implication does not have to be true. References connected to this topic can be found for example in [64, 66, 152]. A big part of the research was devoted to the roots of monotonic functions. In 1961, in the paper [62], Kuczma gave a complete description of the iterative roots of continuous, strictly monotonic self-mappings of a given interval. In particular, each such function has infinitely many roots of a given order. The situation changes completely if we drop the assumption that the monotonicity is strict. Let $\mathcal{F}_n(I)$ be the set of all continuous and non-decreasing self-mappings of a compact interval I having continuous iterative roots of order n . As follows from papers [13] by Blokh and [144] by Simon, for every $n \geq 2$ the set $\mathcal{F}_n(I)$ is small in terms of both measure and category in the space $C(I, I)$ of all continuous self-mappings of I endowed with sup-norm.

However, the paper [44] by Humke and Laczkowich shows that this set is analytic and non-Borel in $C(I, I)$. Nonexistence of roots is typical also for C^1 -smooth functions with two fixed points which was observed by Weinian Zhang in [182].

Further we focus on iterative roots in some selected classes of functions, mainly for mappings which are non-embeddable.

7.1. Iterative roots of piecewise monotonic functions

Denote by $\sqrt[n]{f}$ the set of all continuous roots of n -th order of f . By a *lap* of a piecewise monotone function we mean the maximal interval of monotonicity of this map.

A continuous piecewise strictly monotone map $f: I \rightarrow I$, where I is a closed interval is called a *horseshoe map* if it has more than one lap and each lap is mapped by f onto I . We shall call a horseshoe map *strict* if it has no homtervals (intervals on which all iterates of the map are monotone).

The *type* of a horseshoe map f will be a pair (m, σ) , where m is the number of laps of f and σ indicates whether f is increasing or decreasing on the leftmost lap. The complete solution to the problem of the existence of iterative roots of strict horseshoe maps was given by Blokh, Coven, Misiurewicz and Nitecki.

Theorem 24 ([14]). *Let f be a strict horseshoe map of type (m, σ) .*

- (i) *If $\sqrt[n]{m} \notin \mathbb{N}$, then $\sqrt[n]{f} = \emptyset$.*
- (ii) *If $\sqrt[n]{m} \in \mathbb{N}$, $m(n+1)$ is odd and $\sigma = \downarrow$, then $\sqrt[n]{f} = \emptyset$.*
- (iii) *If $\sqrt[n]{m} \in \mathbb{N}$, $m(n+1)$ is even, then $\sqrt[n]{f}$ has a unique element.*
- (iv) *If $\sqrt[n]{m} \in \mathbb{N}$, $m(n+1)$ is odd and $\sigma = \uparrow$, then $\sqrt[n]{f}$ has exactly two elements.*
- (v) *If $\sqrt[n]{f} \neq \emptyset$, then every map in $\sqrt[n]{f}$ is piecewise monotone.*

Zhang and Yang in 1983 in the paper [179] written in Chinese initiated the study of the problem of iterative roots of piecewise monotonic functions. Their method based on the notion of so called “characteristic interval” was described in the paper mentioned above and then was developed in [85, 183].

Let $F: I \rightarrow \mathbb{R}$ be a continuous function. An interior point $c \in I$ is called a *fort* of F if F is strictly monotonic in no neighborhood of c . Denote by $S(F)$ the set of all forts of F . If F has finitely many forts, then F is piecewise monotonic. By $\mathcal{PM}(I)$ we denote the set of all continuous piecewise monotonic functions with finitely many forts. If $F \in \mathcal{PM}(I)$ then $S(F^n) \subset S(F^{n+1})$ for $n \in \mathbb{N}$. If in addition, there is a $k \geq 1$ such that $S(F^k) = S(F^{k+1})$, then $S(F^k) = S(F^{k+i})$ for every $i \in \mathbb{N}$ (see [84, 183]).

For a given function $F \in \mathcal{PM}(I)$ we define the *nonmonotonicity height* $H(F)$ as the least $k \geq 0$ satisfying $S(F^k) = S(F^{k+1})$ if such a k exists and ∞ otherwise.

Note that for horseshoe mappings the nonmonotonicity height is infinite. However, $H(F) = 0$ if and only if F is monotonic and in this case we have a full description of iterative roots (see e.g. [62]). In [181] Zhang and Zhang gave an algorithm to compute piecewise linear iterative roots of a piecewise linear monotonic mapping with finitely many forts.

In the case $H(F) = 1$ there is a maximal sub-interval of I , denoted by $K(F)$, which covers the range of F so that F is strictly monotonic on it (see [183]). Such a sub-interval of I is unique and is called the *characteristic interval* of F . The problem can be reduced to the monotonic case. If F is increasing (decreasing) on $K(F)$, then F has iterative roots of all orders (all odd orders) (see [183]). In the first case F is also embeddable in a continuous iteration semigroup (see [158]). In [85] something more was proved, namely

Theorem 25. *Every continuous iterative root of a piecewise monotonic function F with $H(F) = 1$ is an extension of an iterative root of F of the same order on the characteristic interval $K(F)$.*

We point out that those iterative roots are not obtained by the same mode of extension and they require the various modes of constructions.

The nonmonotonicity height of iterative roots can not be arbitrary. It depends on the number of forts of F . Put

$$N(F) := \text{card } S(F).$$

We have

Theorem 26 ([81]). *If $H(F) = 1$, $N(F) = n$ and F maps the characteristic interval onto itself then F has no continuous iterative roots f of order n such that $H(f) = n$.*

In the case $H(F) \geq 2$ it follows from Theorem 1 in [183] that F has no continuous iterative roots of order $n > N(F)$. The case $n = N(F)$ was partially solved in the paper [84]. Liu et al. [84] proved that if $f \in \mathcal{PM}(I)$ is an iterative root of order $n = N(F) \geq 2$ of F with $H(F) \geq 2$, then $N(f) = 1$ and f is strictly monotonic on the convex hull of $S(F)$. Moreover, they gave a necessary and sufficient condition for the existence of iterative roots f of order $N(F)$ increasing in $\text{conv}(S(F))$ and they determined these roots.

More information on this topic can be found in the survey paper [48] by Jarczyk.

7.2. Iterative roots of homeomorphisms of the circle

To discuss fully the problem of existence, uniqueness and general construction of continuous iterative roots of homeomorphism $F: \mathbb{S}^1 \rightarrow \mathbb{S}^1$, where $\mathbb{S}^1 := \{z \in \mathbb{C} : |z| = 1\}$ it is necessary to consider the following three cases

- a) F is an orientation-reversing homeomorphism;
- b) F is an orientation-preserving homeomorphism and $\text{Per } F = \emptyset$, where $\text{Per } F$ denotes the set of all periodic or fixed points of F ;
- c) F is an orientation-preserving homeomorphism and $\text{Per } F \neq \emptyset$.

Recall that for every continuous mapping $F: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ there exists a unique (up to a translation by an integer) continuous function $f: \mathbb{R} \rightarrow \mathbb{R}$, called the *lift* of F , such that

$$F(e^{2\pi i x}) = e^{2\pi i f(x)}$$

and

$$f(x+1) = f(x) + k$$

for all $x \in \mathbb{R}$, where k is an integer. The number k is called the *degree* of F . If F is a homeomorphism, then f is also a homeomorphism and $k \in \{-1, 1\}$. We call F *orientation-preserving* if $k = 1$, resp. *orientation-reversing* if $k = -1$, which is equivalent to the fact that f is increasing, resp. f is decreasing.

Case a). Notice that a composition of any number of orientation-preserving homeomorphisms and a composition of an even number of orientation-reversing homeomorphisms are orientation-preserving homeomorphisms, therefore there might be only orientation-reversing iterative roots of odd number. In [99] Mai showed that for every odd number n an orientation-reversing homeomorphism F has iterative roots of order n . One can also show that these roots depend on an arbitrary function.

Now suppose that $F: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is an orientation-preserving homeomorphism, then the limit

$$\alpha(F) := \lim_{n \rightarrow \infty} \frac{f^n(x)}{n} \pmod{1}, \quad x \in \mathbb{R}$$

always exists and does not depend on the choice of x and the lift f . This number is called the *rotation number* of F (see [28]). It is known that $\alpha(F)$ is a rational number if and only if F has a periodic or fixed point (see for example [28]).

Case b). Let $\alpha(F) \notin \mathbb{Q}$, then F does not have periodic points. It is well known that every orientation-reversing homeomorphism and, as a consequence, each of its iterates has a fixed point. Therefore F may have only orientation-preserving iterative roots. Set $L_F := \{F^n(z), n \in \mathbb{Z}\}^d$. This set does not depend on $z \in \mathbb{S}^1$ and L_F either equals $\mathbb{S}^1$ or is a nowhere-dense perfect set (see [28]).

The next results are based on the following generalization of the Poincaré theorem.

Theorem 27. *If $\alpha(F) \notin \mathbb{Q}$ then the Schröder equation*

$$\Phi(F(z)) = s\Phi(z), \quad z \in \mathbb{S}^1,$$

where $s := \exp(2\pi i \alpha(F))$ has a unique continuous solution $\Phi: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ such that $\Phi(1) = 1$. The map Φ has the property that for each $z \in \mathbb{S}^1$, $\Phi^{-1}(z)$ is

either a point or a closed arc of $\mathbb{S}^1$. Moreover, if $L_F = \mathbb{S}^1$ then Φ is a homeomorphism.

For the proof see e.g. [27, 155]. As a simple consequence of the above result we obtain the following.

Theorem 28. *If $L_F = \mathbb{S}^1$ then for every integer $n \geq 2$ a homeomorphism F has exactly n iterative roots G of order n and*

$$G(z) = \Phi^{-1}((\sqrt[n]{s})\Phi(z)), \quad z \in \mathbb{S}^1.$$

In the case $L_F \neq \mathbb{S}^1$ define the set $K_F := \Phi(\mathbb{S}^1 \setminus L_F)$. This set is countable, dense in $\mathbb{S}^1$ and $K_F = sK_F$.

Theorem 29 ([171]). *Let $L_F \neq \mathbb{S}^1$. A homeomorphism F has iterative roots of n -th order if and only if*

$$(\sqrt[n]{s})_m K_F = K_F$$

for an $m \in \{0, 1, \dots, n-1\}$, where $(\sqrt[n]{s})_m := e^{2\pi i \frac{1}{n}(\alpha(F)+m)}$.

Moreover, if the assumptions of Theorem 29 hold true, F has infinitely many iterative roots depending on an arbitrary function. In [171] Zdun also gave the construction of iterative roots by finding homeomorphisms conjugating the roots and the rotation $R(z) = (\sqrt[n]{s})_m z, z \in \mathbb{S}^1$. In [134] another construction of iterative roots for F satisfying the assumptions of Theorem 29 was given. It is based on the method described in [66] (see also [64]), i.e. the iterative root is an extension of a function defined on the set $\mathbb{S}^1 \setminus L_F$. Such iterative roots were also dealt by Mai [99].

Case c). The problem of the existence of orientation-preserving iterative roots of homeomorphisms with periodic points was fully solved by Solarz [140]. There were known some partial results by Zdun [160, 173], by Mai [99], by Jarczyk [47] and by Solarz [135–137, 139]. If $F: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is an orientation-preserving homeomorphism such that $\alpha(F) = \frac{q}{n}$, where q, n are positive integers with $0 < q < n$ and $\gcd(q, n) = 1$, then all periodic points have the same period n (see [100, 138]). Moreover, if $n > 1$ there exists a unique number $p \in \{1, 2, \dots, n-1\}$ satisfying $pq = 1 \pmod{n}$. We denote this number p by $\text{char } F$ (see [173]). If $n = 1$ then $\alpha(F) = 0$ and we define $\text{char } F := 1$.

Firstly, let us focus on orientation-reversing iterative roots. Of course, if F has such iterative roots, then F must have a fixed point. Thus if F is the identity function it has infinitely many orientation-reversing iterative roots of even order, they depend on an arbitrary function and every such iterative root is an involution (see [47]). If F is different from the identity function, then it has orientation-reversing iterative roots if and only if there is a partition of $\mathbb{S}^1$ onto two arcs I_1, I_2 , and there exists an orientation-reversing homeomorphism mapping I_1 onto I_2 (see [99]).

For orientation-preserving iterative roots we have the following result (see [47, 139]).

Theorem 30. *For every integer $m \geq 2$ and every orientation-preserving homeomorphism $F: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ with fixed points and such that $\text{Per } F \neq \mathbb{S}^1$ there exist infinitely many orientation-preserving homeomorphisms $G: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ having fixed points and satisfying $G^m = F$. These solutions depend on an arbitrary function.*

If $\text{Per } F = \mathbb{S}^1$ then the identity function is the only orientation-preserving iterative root of F of order m with fixed points.

Unlike real homeomorphism, circle mappings with periodic points of order $n \geq 1$ may have iterative roots with periodic points of order $n' > 1$. However in such a case n must divide n' (see for example [139]). Therefore the general theorem reads as follows.

Theorem 31 ([140]). *Let $m \geq 2$ and $l \geq 1$ be integers and let $F: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be an orientation-preserving homeomorphism such that $\alpha(F) = \frac{q}{n}$, where $0 \leq q < n$ and $\gcd(q, n) = 1$. F has a continuous and orientation-preserving iterative root of order m with periodic points of order nl if and only if the following conditions are fulfilled:*

- (i) $\frac{m}{l} =: m' \in \mathbb{Z}$ and there is $q' \in \mathbb{Z}_{nl}$ such that $\gcd(q', nl) = 1$ and $q'm' = q \pmod{n}$;
- (ii) *for some $z_0 \in \text{Per } F$ there is a partition of $\overline{[z_0, F^{\text{char } F}(z_0))}$ onto l consecutive disjoint arcs $J_0, \dots, J_{l-1}$ such that $F^n[J_i] = J_i, i \in \mathbb{Z}_l$ and if $l > 1$, then there exist orientation-preserving homeomorphisms $V_i: J_i \rightarrow J_{i+1}, i \in \mathbb{Z}_{l-1}$ satisfying*

$$F^n|_{J_{i+1}} = V_i \circ F^n|_{J_i} \circ V_i^{-1}, \quad i \in \mathbb{Z}_{l-1}.$$

Every iterative root G of F of order m , if exists, is of the following form

$$G(z) := \begin{cases} V^{q'}(G_0(z)), & z \in J_0, \\ V^{q'}(z), & z \in \mathbb{S}^1 \setminus J_0, \end{cases}$$

where $G_0: J_0 \rightarrow J_0$ is an orientation-preserving homeomorphism with fixed points satisfying $G_0^{m'} = F^n|_{J_0}$ and V is an orientation-preserving homeomorphism depending on F, V_i and G_0 so that V^{nl} is the identity function.

The above theorem is a generalization of the results from [47, 135, 139, 173] (see [141]). In particular, if $\text{Per } F = \mathbb{S}^1$ we have that F has iterative roots of all orders and they depend on an arbitrary function (see also [47, 135]).

There were also other attempts to solve the problem in this case. For the details see [137] or for some class of homeomorphism also [99].

7.3. Iterative roots of the homeomorphisms of the plane

We present the methods of constructions of iterative roots of Sperner and Brouwer homeomorphisms (see Definitions 8 and 7). A Sperner homeomorphism f

has homeomorphic iterative roots of order $n > 1$ if and only if f preserves the orientation or if f reverses the orientation and n is odd (see [70]).

Put

$$T_{\frac{1}{n}}(x_1, x_2) := \left(x_1 + \frac{1}{n}, x_2\right), \quad S_{\frac{1}{n}}(x_1, x_2) := \left(x_1 + \frac{1}{n}, -x_2\right), \quad (x_1, x_2) \in \mathbb{R}^2.$$

Theorem 32 ([70]). *Let f be an orientation-preserving Sperner homeomorphism of $\mathbb{R}^2$ onto itself. Then*

- (a) *for every even $n > 0$, g is a continuous iterative root of n -th order of f if and only if it is expressed in either of the forms*

$$g = \varphi^{-1} \circ T_{\frac{1}{n}} \circ \varphi \quad (16)$$

and

$$g = \varphi^{-1} \circ S_{\frac{1}{n}} \circ \varphi, \quad (17)$$

where φ is a homeomorphic solutions of the Abel equation

$$\varphi(f(x)) = \varphi(x) + (1, 0); \quad (18)$$

- (b) *for every odd $n > 1$, the function g is a continuous iterative root of f of n -th order if and only if it has the form (16), where φ is a homeomorphic solution of the Abel equation (18).*

Theorem 33 ([70]). *Let f be an orientation-reversing Sperner homeomorphism of $\mathbb{R}^2$ onto itself. Let n be an odd positive integer greater than 1. Then the function g is a continuous iterative root of n -th order of f if and only if it has the form (17), where φ is a homeomorphic solution of the equation $\varphi(f(x)) = S_0(\varphi(x)) + (1, 0)$, where $S_0(x_1, x_2) = (x_1, -x_2)$.*

A direct construction of all iterative roots of a Sperner homeomorphism was given in [68]. Moreover, a construction of all Brouwerian continuous iterative roots of a Brouwer homeomorphism f embeddable in a flow is described in [79]. It is based on the previous construction from [68] and on the facts that the restrictions of a Brouwer homeomorphism f to the maximal parallelizable regions M_α for $\alpha \in A$ are orientation-preserving Sperner homeomorphisms and that $\bigcup_{\alpha \in A} M_\alpha = \mathbb{R}^2$ (see Theorems 11, 12 and 13 in Sect. 4). Theorem 33 gives us the form of all continuous iterative roots g_α of $f|_{M_\alpha}$. To obtain a Brouwer homeomorphism g satisfying the equation $g^n = f$, we can use the properties of the cover M_α , $\alpha \in A$ of the plane presented at the end of Section 3 based on the concept of the Kaplan diagram. This cover is partially ordered. The solution g of the equation $g^n = f$ is obtained by gluing solutions g_α of $g_\alpha^n = f|_{M_\alpha}$ defined on the regions of the cover M_α . With an arbitrary choice of iterative roots g_α of $f|_{M_\alpha}$ the function g can be discontinuous only on $\text{bd } M_\alpha \cap M_\beta$, where M_β are successors of M_α with respect to the partial order defined in the cover. Therefore to obtain the continuity of g , condition

$$\lim_{k \rightarrow \infty} g_\alpha(x_k) = g_\beta(x)$$

for each $x \in \text{bd } M_\alpha \cap M_\beta$ and all sequences $(x_k)_{k \in \mathbb{Z}^+}$ of elements of M_α such that $\lim_{k \rightarrow \infty} x_k = x$ should be satisfied, where g_α and g_β are iterative roots of f restricted to M_α and M_β , respectively.

7.4. Some strange iterative roots of bijections

Let $\mathcal{R}$ be a family of subsets of $X \times X$ such that

$$\text{card } \mathcal{R} \leq \text{card } X$$

and

$$\text{card } \{y \in X : \text{card } R^y = \text{card } X\} = \text{card } X \quad \text{for } R \in \mathcal{R}.$$

Let R^y denote a horizontal section of R .

We say that $\varphi: X \rightarrow X$ has a big graph with respect to $\mathcal{R}$ if φ meets every set of $\mathcal{R}$.

Let L_k denote the number of k -cycles of f and L_0 denote the number of infinite orbits of f . Bartłomiejczyk proved the following result.

Theorem 34 ([10]). *Let a bijection $f: X \rightarrow X$ have an iterative root of order n . If there exists $k_0 \in \mathbb{N}$ such that $\text{card } L_{k_0} = \text{card } X$ and*

$$\sum_{k \neq k_0} L_k < \text{card } X,$$

then f has an iterative root with a big graph.

7.5. Set-valued iterative roots of bijections

It turns out that even very simple and nice functions, such as the so-called hat function $f(x) = \min\{2x, 2 - 2x\}$ for $x \in [0, 1]$ or the celebrated parabola $y = 4x(1 - x)$ for $x \in [0, 1]$ can have no roots. Assume that $f: X \rightarrow X$ is a function which does not have iterative roots. Then we can put the following question: Is there a set-valued function $G: X \rightarrow 2^X$ satisfying suitable conditions such that

$$f(x) \in G^n(x), \quad x \in X,$$

where $G^0(x) = \{x\}$ and $G^{k+1}(x) := \bigcup_{y \in G^k(x)} G(y)$?

Without any assumption on the set-valued function G the problem is trivial since the set-valued functions $G(x) = \{x, f(x)\}$ and $G(x) = X$ are the iterative roots of every order of the function f . The problem is to find such a function G , for which the set $G(x)$ is not uniform and relatively small in the sense of inclusion. The case where f is a bijection was considered by Powierża in papers [110, 112]. He gave a construction of a set-valued iterative root. Such a root turns out to be single-valued for bijections which have an iterative root. This result generalizes the classical result of Łojasiewicz (see [86]).

The existence of minimal solutions in the sense of inclusion were studied by Jarczyk and Powierża in [50]. Moreover, they obtained several results indicating that the smallest set-valued iterative root of a given order does not exist. In papers [51, 82] it is shown that the phenomenon of the lack of iterative roots appears also for some set-valued functions with exactly one value not being a singleton. Even assumptions such as continuity or strict monotonicity on the single-valued parts of such a set-valued function does not guarantee the existence of its square roots.

Recall that a set-valued function $F : X \rightarrow 2^X$ has a square iterative root G if $F(x) = G^2(x) =: \bigcup_{y \in G(x)} G(y)$, $x \in X$ (see [82, 83]). Let us quote the following result.

Let c be a fixed element of a set X . Let $F_c(X)$ stands for the set of all set-valued functions $f : X \rightarrow 2^X$ such that $\text{card } f(c) > 1$ and $f(x)$ is a singleton whenever $x \in X \setminus \{c\}$.

Theorem 35 ([82]). *Let $f \in F_c(X)$. If $\{c\}$ is a value of f , then the set-valued function f has no square iterative roots, one-to-one on the set $f(c)$. If, in addition, f is one-to-one on $f(c)$, then f has no square iterative roots at all.*

If $f(c) = \{c, x_0\}$ with some $x_0 \in X$ satisfying $f(x_0) \neq \{x_0\}$, then f has no square iterative roots.

Many other theorems on this topic were proved by Powierża, Jarczyk, Jarczyk, Li and Zhang (see [50, 51, 110–113]). They considered mainly the case where the values of G consist of one or two elements.

A topological approach to the square iterative roots of an upper semi-continuous set-valued function defined on an interval can be found in papers [82, 83].

7.6. Stability of iterative roots

The stability of iterative roots is important in the numerical computation of iterative roots. Known results show that under some conditions iterative roots of strictly monotonic self-mappings are C^0 stable in both the local sense and the global sense. Let I be a compact interval. Let $C^r(I, I) := \{h \in C^r(I) : h(I) \subset I\}$ and let $C^r(I)$ be equipped with the norm $\|h\|_r := \sup_{x \in I} |h(x)| + \dots + \sup_{x \in I} |h^{(r)}(x)|$.

Applying the continuity of the iteration operator $\mathcal{F}_n(f) := f^n$ Zhang and Zhang [185] proved that if $\lim_{n \rightarrow \infty} F_m = F$ in C^0 , then we can find an iterative root f_m of F_m of order n for each $m \in \mathbb{N}$ such that the sequence $(f_m)_{m \in \mathbb{N}}$ tends to the iterative root of F of order n . This root is associated to a given initial function. This method allows us to find approximate roots. In [181] an algorithm determining the iterative roots of the polygonal functions was given.

In the C^1 space iterative roots are only locally stable but globally unstable. These results were proved by Zhang et al. [184]. They obtained the following

result. Let $I = [0, 1]$. Put $\mathcal{H}(\lambda) := \{h \in C^2(I, I) : h(0) = 0, h'(0) = \lambda, h(x) < x, h'(x) > 0, x \in (0, 1]\}$.

Theorem 36 (On local C^1 stability, [184]). *Let $F \in \mathcal{H}(\lambda)$ with some $\lambda \in (0, 1)$ and let $(F_m)_{m \in \mathbb{N}}$ be a sequence of functions from $\mathcal{H}(\lambda)$. If*

$$\lim_{m \rightarrow \infty} \|F_m - F\|_2 = 0$$

then

$$\lim_{m \rightarrow \infty} \|f_m - f\|_1 = 0,$$

where f and f_m are unique C^1 iterative roots of n -th order of F and F_m , respectively.

Theorem 37 (On global C^1 unstability, [184]). *For any $r \in \mathbb{N}$ and any function $F \in C^r(I, I)$ satisfying $F(0) = 0$, $F'(0) \in \mathbb{R} \setminus \{0, 1\}$, $F(1) = 1$, $F'(1) \in \mathbb{R} \setminus \{0, 1\}$, $F(x) \neq x$ and $F'(x) > 0$, $x \in (0, 1)$, there is a sequence $(F_m)_{m \in \mathbb{N}}$ of functions from $C^1(I, I)$ such that $\lim_{m \rightarrow \infty} \|F_m - F\|_r = 0$ and has no C^1 iterative roots of n -th order for any integer $n \geq 2$.*

8. The structure of iteration groups of homeomorphisms on an interval

For the complete description of the structure of iteration groups we introduce the following auxiliary notations and definitions. Let V be a divisible subgroup of the additive group $(\mathbb{R}, +)$ such that $1 \in V$ and J be an open interval. Let $\mathcal{F}(J, V) = \{f^t : J \rightarrow J, t \in V\}$ be an iteration group of homeomorphisms on J over V . Let us recall that if $V = \mathbb{R}$, then $\mathcal{F}(J, \mathbb{R})$ is called an iteration group. A family of mappings is said to be *disjoint* if the graphs of any two distinct elements belonging to the family do not intersect.

Let us emphasize that in this section there is no regularity with respect to the iterative parameter t assumed on the group.

Let $I = (a, b)$ and $\mathcal{F}(I, V)$ be a disjoint iteration group over V . The sets of all cluster points of the orbits $\{f^t(x), t \in U\}$ do not depend on $x \in I$ and the choice of a divisible subgroup $U \subset V$ (see [167]). Denote this set by $L_{\mathcal{F}}$. The set $L_{\mathcal{F}}$ is either a nowhere dense perfect set such that $a, b \in L_{\mathcal{F}}$ or $L_{\mathcal{F}} = [a, b]$ (see [167]).

Let $L \subset [a, b]$ be a perfect and nowhere dense set such that $a, b \in L$ and let $J_{\alpha}, \alpha \in \mathbb{Q}$ be open disjoint intervals such that

$$(a, b) \setminus L = \bigcup_{\alpha \in \mathbb{Q}} J_{\alpha} \quad \text{and} \quad J_{\alpha} < J_{\beta} \Leftrightarrow \alpha < \beta.$$

Let $c : V \rightarrow \mathbb{R}$ be an additive function such that $\text{card Im } c = \aleph_0$ and Φ be an order preserving bijection of $\mathbb{Q}$ onto $\text{Im } c + T$, where T is an at most countable subset of $\mathbb{R}$. Denote by $b_{\alpha, t}$ for $\alpha \in \mathbb{Q}$ and $t \in V$ the affine, increasing functions

defined on J_α such that $b_{\alpha,t}[J_\alpha] = J_{\Phi^{-1}(\Phi(\alpha)+c(t))}$. Put $p^t(x) := \sup\{b_t(u), u < x\}$, where $b_t(u) := b_{\alpha,t}(u)$ for $u \in J_\alpha$.

The family $\mathcal{P}(L, T, c, \Phi) := \{p^t, t \in V\}$ is a disjoint iteration group over V of piecewise linear continuous functions such that $L_{\mathcal{P}(L, T, c, \Phi)} = L$.

Theorem 38 ([168, 169]). *Let $\mathcal{F}(I, V)$ be a disjoint iteration group over V .*

1. *If $L_{\mathcal{F}} = cl I$ then $\mathcal{F}(I, V)$ is conjugated to a group of translations $\{\text{id} + d(t), t \in V\}$ for $t \in V$, where $d: V \rightarrow \mathbb{R}$ is an additive function, that is $f^t = h^{-1} \circ (\text{id} + d(t)) \circ h$ for a homeomorphism $h: I \rightarrow \mathbb{R}$.*
2. *If $L_{\mathcal{F}} \neq cl I$ then $\mathcal{F}(I, V)$ is conjugated to a group $\mathcal{P}(L, T, c, \Phi)$ for some T, c, Φ and $L = L_{\mathcal{F}}$, that is $f^t = \gamma^{-1} \circ p^t \circ \gamma, t \in V$ for a homeomorphism $\gamma: I \rightarrow I$ such that $\gamma(x) = x$ for $x \in L_{\mathcal{F}}$.*

The mentioned sets L, T , the functions h, Φ, d, c and the above formulas determine all disjoint iteration groups.

In paper [33] Farzadfarad and Robati generalized this result for a more general structure than iteration groups. Namely, the group of continuous self-mappings of I whose graphs are disjoint. They described the general structure of such groups (see also [31]).

Now we consider the structure of an arbitrary iteration group $\mathcal{F}(I, \mathbb{R})$.

For any group $\mathcal{F}(I, V)$ there exists a family of pairwise disjoint open intervals $I_\alpha, \alpha \in M$ such that $f^t[I_\alpha] = I_\alpha, t \in V$ and $f^t(x) = x, x \in I \setminus \bigcup_{\alpha \in M} I_\alpha, t \in V$ and there is no $x_0 \in I_\alpha$ such that $f^t(x_0) = x_0, t \in V$.

Every iteration group $\mathcal{F}(J, V)$, where $J \in \{I_\alpha, \alpha \in M\}$ satisfies one of the following conditions:

- (I) there exists a $t \in V$ such that $f^t(x) \neq x, x \in J$,
- (II) for every $t \in V, f^t$ has a fixed point in J and the family of functions $F(J, V)$ has no common fixed point.

Every group $\mathcal{F}(J, V)$ of type (I) can be built by a special compilation of some disjoint iteration groups $\mathcal{F}(J_\omega, U^\top), \omega \in \mathbb{Q}$, where U is a linear subspace of V over $\mathbb{Q}, U^\top \oplus U = V$ and J_ω are some open pairwise disjoint subintervals of J such that $cl J \setminus \bigcup_{\omega \in \mathbb{Q}} J_\omega$ is a nowhere dense perfect set.

Furthermore, every group of type (II) is built by means of a countable family of iteration groups of type (I). More precisely, there exist sequences of intervals $\{K_n\}$ and divisible subgroups $\{V_n\}$ such that $I_n \subset I_{n+1}, V_n \subset V_{n+1}, J = \bigcup_{n=1}^\infty J_n, V = \bigcup_{n=1}^\infty V_n$ and $\mathcal{F}(J, V) = \bigcup_{n=1}^\infty \mathcal{F}(K_n, V_n)$, where $\mathcal{F}(K_n, V_n)$ are disjoint iteration groups of type (I).

The exact description of the structure of iteration groups of homeomorphisms is given in [170] and, with some additional assumptions, in [149, 150].

The structure of iteration semigroups is still unknown.

9. Iteration groups of homeomorphisms of the circle

We discuss here the structure of disjoint groups. Recall that in the circle case a group is called disjoint if each element of the group is either the identity mapping or a mapping with no fixed points. We determine the normal form and we present the general construction of such a group. Let $\mathcal{T} := \{T^t: \mathbb{S}^1 \rightarrow \mathbb{S}^1, t \in \mathbb{R}\}$ be a disjoint iteration group or let the rotation number of at least one iterate be irrational. The set $L_{\mathcal{T}} := \{T^t(z), t \in \mathbb{R}\}^d$ of limit points of the orbit does not depend on $z \in \mathbb{S}^1$ and either $L_{\mathcal{T}}$ is a non-empty perfect and nowhere dense subset of $\mathbb{S}^1$ or $L_{\mathcal{T}} = \mathbb{S}^1$ or $L_{\mathcal{T}} = \emptyset$ (see [5, 24]). In the last case the iteration group is finite (see [24]). The basic fact which is used in the description of an infinite iteration group is the following.

Theorem 39 ([5, 24]). *If an infinite iteration group $\mathcal{T} := \{T^t: \mathbb{S}^1 \rightarrow \mathbb{S}^1, t \in \mathbb{R}\}$ is disjoint or at least one iterate is not periodic, then there exists a unique pair (Φ, c) such that $\Phi: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is a continuous mapping of degree 1 with $\Phi(1) = 1$ and $c: \mathbb{R} \rightarrow \mathbb{S}^1$ for which*

$$\Phi(T^t(z)) = c(t)\Phi(z), \quad z \in \mathbb{S}^1, t \in \mathbb{R}. \quad (19)$$

The function c is given by $c(t) = e^{2\pi i \alpha(T^t)}$ for $t \in \mathbb{R}$ and is a homomorphic mapping, where $\alpha(T^t)$ is the rotation number of T^t . The mapping Φ is increasing and $\Phi[L_{\mathcal{T}}] = \mathbb{S}^1$. Moreover, Φ is a homeomorphism if and only if $L_{\mathcal{T}} = \mathbb{S}^1$.

Hence it is obvious that if $L_{\mathcal{T}} = \mathbb{S}^1$, then the iteration group $\mathcal{T}$ is conjugated to a group of rotations $R^t = c(t)\text{id}$ for a homomorphism $c: \mathbb{R} \rightarrow \mathbb{S}^1$.

Now let $L_{\mathcal{T}}$ be a nowhere dense perfect set. We have the decomposition $\mathbb{S}^1 \setminus L_{\mathcal{T}} = \bigcup_{\omega \in \mathbb{Q}} I_{\omega}$, where I_{ω} are open pairwise disjoint arcs. The pair of mappings (Φ, c) determined in Theorem 39 has the following properties

1. for every $\omega \in \mathbb{Q}$ the mapping Φ is constant on I_{ω} ,
2. for any distinct $\nu, \nu' \in \mathbb{Q}$, $\Phi[I_{\nu}] \cap \Phi[I_{\nu'}] = \emptyset$,
3. the sets $\text{Im } c$ and $K_{\mathcal{T}} := \Phi[\mathbb{S}^1 \setminus L_{\mathcal{T}}]$ are countable and dense in $\mathbb{S}^1$,
4. $K_{\mathcal{T}} \cdot \text{Im } c = K_{\mathcal{T}}$.

The above properties let us define the bijection $\tilde{\Phi}: \mathbb{Q} \rightarrow K_{\mathcal{T}}$ and the mapping $\Upsilon: \mathbb{Q} \times \mathbb{R} \rightarrow \mathbb{Q}$ as follows

$$\tilde{\Phi}(\omega) := \Phi[I_{\omega}], \quad \Upsilon(\omega, t) := \tilde{\Phi}^{-1}(\tilde{\Phi}(\omega)c(t)), \quad \omega \in \mathbb{Q}, t \in \mathbb{R}.$$

The mapping Υ is needed for the description of a special canonical piecewise linear group $\mathcal{P}$ which is conjugated to the group $\mathcal{T}$.

Theorem 40 ([5, 24]). *If an infinite iteration group $\mathcal{T} := \{T^t: \mathbb{S}^1 \rightarrow \mathbb{S}^1, t \in \mathbb{R}\}$ is disjoint or at least one iterate is not periodic and $L_{\mathcal{T}} \neq \mathbb{S}^1$, then there exists a unique disjoint iteration group $\mathcal{P} = \{P^t: \mathbb{S}^1 \rightarrow \mathbb{S}^1, t \in \mathbb{R}\}$ such that for any $\omega \in \mathbb{Q}, t \in \mathbb{R}$, the mapping P^t is linear on I_{ω} and $P^t[I_{\omega}] = I_{\Upsilon(\omega, t)}$. Moreover $L_{\mathcal{T}} = L_{\mathcal{P}}$.*

If $\mathcal{T}$ is a disjoint group, then $\mathcal{T}$ is conjugated to the group $\mathcal{P}$, that is there exists a homeomorphism $\Gamma: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ such that

$$T^t = \Gamma^{-1} \circ P^t \circ \Gamma, \quad t \in \mathbb{R}$$

and $\Gamma(z) = z$ for $z \in L_{\mathcal{T}}$.

The above group $\mathcal{P}$ is also called a *normal form* of the irregular group $\mathcal{T}$.

To give a general construction of all disjoint iteration groups we introduce on $\mathbb{S}^1$ the following order relation. For any $v, w, z \in \mathbb{S}^1$ there exist unique $t_1, t_2 \in [0, 1)$ such that $we^{2\pi it_1} = z$ and $we^{2\pi it_2} = v$, so we can put

$$v \preceq w \preceq z \Leftrightarrow t_1 \leq t_2 \text{ or } t_2 = 0.$$

- Let L be a perfect nowhere dense subset of $\mathbb{S}^1$ and I_q for $q \in \mathbb{Q}$ be open pairwise disjoint arcs such that $\mathbb{S}^1 \setminus L = \bigcup_{q \in \mathbb{Q}} I_q$.
- Let $M \subset \bigcup_{q \in \mathbb{Q}} I_q$ be such that $\text{card}(M \cap I_q) = 1$ for $q \in \mathbb{Q}$. For any $\alpha \in M$ denote by I^α the arc I_q such that $\alpha \in I_q$.
- Fix $z_M \in \bigcup_{\alpha \in M} \text{cl } I^\alpha$ and define $\alpha \preceq_M \beta$ if and only if $z_M \preceq \alpha \preceq \beta$, $\alpha, \beta \in M$.
- Let $c: \mathbb{R} \rightarrow \mathbb{S}^1$ be a homomorphic mapping with $\text{card } \text{Im } c = \aleph_0$.
- Take a non-empty subset A of $\mathbb{S}^1$ such that $\text{card } A \leq \aleph_0$ and put $K := \text{Im } c \cdot A$.
- Choose $z_K \in \mathbb{S}^1 \setminus K$ and set $z_1 \preceq_K z_2$ if and only if $z_K \preceq z_1 \preceq z_2$, $z_1, z_2 \in K$.
- Let $\Phi: (M, \preceq_M) \rightarrow (K, \preceq_K)$ be an order preserving bijection.

Then for every $\alpha \in M$ and $t \in \mathbb{R}$ there exists a unique linear increasing mapping $P_{\alpha,t}$ defined on I^α such that $P_{\alpha,t}[I^\alpha] = I^{\Phi^{-1}(\Phi(\alpha)c(t))}$. Moreover, for every $t \in \mathbb{R}$ there exists a unique continuous extension P^t of the mappings $P_{\alpha,t}$, $\alpha \in M$ on $\mathbb{S}^1$. The family $\{P^t: \mathbb{S}^1 \rightarrow \mathbb{S}^1, t \in \mathbb{R}\}$ is a disjoint iteration group. The formula

$$T^t = \Gamma^{-1} \circ P^t \circ \Gamma, \quad t \in \mathbb{R},$$

where $\Gamma: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is an arbitrary homeomorphism such that $\Gamma(z) = z$ for $z \in L$ defines all disjoint iteration groups such that $L_{\mathcal{T}} = L$. The construction presented above was given by Ciepliński [25].

In [20] Ciepliński also determined all disjoint embeddings of the circle homeomorphisms without periodic points as well as homeomorphisms such that $F^n = \text{id}$ for an $n \in \mathbb{N}$.

The problem of conjugacy and semi-conjugacy of disjoint iteration groups was solved in papers [19, 21, 27].

The constructions of more general groups which are not disjoint but one of the iterates has no periodic points can be found in Ciepliński's papers [22, 23].

10. Approximately iterable functions

Let I be a compact interval. We say that a continuous function $f: I \rightarrow I$ is said to be *iterable* if there exists a continuous iteration semigroup $\{f^t, t > 0\}$ such that $f^1 = f$, i.e. f is embeddable in a continuous iteration semigroup. The necessary and sufficient conditions for the iterability of continuous self-mappings of I were given by Zdun [159] (see also Theorem (3.3.31) in [152]).

Iterability is a rare property. In many problems coming from dynamic systems, embeddability can be replaced by a slightly more general concept of “near-embeddability”.

Inspired by a problem posed by Jen (see [153], Problem (3.1.12)) Jarczyk introduced in [46] the notion of “near embeddability”.

Definition 9. A continuous function $f: I \rightarrow I$ is “almost iterable” if there exists an iterable function $g: I \rightarrow I$ such that

$$\lim_{n \rightarrow \infty} (f^n(x) - g^n(x)) = 0 \quad (20)$$

for every $x \in I$ and the convergence is uniform on every component of $[a_f, b_f] \setminus \text{Fix } f$, where $a_f := \inf \text{Fix } f$, $b_f := \sup \text{Fix } f$ and $\text{Fix } f$ denotes the set of all fixed points of f .

Clearly, every iterable function is almost iterable. The converse is not true as it follows from Example in [46]. Jarczyk also gave several characterizations of “almost iterability”. One of them reads as follows.

Theorem 41 ([46]). *A continuous function $f: I \rightarrow I$ is almost iterable if and only if the function $f|_{[a_f, b_f]}$ increases and every interval, where it is constant, contains a fixed point of f , and if a_f, b_f are interior points of I , f has no periodic point of order 2.*

The concept of “near iterability” was later investigated by Przebieracz (see [114, 115, 117]). She gave several definitions of “near embeddability”, some of which turned out to be generalizations of Definition 9.

A continuous function $f: I \rightarrow I$ is “near iterable”

- (I) if there exists an iterable function $g: I \rightarrow I$ such that (20) holds true for every $x \in I$ (*weak almost iterability*),
- (II) if there exists an iterable function $g: I \rightarrow I$ such that (20) holds true for every $x \in I \setminus M$, where $M \subset I$ is such that $\text{int} M = \emptyset$ (*M-weak almost iterability*),
- (III) if for every $\varepsilon > 0$ there exist $n_0 \in \mathbb{N}$ and an iterable function g such that

$$|f^n(x) - g^n(x)| < \varepsilon, \quad n > n_0, \quad x \in I$$

(*approximate iterability*),

(IV) if for every $\varepsilon > 0$ there exists an iterable function g such that

$$|f^n(x) - g^n(x)| < \varepsilon, \quad n \in \mathbb{N}, \quad x \in I.$$

“Near iterability” in the sense of definition (II) is a generalization of (I) and both generalize the notion of almost iterability. Moreover, approximate iterability generalizes (IV) and the notion of almost iterability, but the last two are not comparable (see [115]).

Definition (IV) is equivalent to

(IV') for every $\varepsilon > 0$ there exists an iterable function $g: I \rightarrow I$ such that

$$|f(x) - g(x)| < \varepsilon, \quad x \in I,$$

which means that f is in the closure of the set of iterable self-mappings of I (see [115]). Other conditions equivalent to (IV) may be found in [115].

Approximate iterability (III) may be characterized as follows.

Theorem 42 ([114]). *A continuous function $f: I \rightarrow I$ is approximately iterable if and only if the function $f|_{[a_f, b_f]}$ increases and f has no periodic point of order 2.*

Finally, the fact that f has no periodic point of order 2 is also the necessary condition for weak almost iterability (I) and M -weak almost iterability (II). For the characterizations of such functions see [117].

11. Set-valued iteration semigroups

Let X be a non-empty set, $n(X) := \{A \subset X : A \neq \emptyset\}$ and if X is a normed space let $b(X)$ be the set of all bounded members of $n(X)$, $c(X)$ be a set of all compact elements of $n(X)$ and $cc(X)$ be the family of all convex subsets of $c(X)$. Let moreover $c(X)$ be endowed with the Hausdorff metric. For the properties of the Hausdorff metric see for example [17].

A family of set-valued functions $\{F^t: X \rightarrow n(X), t > 0\}$ is said to be a *set-valued iteration semigroup* if $F^{u+v}(x) = F^u \circ F^v(x)$ for $x \in X, u, v > 0$, where $F^u \circ F^v$ is defined by (15).

The fundamental problem lies in the question if there exists a semigroup $\{f^t: X \rightarrow X, t > 0\}$ of single valued functions of suitable regularity called *selections*, such that

$$f^t(x) \in F^t(x), \quad x \in X, t > 0.$$

In [125] Smajdor gave a positive answer to this question in the case where X is a subset of a separable uniformly convex Banach space E and $F^t: X \rightarrow cc(X)$ have a kind of monotonicity properties. Let π be a real function defined on X . A set-valued function $F: X \rightarrow X$ is said to be π -increasing if for every $x, y \in X$ with $\pi(x) \leq \pi(y)$ and every $w \in F(y)$ there exists a $u \in F(x)$ such that $\pi(u) \leq \pi(w)$.

Theorem 43 ([125]). *Let X be a non-empty convex subset of a normed linear space and $\pi: X \rightarrow \mathbb{R}$ be strictly convex and lower semicontinuous. If $\{F^t: X \rightarrow cc(X), t > 0\}$ is an iteration semigroup of a π -increasing set-valued function, then there exists an iteration semigroup $\{f^t: X \rightarrow X, t > 0\}$ such that $f^t(x) \in F^t(x)$ for $x \in X, t > 0$ and*

$$\pi(f^t(x)) = \inf\{\pi(y) : y \in F^t(x)\}, \quad x \in X, \quad t > 0.$$

If π is continuous, then the continuity of the mappings $x \mapsto F^t(x)$ and $t \mapsto F^t(x)$ implies the continuity of $x \mapsto f^t(x)$ and $t \mapsto f^t(x)$.

If $\pi(x) = \|x - v\|$ for a given $v \in X$, then we get the same assertion for iteration semigroups of set-valued functions with convex closed values.

A similar problem was also considered by Olko for a uniformly continuous iteration semigroup of linear set-valued functions $\{F^t: X \rightarrow c(X), t \geq 0\}$, where X is a cone with a finite basis. In [105] she gave some conditions which imply the existence of exactly one iteration semigroup of linear selections of F^t .

In her later paper she proved the following result.

Theorem 44 ([107]). *Let X be an open convex cone in a Banach space E . Let $\{F^t: X \rightarrow c(X), t \geq 0\}$ be an iteration semigroup of linear continuous set-valued functions satisfying $F^0 = \text{id}$. Let one of the following conditions be fulfilled:*

- 1) $\lim_{t \rightarrow 0} \|F^t - \text{id}\| = 0$, i.e. $\{F^t, t \geq 0\}$ is uniformly continuous,
- 2) there exists a set-valued function $G: C \rightarrow b(X)$ such that

$$\lim_{t \rightarrow 0} \frac{1}{t} (F^t(x) - x) = G(x), \quad x \in C,$$

i.e. $\{F^t, t \geq 0\}$ has an infinitesimal operator.

Then for every $t_0 > 0, x_0 \in C$ and every y_0 —extreme point of $F^{t_0}(x_0)$ there exists exactly one iteration semigroup $\{f^t, t \geq 0\}$ of linear selections f^t of F^t with the property $f^{t_0}(x_0) = y_0$.

Moreover, f^t is extreme for $t \in [0, t_0]$ and there is a unique linear and continuous operator $g: E \rightarrow E$ such that $f^t = e^{tg}$ for $t \geq 0$.

In [106] Olko proved that the concavity of the semigroup of linear continuous set-valued functions also implies the existence of an iteration semigroup of linear selections.

Another aspect being investigated is the existence of majorizing iteration semigroups for a given semigroup. Let $\{F^t: X \rightarrow n(X), t > 0\}$ and $\{G^t: X \rightarrow n(X), t > 0\}$ be two iteration semigroups, we say that $\{G^t, t > 0\}$ majorizes $\{F^t, t > 0\}$ if $F^t(x) \subset G^t(x)$ for $t > 0$ and $x \in X$. A set-valued iteration semigroup $\{F^t, t > 0\}$ is said to be increasing (decreasing) if $F^t(x) \subset F^s(x)$ for $t < s$ ($s < t$) and $x \in X$. It is shown in [125] that every iteration semigroup is majorized by an increasing iteration semigroup. Moreover, the following result holds.

Theorem 45 ([125]). *Let X be a non-empty set and $\{F^t, t > 0\}$ be an iteration semigroup from X into X such that for every $x, y \in X$ the sets $\{t > 0 : y \in F^t(x)\}$ are closed intervals. Then this iteration semigroup is an intersection of an increasing iteration semigroup and a decreasing one.*

There are given two constructions of the smallest increasing iteration semigroup majorizing a given one.

Olko [104] proved that if X is an open convex cone in a separable Banach space and a measurable semigroup $\{F^t : \bar{X} \rightarrow cc(X), t \geq 0\}$ of linear continuous mappings such that $F^0 = \text{id}$ and $F^t(x) - \{x\} \subset \bar{X}$ for $x \in \bar{X}$ and $t \geq 0$ has an infinitesimal operator H , then it can be majorized by a semigroup of exponential type, i.e.

$$F^t(x) \subset G^t(x) := \sum_{i=1}^{\infty} \frac{t^i}{i!} H^i(x), \quad x \in X, t \geq 0.$$

Let $\{F^t, t > 0\}$ be a given one-parameter family of set-valued functions. The problem is to give the necessary and sufficient conditions under which such a family is a set-valued iteration semigroup. The solution to this problem was given:

- in [126] for a family $\{F^t : X \rightarrow cc(X), t \geq 0\}$ of Jensen set-valued functions with $\sup\{\text{diam} F^t(x) : x \in S\} < \infty, t \geq 0$ and X being a closed convex cone in a normed space,
- in [127] for a family $\{F^t : X \rightarrow c(X); t > 0\}$ of midpoint convex set-valued maps, where X is a locally convex vector space,
- in [128] (resp. [129]) for a family $\{F^t : X \rightarrow cc(X), t > 0\}$ being an increasing iteration semigroup of continuous Jensen set-valued functions (resp. a concave iteration semigroup of continuous Jensen functions), here X is a closed convex cone with a non-empty interior in a real separable Banach space,
- in [108, 132] for the family

$$\left\{ \sum_{i=1}^{\infty} \frac{t^i}{i!} G^i(x), t \geq 0 \right\}, \quad (21)$$

where X is a closed convex cone in a Banach space and $G : X \rightarrow cc(X)$ is a given linear continuous set-valued function with $0 \in G(x), x \in X$.

Recently Piszczek proved the following significant result.

Theorem 46 ([108]). *Let K be a closed convex cone with a non-empty interior in a Banach space and let $G : K \rightarrow cc(K)$ be a continuous additive set-valued function. Assume that $F_t(x) := \sum_{i=1}^{\infty} \frac{t^i}{i!} G^i(x)$, for $x \in K$ and $t \geq 0$. The family $\{F_t : t \geq 0\}$ is an iteration semigroup if and only if*

$$G \circ F_t = T_t \circ G \quad \text{for } t \geq 0.$$

Some characterizations of Hukuhara's differentiable iteration semigroups of some set-valued functions were also given (see [130, 131]).

A natural generalization of iteration semigroups of set-valued functions are expanding and collapsing iteration semigroups. The family $\{F^t: X \rightarrow n(X), t > 0\}$ is called an *expanding iteration semigroup*, resp. *collapsing iteration semigroup* if it fulfils the following inclusion $F^t \circ F^s(x) \subset F^{t+s}(x)$ for $x \in X$ and $s, t \geq 0$, resp. $F^{t+s}(x) \subset F^t \circ F^s(x)$ for $x \in X$ and $s, t \geq 0$ (see. [87, 88]). In 1995 Olko (see [109]) showed that if $G: X \rightarrow cc(X)$ is a linear and continuous function defined on a closed and convex cone X in a Banach space, then (21) is an expanding iteration semigroup (it is a semigroup if X is a closed cone in $\mathbb{R}$).

Later Łydzińska [87] gave the necessary and sufficient conditions for a family

$$M := \{F^t, F^t(x) = A^{-1}(A(x) + \min\{t, \sup A(X) - A(x)\}), x \in X, t \geq 0\},$$

where X is an arbitrary set, $A: X \rightarrow n(X)$ and $A^{-1}(V) := \{x \in X : A(x) \cap V \neq \emptyset\}, V \in n(X)$ to be a collapsing semigroup. Łydzińska also investigated expanding iteration semigroups. She gave in [88] the sufficient conditions which imply that M is an expanding semigroup. Moreover, in [89] she proved that the expansion of M implies collapse, i.e. every expanding iteration semigroup turns out to be a semigroup. Next Łydzińska studied the lower semicontinuity of the mappings $x \mapsto F^t(x)$ and $t \mapsto F^t(x), x \in X, t \geq 0$, where X is some topological space and $F^t \in M$. Recall that a function $F: X \rightarrow n(Y)$ is *lower semicontinuous* if $F^{-1}(U)$ is open in X for every open set $U \subset Y$. Generally, the continuity of A and lower semicontinuity of $x \mapsto F^t(x)$ imply the lower semicontinuity of $t \mapsto F^t(x)$, but in some connected topological spaces these notions are equivalent. For details and a characterization of a lower semicontinuity see [90].

12. Iterations of mean-type mappings

In this section we return to the subject of one dimensional dynamics and we concentrate on a particular dynamical system in the $\mathbb{R}^N$ space for $N \geq 2$, namely, iterations of mean-type mappings and invariant means. Matkowski showed that, under some general conditions, the sequence of iterates of every mean-type mapping on a finite dimensional cube converges to a unique invariant mean-type mapping.

Let us recall some necessary definitions.

Let I be an interval. A function $M: I^N \rightarrow \mathbb{R}$ is said to be a mean if for all $(x_1, \dots, x_N) \in I^N$

$$\min(x_1, \dots, x_N) \leq M(x_1, \dots, x_N) \leq \max(x_1, \dots, x_N).$$

A mean is called strict if the above inequalities are sharp whenever $x_i \neq x_j$ for some $i, j \in 1, \dots, N, i \neq j$.

A function $\mathbf{M}: I^N \rightarrow I^N$, $\mathbf{M} = (M_1, \dots, M_N)$ is called a mean-type mapping if each coordinate function M_i is a mean. The fundamental result on iteration sequences $\{\mathbf{M}^n\}$ is the following.

Theorem 47 ([95]). *If $\mathbf{M}: I^N \rightarrow I^N$ is a continuous mean-type mapping such that at most one of the coordinate means M_i is not strict, then:*

1. *every iterate $\mathbf{M}^n$ is a mean-type mapping,*
2. *there is a continuous mean $K: I^N \rightarrow I$ such that the sequence $\mathbf{M}^n$ converges to a mean-type mapping $\mathbf{K} = (K_1, \dots, K_N)$ such that $K_1 = \dots = K_N =: K$,*
3. *$\mathbf{K}$ is $\mathbf{M}$ invariant, i.e.*

$$\mathbf{K} \circ \mathbf{M} = \mathbf{K},$$

4. *a continuous $\mathbf{M}$ -invariant mean-type mapping is unique.*

In his later paper [98] Matkowski generalized the condition “at most one of the means is not strict” to the more symmetric condition:

$$\min(\mathbf{x}) + \max(\mathbf{M}(\mathbf{x})) < \min(\mathbf{M}(\mathbf{x})) + \max(\mathbf{x}),$$

for all $\mathbf{x} = (x_1, \dots, x_p)$ not on the diagonal.

The natural generalizations of this topic are continuous iteration semigroups of mean-type mappings. The presented result concerns only the two dimensional case but its ideas can be extended to any higher-dimensional space.

Theorem 48 ([96]). *If $\{\mathbf{M}^t: I^2 \rightarrow I^2, t > 0\}$ is a continuous iteration semigroup of mean-type mappings, then there exists a unique continuous mean $K: I^2 \rightarrow I$ such that for every $t > 0$, K is $\mathbf{M}^t$ invariant, i.e.*

$$K \circ \mathbf{M}^t = K, \quad t > 0$$

and K is strict.

In the same paper a form of iterations groups of quasi-arithmetic weighted mean-type mappings was determined.

Jarczyk and Matkowski determined the form of all semigroups in which a given mean-type mapping can be embedded and they also established the necessary and sufficient conditions for embeddability.

Theorem 49 ([49]). *Let M, N be homogeneous symmetric strict means in $(0, \infty)$ and let K be a mean invariant with respect to (M, N) . Assume that (M, N) is embeddable in a continuous iteration semigroup $\{(M^t, N^t), t > 0\}$ of homogeneous symmetric strict mean-type self-mappings of $(0, \infty)^2$. Then there exist numbers $a, b \in [0, \infty]$ such that*

$$a \leq b, \quad 1 \in \{a, b\} \quad \text{and} \quad \frac{M(x, 1)}{N(x, 1)} \in [a, b], \quad x \in (0, \infty), \quad (22)$$

a continuous function $e: (0, \infty) \rightarrow (0, \infty)$ satisfying the conditions

$$e((0, \infty)) = [a, b] \cap (0, \infty), \quad (23)$$

$$e(x) = x, \quad x \in [a, b] \cap (0, \infty), \quad (24)$$

$$e(1/x) = e(x), \quad x \in (0, \infty), \quad (25)$$

and such that

$$K(x, 1) < K(y, 1) < K(1/x, 1), \\ x \in (0, 1), \quad y \in (\min\{e(x), 1/e(x)\}, \max\{e(x), 1/e(x)\}), \quad (26)$$

and a continuous strictly monotonic function $\alpha: [a, b] \rightarrow [-\infty, \infty]$ satisfying the condition

$$\alpha\left(\frac{M(x, 1)}{N(x, 1)}\right) = \min\{\alpha(e(x)) + 1, \alpha(1)\}, \quad x \in (0, \infty), \quad (27)$$

and such that

$$\alpha \text{ takes the greatest value at } 1 \quad (28)$$

and

$$M^t(x, y) = \frac{K(x, y)}{K\left(1/F^t\left(\frac{x}{y}\right), 1\right)}, \quad N^t(x, y) = \frac{K(x, y)}{K\left(F^t\left(\frac{x}{y}\right), 1\right)} \quad (29)$$

for all $t, x, y \in (0, \infty)$, where

$$F^t(x) = \alpha^{-1}(\min\{\alpha(e(x)) + t, \alpha(1)\}) \quad (30)$$

for all $t, x \in (0, \infty)$.

Conversely: if $a, b \in [0, \infty]$ satisfy condition (22), $e: (0, \infty) \rightarrow (0, \infty)$ is a continuous function and $\alpha: [a, b] \rightarrow [-\infty, \infty]$ is a continuous strictly monotonic function satisfying condition (27) and such that conditions (23)–(26) and (28) hold, then formulas (29) and (30) define a continuous iteration semigroup (M^t, N^t) of homogeneous symmetric strict mean-type self-mappings of $(0, \infty)^2$ with $M^1 = M$ and $N^1 = N$.

Theorem 50 ([49]). Let M, N be homogeneous symmetric strict means in $(0, \infty)$ and let K be a mean invariant with respect to (M, N) . Put $f := M(\cdot, 1)/N(\cdot, 1)$.

The mapping (M, N) is iterable in the class of homogeneous symmetric strict mean-type self-mappings of $(0, \infty)^2$ if and only if,

- (i) there exist numbers $a, b \in [0, \infty]$ such that (22) holds and a continuous function $e: (0, \infty) \rightarrow (0, \infty)$ satisfying conditions (23)–(25),
- (ii) $a \leq f(a+) \leq f(x) \leq f(b-) \leq b$, $x \in (0, \infty)$, the function $f|_{[a, b] \cap (0, \infty)}$ is increasing, and there is at most one interval of constancy of f , which, in addition, contains 1; moreover,

$$f(e(x)) = f(x), \quad x \in (0, \infty),$$

(iii) condition (26) holds.

13. Hayers–Ulam stability of the translation equation

The stability properties of functional equations have attracted the attention of many mathematicians. Recently the problem of Hayers–Ulam stability of the translation equation, i.e. Eq. (2), was studied. Recall that Eq. (2) is stable in the class $\mathcal{F}$ if for every function H in $\mathcal{F}$ satisfying (2) with some error there is a solution of (2) in $\mathcal{F}$ close to H . Namely, let (X, ρ) be a metric space, and $(G, +)$ be at least a semigroup, we call (2) stable if for each $\epsilon > 0$ there is a $\delta > 0$ such that for each function $H: X \times G \rightarrow X$ belonging to the class $\mathcal{F}$ satisfying

$$\rho(H(H(x, s), t), H(x, s + t)) < \delta, \quad x \in X, s, t \in G \quad (31)$$

there exists a solution $F: X \times G \rightarrow X$ of (2) in $\mathcal{F}$ satisfying

$$\rho(H(x, t), F(x, t)) < \epsilon \quad (32)$$

for $x \in X, t \in G$.

In 2006 in [91] Mach and Moszner proved that if $(G, +)$ is a monoid and H satisfies one of the following conditions:

- $H(x, \cdot)$ is a bijection for a certain $x \in X$,
- $H(x_0, \cdot)$ is an injection for a certain $x_0 \in X$ and $H(x_0, G) = H(X, 0)$,

then (2) is stable in the above classes with $\delta = \epsilon$, resp. $\delta = \frac{1}{2}\epsilon$. The function F in the case of stability of equation (2) does not depend on ϵ and may not be unique for a function H satisfying (31). The authors also proved that the system of equations consisting of the translation equation and the identity condition $F(x, 0) = x, x \in X$, where 0 is the neutral element in G , is not stable in general. Recall that every continuous solution of such a system is called a dynamical system on X or with the notation $F^t(x) = F(x, t)$ a continuous iteration group with $F^0 = \text{id}$.

Now assume that $X = I \subset \mathbb{R}$ is an interval and $G = \mathbb{R}$. Chudziak [18] obtained the following result in the class of mappings from $I \times \mathbb{R}$ into I .

Theorem 51 ([18]). *Let I be an open real interval. The translation equation is stable in the Hyers–Ulam sense in the class of functions*

$$\mathcal{F}_{cs} := \{F: I \times \mathbb{R} \rightarrow I: F(x_0, \cdot) \text{ is a continuous surjection for some } x_0 \in I\}.$$

More precisely, for every $\epsilon > 0$ and $\delta = \frac{1}{9}\epsilon$, for every $H \in \mathcal{F}_{cs}$ satisfying (31) there exists a homeomorphism $g: I \rightarrow \mathbb{R}$ such that

$$|H(x, t) - F^t(x)| < \epsilon, \quad x \in I, t \in \mathbb{R},$$

where $F^t(x) = g(t + g^{-1}(x))$ for $x \in I, t \in \mathbb{R}$.

Przebiezac [118] obtained the stability of (2) in the class of functions from $I \times R$ into I which are continuous with respect to each variable (with $\delta = \frac{1}{10}\epsilon$) and in the class of continuous, surjective dynamical systems $H: I \times R \rightarrow I$ (with $\delta = \frac{1}{9}\epsilon$).

Theorem 52 ([118]). *Let I be an interval. Suppose that $H: \mathbb{R} \times I \rightarrow I$ is continuous with respect to each variable and satisfies*

$$|H(s, H(t, x)) - H(t + s, x)| \leq \delta, \quad x \in I, \quad s, t \in \mathbb{R}. \quad (33)$$

Then there exists a continuous iteration group $\{F^t, t \in \mathbb{R}\}$ such that

$$|F^t(x) - H(t, x)| \leq 10\delta, \quad x \in I, \quad t \in \mathbb{R}.$$

If, additionally, H is surjective, then there exists a continuous iteration group $\{F^t, t \in \mathbb{R}\}$ with $F^0 = \text{id}$ such that

$$|F^t(x) - H(t, x)| \leq 9\delta, \quad x \in I, \quad t \in \mathbb{R}.$$

She also showed that despite the stability of (2) and the stability of $F(x, 0) = x, x \in I$, the system of these equations is not stable in general. More precisely, for every proper interval $I \neq \mathbb{R}$ there exists $\varepsilon > 0$ such that for every $\delta > 0$ there exists H satisfying (33) with the property that for every continuous iteration group $\{F^t, t \in \mathbb{R}\}$ with $F^0 = \text{id}$ there exist t_0 and x_0 such that $|H(t_0, x_0) - F^{t_0}(x_0)| > \varepsilon$.

Earlier, in [116], she proved that in the class of continuous functions (with respect to each variable) equation (2) with $X = I$ and $G = (0, +\infty)$ is stable but not on the whole I .

Theorem 53 ([116]). *For every $\epsilon, \zeta > 0$ there exists $\delta > 0$ such that for every continuous function $H: I \times (0, +\infty) \rightarrow I$ satisfying (31) there exist $L \subset I, |L| < \zeta$ and a continuous solution $F: I \times (0, +\infty) \rightarrow I$ of (31) such that (32) holds true for $x \in \text{cl}(I \setminus L)$ and $t \in (0, +\infty)$.*

In [116] one can find the assumptions on H under which equation (2) is stable on the whole I .

Moreover, Jabłoński and Reich [45] proved that under some assumptions on the abelian group $(G, +)$ the translation equation is stable in the ring of formal power series $\mathbb{K}[[Y]]$, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. In this case $F(Y, t) = \sum_{i=1}^{\infty} c_i(t)Y^i$, where $c_1: G \rightarrow \mathbb{K} \setminus \{0\}$ and $c_i: G \rightarrow \mathbb{K}$ for $i > 1$.

More information on the stability of the translation equation may be found in the survey paper by Moszner [103].

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