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Aequationes Mathematicae

On multivalued iteration semigroups

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Abstract. We will give a necessary and sufficient condition for the family $\{F_t: \ t \geq 0\}$ of multifunctions $F_t(x) = \sum_{i=0}^{\infty} \frac{t^i}{i!} G^i(x)$, where G is a continuous and additive multifunction, to be an iteration semigroup.

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In the paper Smajdor [10] showed that the condition

$$G + tG^2 = (I + tG) \circ G, \quad t \geq 0,$$

where I is the identical map and $\circ$ denotes superposition, is a necessary and sufficient condition under which the family $\{F_t: \ t \geq 0\}$ of multifunctions $F_t(x) = \sum_{i=0}^{\infty} \frac{t^i}{i!} G^i(x)$, where G is a continuous and additive multifunction, is an iteration semigroup, with the assumption $0 \in G(x)$ for all $x \in K$. We want to present another one without the assumption that $0 \in G(x)$ and its correlation.

Throughout this paper all vector spaces are supposed to be real. Let X be a vector space. We define

$$A + B := \{a + b: \ a \in A, \, b \in B\}, \quad tA := \{ta: \ a \in A\},$$

where $A, B \subset X$ and $t \in \mathbb{R}$.

A subset K of X is called a *cone* if $tK \subset K$ for all positive t. A cone is said to be *convex* if it is a convex set.

Let X and Y be two vector spaces and let $K \subset X$ be a convex cone. A set-valued function $F\colon K \to n(Y)$, where $n(Y)$ denotes the family of all nonempty subsets of Y, is called *additive* if

$$F(x + y) = F(x) + F(y)$$

for all $x, y \in K$. If additionally F satisfies

$$F(tx) = tF(x)$$

for all $x \in K$ and $t \geq 0$, then F is *linear*.

Let $c(Y)$ denote the family of all nonempty compact subsets of a normed space Y. The continuity of a multifunction with compact values denotes continuity with respect to the Hausdorff metric d.

Let X, Y be normed spaces and K be a closed convex cone in X. The *norm* $\|F\|$ of a continuous additive multifunction $F\colon K \to n(Y)$ is the smallest element of the set $\{M > 0 : \ \|F(x)\| \leq M\|x\|, \, x \in K\}$.

Lemma 1. *Let K be a closed convex cone with a nonempty interior in a Banach space and let Y be a normed space. Suppose that $F_n\colon K \to c(Y)$, $n \in \mathbb{N}$, are continuous additive set-valued functions. If*

$$\lim_{n\to\infty} F_n(x) = F(x) \quad \textit{for } x \in K,$$

then F is continuous and additive.

Proof. It is clear that F is linear.

As $(F_n)_{n\in\mathbb{N}}$ is convergent to F, the set

$$\bigcup_{n=1}^{\infty} F_n(x)$$

is bounded for every $x \in K$. By Theorem 3 in Ref. [12] there exists a positive constant M such that

$$\|F_n\| \leq M \quad \text{for } n = 1, 2, \ldots. \tag{1}$$

Moreover, by Lemma 5 in Ref. [11] there exists $M_0 > 0$ such that

$$d\left(F_n(x), F_n(y)\right) \leq M_0\|F_n\|\|x - y\|, \quad x, y \in K. \tag{2}$$

Let $\epsilon > 0$, $x, y \in K$ and $\|x - y\| < \frac{\epsilon}{3M_0M}$. There exists $n \in \mathbb{N}$ such that

$$d\left(F(x), F_n(x)\right) < \frac{\epsilon}{3} \quad \text{and} \quad d\left(F(y), F_n(y)\right) < \frac{\epsilon}{3}.$$

Thus, by (1) and (2) we have

$$\begin{aligned} d(F(x), F(y)) &\leq d(F(x), F_n(x)) + d(F_n(x), F_n(y)) + d(F_n(y), F(y)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

This means that F is continuous.

Lemma 2. (Lemma 7 in Ref. [6]) *Let K be a closed convex cone with a nonempty interior in a Banach space and let Y be a normed space. Suppose that $F, F_n\colon K \to c(Y)$ are continuous additive set-valued functions. If*

$$\lim_{n\to\infty} F_n(x) = F(x) \quad \textit{for } x \in K,$$

then the sequence $(F_n)_{n\in\mathbb{N}}$ uniformly converges to F on every $D \in c(K)$.

Lemma 3. (Lemma 4 in Ref. [8]) *Let D be a nonempty set and Y be a normed space. If $F, F_n \colon D \to c(Y)$ are set-valued functions and the sequence $(F_n)_{n\in\mathbb{N}}$ uniformly converges to F on D, then*

$$\lim_{n\to\infty} F_n(D) = F(D).$$

Lemma 4. (Theorem 2 in Ref. [2]) *Let (X, ρ_X) and (Y, ρ_Y) be two metric spaces and let d_X and d_Y be the Hausdorff metrics derived from ρ_X and ρ_Y, respectively. If $F \colon X \to n(Y)$ is a set-valued function and M is a positive constant such that*

$$d_Y(F(x), F(y)) \leq M\rho_X(x, y)$$

for all $x, y \in X$, then

$$d_Y(F(A), F(B)) \leq M d_X(A, B)$$

for every nonempty subsets A, B of X.

The *superposition* $G \circ F$ of two multifunctions $F, G \colon K \to n(K)$ is defined as follows

$$(G \circ F)(x) = G(F(x)) = \bigcup_{y\in F(x)} G(y).$$

Lemma 5. *Let K be a closed convex cone with a nonempty interior in a Banach space. Suppose that $F_n, F, G_n, G \colon K \to c(K)$, $n \in \mathbb{N}$, are continuous additive set-valued functions. If $\lim_{n\to\infty} F_n(x) = F(x)$ and $\lim_{n\to\infty} G_n(x) = G(x)$ for $x \in K$, then*

$$\lim_{n\to\infty} F_n(G_n(x)) = F(G(x))$$

for $x \in K$.

Proof. Fix $x \in K$. From Lemma 5 in Ref. [11] and Lemma 4 there exists $M_0 > 0$ such that

$$\begin{aligned} d(F_n(G_n(x)), F(G(x))) &\leq d(F_n(G_n(x)), F_n(G(x))) + d(F_n(G(x)), F(G(x))) \\ &\leq M_0\|F_n\| d(G_n(x), G(x)) + d(F_n(G(x)), F(G(x))). \end{aligned}$$

By the same argument as in the proof of Lemma 1 there exists a positive constant M such that

$$\|F_n\| \leq M \quad \text{for } n = 1, 2, \ldots.$$

Moreover, by Lemma 2, the sequence $(F_n)_{n\in\mathbb{N}}$ is uniformly convergent to F on every nonempty compact subset of K. Thus, from Lemma 3,

$$\lim_{n\to\infty} d(F_n(G_n(x)), F(G(x))) = 0.$$

Let K be a convex cone in a normed space and let $cc(K)$ stand for the family of all compact convex members of $n(K)$. The *Hukuhara difference* $A - B$

of $A, B \in cc(K)$ is a set $C \in cc(K)$ such that $A = B + C$. By Rådström's Cancellation Lemma [5] it follows that if this difference exists, then it is unique.

Lemma 6. (Lemma 3 in Ref. [7]) *Let X and Y be two normed vector spaces and let K be a closed convex cone in X. Assume that $F\colon K \to cc(K)$ is a continuous additive set-valued function and $A, B \in cc(K)$. If there exists the difference $A - B$, then there exists $F(A) - F(B)$ and $F(A) - F(B) = F(A - B)$.*

Let $F, G\colon K \to cc(K)$. We can define the multifunctions $F + G$ and $F - G$ on K as follows

$$(F + G)(x) := F(x) + G(x) \quad \text{for } x \in K$$

and

$$(F - G)(x) := F(x) - G(x)$$

if the Hukuhara differences $F(x) - G(x)$ exist for all $x \in K$.

Lemma 7. (Lemma 2 in Ref. [4]) *For each set $A \subset K$ the inclusion*

$$(F + G)(A) \subset F(A) + G(A)$$

holds. Moreover, if there exist the Hukuhara difference $F(A) - G(A)$ and the multifunction $F - G$, then

$$F(A) - G(A) \subset (F - G)(A).$$

For a multifunction $F\colon [a, b] \to cc(X)$ such that there exist the Hukuhara differences $F(t) - F(s)$ as $a \le s \le t \le b$, the *Hukuhara derivative* at $t \in (a, b)$ is defined by the formula

$$DF(t) = \lim_{h \to 0^+} \frac{F(t + h) - F(t)}{h} = \lim_{h \to 0^+} \frac{F(t) - F(t - h)}{h},$$

whenever both of these limits exist with respect to the Hausdorff metric d in $cc(K)$ derived from the norm in X (see Ref. [1]). Moreover,

$$DF(a) = \lim_{s \to a^+} \frac{F(s) - F(a)}{s - a}, \quad DF(b) = \lim_{s \to b^-} \frac{F(b) - F(s)}{b - s}.$$

Lemma 8. (Lemma 5 in Ref. [9]) *If $F, G\colon [a, b] \to cc(X)$ are two differentiable multifunctions such that $DF(t) = DG(t)$ for $t \in [a, b]$ and $F(a) = G(a)$, then*

$$F(t) = G(t) \quad \textit{for } t \in [a, b].$$

Let X be a Banach space and let $[a, b] \subset \mathbb{R}$. If a multifunction $F\colon [a, b] \to cc(X)$ is continuous, then there exists the Riemann integral of F (see Ref. [1]). We need the following properties of the Riemann integral.

Lemma 9. ([1] p. 211) *If $F\colon [a, b] \to cc(X)$ is continuous, then*

$$\left\| \int_a^b F(t)\, dt \right\| \le \int_a^b \| F(t) \|\, dt.$$

Lemma 10. (Lemma 7 in Ref. [8]) *Let K be a convex cone in X. If $F\colon K \to cc(X)$ is continuous and additive, $G\colon [a,b] \to cc(K)$ is continuous, then*

$$\int_a^b F(G(t))\,dt = F\left(\int_a^b G(t)\,dt\right).$$

Lemma 11. (Lemma 4 in Ref. [11]) *If $F\colon [a,b] \to cc(X)$ is continuous and $H(t) = \int_a^t F(u)\,du$, then $DH(t) = F(t)$ for $a \le t \le b$.*

Lemma 12. *If $F\colon [0,+\infty) \to cc(X)$ is continuous, then*

$$\int_0^t \left(\frac{(t-u)^n}{n!}\int_0^u F(s)\,ds\right)du = \int_0^t \frac{(t-u)^{n+1}}{(n+1)!}F(u)\,du \tag{3}$$

for $t \ge 0$ and $n = 0, 1, \ldots$.

Proof. For every nonnegative integer n we define

$$\phi_n(t) = d\left(\int_0^t \left(\frac{(t-u)^n}{n!}\int_0^u F(s)\,ds\right)du, \int_0^t \frac{(t-u)^{n+1}}{(n+1)!}F(u)\,du\right), \quad t \ge 0.$$

For $n = 0$

$$\phi_0(t) = d\left(\int_0^t \left(\int_0^u F(s)\,ds\right)du, \int_0^t (t-u)F(u)\,du\right), \quad t \ge 0$$

and according to Lemma 12 in Ref. [3], we have $\phi_0 \equiv 0$.

Fix $n \in \mathbb{N}$ and we suppose that $\phi_i \equiv 0$ for all $i \le n$. By the properties of the Riemann integral we obtain, for $0 < k < 1$, the following equalities

$$\begin{aligned}
&\int_0^{t+k} \left(\frac{(t+k-u)^{n+1}}{(n+1)!}\int_0^u F(s)\,ds\right)du \\
&\quad = \int_0^t \left(\frac{(t-u)^{n+1}}{(n+1)!}\int_0^u F(s)\,ds\right)du + \sum_{i=1}^{n+1}\frac{k^i}{i!}\int_0^t \left(\frac{(t-u)^{n+1-i}}{(n+1-i)!}\int_0^u F(s)\,ds\right)du \\
&\qquad + \int_t^{t+k} \left(\frac{(t+k-u)^{n+1}}{(n+1)!}\int_0^u F(s)\,ds\right)du
\end{aligned}$$

and

$$\int_0^{t+k} \frac{(t+k-u)^{n+2}}{(n+2)!} F(u)\, du$$
$$= \int_0^t \frac{(t-u)^{n+2}}{(n+2)!} F(u)\, du$$
$$+ \sum_{i=1}^{n+2} \frac{k^i}{i!} \int_0^t \frac{(t-u)^{n+2-i}}{(n+2-i)!} F(u)\, du + \int_t^{t+k} \frac{(t+k-u)^{n+2}}{(n+2)!} F(u)\, du.$$

Using the last equality and the induction assumption we conclude that

$$\frac{\phi_{n+1}(t+k) - \phi_{n+1}(t)}{k}$$
$$\le \left\| \frac{1}{k} \int_t^{t+k} \left(\frac{(t+k-u)^{n+1}}{(n+1)!} \int_0^u F(s)\, ds \right) du \right\|$$
$$+ \left\| \frac{1}{k} \int_t^{t+k} \frac{(t+k-u)^{n+2}}{(n+2)!} F(u)\, du \right\| + \left\| \int_0^t \frac{k^{n+1}}{(n+2)!} F(u)\, du \right\|.$$

Since

$$\left\| \frac{1}{k} \int_t^{t+k} \left(\frac{(t+k-u)^{n+1}}{(n+1)!} \int_0^u F(s)\, ds \right) du \right\| \le \frac{(t+k)k^{n+1}}{(n+2)!} M$$

and

$$\left\| \frac{1}{k} \int_t^{t+k} \frac{(t+k-u)^{n+2}}{(n+2)!} F(u)\, du \right\| \le \frac{k^{n+2}}{(n+3)!} M,$$

where $M = \sup\{\|F(s)\| :\ 0 \le s \le t+1\}$, it follows that

$$\liminf_{k \to 0^+} \frac{\phi_{n+1}(t+k) - \phi_{n+1}(t)}{k} \le 0.$$

Moreover, ϕ_{n+1} is continuous, nonnegative and $\phi_{n+1}(0) = 0$, whence applying the corollary from the Zygmund Lemma we obtain $\phi_{n+1} \equiv 0$. This means that (3) holds for every $n \in \mathbb{N}$ and $t \ge 0$.

Let K be a nonempty set. A family $\{F_t :\ t \ge 0\}$ of set-valued functions $F_t \colon K \to n(K)$ is said to be an *iteration semigroup* if

$$F_t \circ F_s = F_{t+s}$$

for all $t, s \ge 0$.

Let K be a convex cone in a normed space. An iteration semigroup $\{F_t : t \geq 0\}$ of set-valued functions $F_t \colon K \to cc(K)$ is called *differentiable* if all set-valued functions $t \mapsto F_t(x)$, $x \in K$, have the Hukuhara derivative on $[0, +\infty)$.

Theorem 1. *Let K be a closed convex cone with a nonempty interior in a Banach space. If $\{F_t : t \geq 0\}$ is a differentiable iteration semigroup of continuous additive multifunctions $F_t \colon K \to cc(K)$ with $F_0(x) = \{x\}$, then*

(i) *$DF_t(x) = F_t(G(x))$ for all $x \in K$, $t \geq 0$, where $DF_t(x)$ denotes the Hukuhara derivarive of $F_t(x)$ with respect to t and $G(x) = DF_t(x)|_{t=0} = \lim_{h \to 0^+} \frac{F_h(x) - x}{h}$ is continuous and additive,*

(ii) *$F_t(x) = x + \int_0^t F_u(G(x))\, du$ for all $x \in K$, $t \geq 0$,*

(iii) *For all $x \in K$ and $t \geq 0$*

$$F_t(x) \subset \sum_{i=0}^{\infty} \frac{t^i}{i!} G^i(x).$$

If additionally $F_t \circ G = G \circ F_t$ for $t \geq 0$, then

$$F_t(x) = \sum_{i=0}^{\infty} \frac{t^i}{i!} G^i(x).$$

Proof. (i) It follows immediately from Theorem 1 in Ref. [6] and Lemma 1.

(ii) Let us fix $x \in K$. We observe, by Lemmas 2 and 3, that $t \mapsto F_t(G(x))$ is continuous, so it is integrable. According to (i) and Lemmas 8, 11 we get

$$F_t(x) = x + \int_0^t F_u(G(x))\, du, \quad t \geq 0.$$

(iii) Let us fix $x \in K$ and $t \geq 0$. At first we show the following inclusion

$$F_t(G(x)) \subset G(F_t(x)). \tag{4}$$

By Lemmas 2, 3, 6 and 7 we obtain

$$\begin{aligned} F_t(G(x)) &= F_t\left(\lim_{h \to 0^+} \frac{F_h(x) - x}{h}\right) \\ &= \lim_{h \to 0^+} \frac{F_t(F_h(x)) - F_t(x)}{h} \\ &= \lim_{h \to 0^+} \frac{F_h(F_t(x)) - F_t(x)}{h} \\ &\subset \lim_{h \to 0^+} \left(\frac{F_h - I}{h}\right)(F_t(x)) \\ &= G(F_t(x)). \end{aligned}$$

From this, (ii) and Lemma 10 we have

$$F_t(x) = x + \int_0^t F_u(G(x))\,du \subset x + G\left(\int_0^t F_u(x)\,du\right).$$

If we apply (ii) to F_u in the last inclusion, we conclude that

$$\begin{aligned} F_t(x) &\subset x + G\left(\int_0^t \left(x + \int_0^u F_s(G(x))\,ds\right)\,du\right) \\ &\subset x + tG(x) + G^2\left(\int_0^t \left(\int_0^u F_s(x)\,ds\right)\,du\right). \end{aligned}$$

Using Lemma 12 we have

$$F_t(x) \subset x + tG(x) + G^2\left(\int_0^t (t-u)F_u(x)\,du\right).$$

Repeating the same procedure we obtain

$$\begin{aligned} F_t(x) &\subset x + tG(x) + \frac{t^2}{2!}G^2(x) + \cdots + \frac{t^n}{n!}G^n(x) \\ &\quad + G^{n+1}\left(\int_0^t \frac{(t-u)^n}{n!}F_u(x)\,du\right). \end{aligned} \tag{5}$$

It remains to prove that

$$\lim_{n\to\infty} G^{n+1}\left(\int_0^t \frac{(t-u)^n}{n!}F_u(x)\,du\right) = \{0\}. \tag{6}$$

There exists $M > 0$ such that $\|F_u(x)\| \le M$ for $u \in [0,t]$. Thus, by Lemma 9, we see that

$$\begin{aligned} &\left\|G^{n+1}\left(\int_0^t \frac{(t-u)^n}{n!}F_u(x)\,du\right)\right\| \\ &\quad \le \|G\|^{n+1}\int_0^t \frac{(t-u)^n}{n!}\|F_u(x)\|\,du \\ &\quad \le \|G\|^{n+1}M\int_0^t \frac{(t-u)^n}{n!}\,du \end{aligned}$$

$$= \|G\|^{n+1} M \frac{t^{n+1}}{(n+1)!}$$
$$= M \frac{(t\|G\|)^{n+1}}{(n+1)!}.$$

Since $\lim_{n\to\infty} M \frac{(t\|G\|)^{n+1}}{(n+1)!} = 0$, we have (6). Therefore, by (5),

$$F_t(x) \subset \sum_{i=0}^{\infty} \frac{t^i}{i!} G^i(x).$$

By similar considerations the reader can prove that if $F_t \circ G = G \circ F_t$ for $t \geq 0$, then

$$F_t(x) = \sum_{i=0}^{\infty} \frac{t^i}{i!} G^i(x).$$

Lemma 13. *Let K be a closed convex cone with a nonempty interior in a Banach space. If $G\colon K \to cc(K)$ is a continuous additive multifunction and $F_t(x) = \sum_{i=0}^{\infty} \frac{t^i}{i!} G^i(x)$, then $t \mapsto F_t(x)$, $x \in K$, is differentiable and*

$$DF_t(x) = G(F_t(x)), \quad x \in K,\ t \geq 0.$$

Proof. Let us fix $x \in K$, $t \geq 0$ and $h > 0$. We have the equalities

$$\frac{F_{t+h}(x) - F_t(x)}{h} = \lim_{n\to\infty} \sum_{i=0}^{n} \frac{1}{i!} \frac{(t+h)^i - t^i}{h} G^i(x) = \sum_{i=0}^{\infty} \frac{1}{i!} \frac{(t+h)^i - t^i}{h} G^i(x).$$

We will show that the series $\sum_{i=0}^{\infty} \frac{1}{i!} \frac{(t+h)^i - t^i}{h} G^i(x)$ is uniformly convergent. Let $h \in (0,1)$ and let r be a positive number such that $t + 1 \leq r$. Then

$$\left\| \frac{1}{i!} \frac{(t+h)^i - t^i}{h} G^i(x) \right\| \leq \frac{1}{i!} \|G\|^i \|x\| \left((t+h)^{i-1} + t(t+h)^{i-2} + \cdots + t^{i-1} \right)$$
$$\leq \frac{1}{i!} \|G\|^i \|x\| i r^{i-1}.$$

Since the series $\sum_{i=1}^{\infty} \frac{(\|G\| r)^{i-1}}{(i-1)!} \|G\| \|x\|$ is convergent, it follows that the series $\sum_{i=1}^{\infty} \frac{1}{i!} \frac{(t+h)^i - t^i}{h} G^i(x)$ is uniformly convergent for $h \in (0,1)$. Therefore, we can write

$$\lim_{h\to 0^+} \frac{F_{t+h}(x) - F_t(x)}{h} = \sum_{i=1}^{\infty} \frac{t^{i-1}}{(i-1)!} G^i(x)$$
$$= G\left(\sum_{i=1}^{\infty} \frac{t^{i-1}}{(i-1)!} G^{i-1}(x) \right) = G(F_t(x)).$$

In a similar way we obtain

$$\lim_{h\to 0^+}\frac{F_t(x)-F_{t-h}(x)}{h}=G(F_t(x)).$$

Whence

$$DF_t(x)=G(F_t(x))\quad \text{for } x\in K,\ t\geq 0.$$

Theorem 2. *Let K be a closed convex cone with a nonempty interior in a Banach space and let $G\colon K\to cc(K)$ be a continuous additive multifunction. Assume that $F_t(x)=\sum_{i=0}^{\infty}\frac{t^i}{i!}G^i(x)$ for $x\in K$ and $t\geq 0$. The family $\{F_t:\ t\geq 0\}$ is an iteration semigroup if and only if*

$$F_t\circ G=G\circ F_t \tag{7}$$

for $t\geq 0$.

Proof. Suppose that the family $\{F_t:\ t\geq 0\}$ is an iteration semigroup. By Lemma 13 this family is differentiable and $DF_t(x)=G(F_t(x))$ for $x\in K$ and $t\geq 0$. On the other hand, from Theorem 1, we have $DF_t(x)=F_t(G(x))$ for $x\in K$ and $t\geq 0$. Thus, $F_t\circ G=G\circ F_t$ for $t\geq 0$.

Now, we assume that (7) holds. Let $x\in K$ and $t\geq 0$. We observe that

$$F_t(G(x))=\left(\sum_{i=0}^{\infty}\frac{t^i}{i!}G^i\right)(G(x))\subset (I+tG)(G(x))+\sum_{i=2}^{\infty}\frac{t^i}{i!}G^{i+1}(x)$$

and

$$G(F_t(x))=G\left(\sum_{i=0}^{\infty}\frac{t^i}{i!}G^i(x)\right)=G(x)+tG^2(x)+\sum_{i=2}^{\infty}\frac{t^i}{i!}G^{i+1}(x).$$

Since $F_t(G(x))=G(F_t(x))$, we have

$$G(x)+tG^2(x)+\sum_{i=2}^{\infty}\frac{t^i}{i!}G^{i+1}(x)\subset (I+tG)(G(x))+\sum_{i=2}^{\infty}\frac{t^i}{i!}G^{i+1}(x).$$

Thus,

$$G(x)+tG^2(x)\subset (I+tG)(G(x)).$$

The inverse inclusion is obvious (see Lemma 7), therefore

$$G(x)+tG^2(x)=(I+tG)(G(x)).$$

Similarly, we obtain

$$G(F_t^n(x))=F_t^n(G(x))\quad \text{for } n\in\mathbb{N},$$

where $F_t^n(x) = \sum_{i=0}^{n} \frac{t^i}{i!} G^i(x)$. Thus,

$$\begin{aligned} F_t^n(F_s^n(x)) &= \left(\sum_{j=0}^{n} \frac{t^j}{j!} G^j\right)\left(\sum_{i=0}^{n} \frac{s^i}{i!} G^i(x)\right) = \sum_{j=0}^{n}\sum_{i=0}^{n} \frac{t^j s^i}{j!i!} G^{i+j}(x) \\ &= \sum_{p=0}^{n}\sum_{q=0}^{p} \frac{t^{p-q}s^q}{(p-q)!q!} G^p(x) + R_n = \sum_{p=0}^{n} \frac{(t+s)^p}{p!} G^p(x) + R_n \\ &= F_{t+s}^n(x) + R_n, \end{aligned}$$

where $R_n = \sum_{p=n+1}^{2n} \sum_{q=p-n}^{n} \frac{t^{p-q}s^q}{(p-q)!q!} G^p(x)$. We see that

$$\begin{aligned} \|R_n\| &\leq \sum_{p=n+1}^{2n} \sum_{q=p-n}^{n} \frac{t^{p-q}s^q}{(p-q)!q!} \|G\|^p \|x\| \\ &= \sum_{p=n+1}^{2n} \frac{1}{p!}(t+s)^p \|G\|^p \|x\|. \end{aligned}$$

Therefore

$$\lim_{n\to\infty} R_n = \{0\}.$$

From Lemma 5 we get

$$F_t(F_s(x)) = F_{t+s}(x).$$

In the end we show the relation between our condition and the condition in Ref. [10].

Theorem 3. *Let K be a closed convex cone with a nonempty interior in a Banach space. Assume that $G\colon K \to cc(K)$ is a continuous additive multifunction and $F_t(x) = \sum_{i=0}^{\infty} \frac{t^i}{i!} G^i(x)$ for $x \in K$ and $t \geq 0$.*

1. *If $F_t \circ G = G \circ F_t$ for $t \geq 0$, then $(I + tG) \circ G = G \circ (I + tG)$, $t \geq 0$.*
2. *If $(I + tG) \circ G = G \circ (I + tG)$ and $0 \in G(x)$ for $t \geq 0$, $x \in K$, then $F_t \circ G = G \circ F_t$ for $t \geq 0$.*

Proof. 1) It follows immediately from the proof of Theorem 2.

2) According to Theorem 1 in Ref. [10] the family $\{F_t : t \geq 0\}$ is an iteration semigroup. Thus, by Theorem 2 the equality

$$F_t \circ G = G \circ F_t$$

holds for all $t \geq 0$.

Example. Let $K = [0, +\infty)$ and let $\{F_t : t \geq 0\}$ be a family of multifunctions $F_t(x) = \left[xe^{at}, xe^{bt}\right]$, as $0 \leq a \leq b$. Then this family is a differentiable iteration semigroup, $F_t(x) = \sum_{i=0}^{\infty} \frac{t^i}{i!} G^i(x)$, where $G(x) = [ax, bx]$ and $F_t \circ G = G \circ F_t$ for $t \geq 0$.

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