On a composite functional equation fulfilled by modulus of an additive function

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Dedicated to Janos Aczel on the occasion of his birthday.

Abstract. We deal with the problem of determining general solutions $f: \mathbb{R} \rightarrow \mathbb{R}$ of the following composite functional equation introduced by Fechner:

$$f(f(x) - f(y)) = f(x + y) + f(x - y) - f(x) - f(y).$$

Our result gives a partial answer to this problem under some assumptions upon $f(\mathbb{R})$. We are applying a theorem of Simon and Volkmann concerning a certain characterization of modulus of an additive function. A new proof of their result is also presented.

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1. Introduction

In this note we are dealing with the composite functional equation

$$f(f(x) - f(y)) = f(x + y) + f(x - y) - f(x) - f(y) \quad (*)$$

introduced by Fechner [1] in connection with the elementary identity

$$||x| - |y|| = |x + y| + |x - y| - |x| - |y|, \quad (1.1)$$

which goes back to Tarski [5]. In his paper Fechner established all solutions $f: \mathbb{R} \rightarrow \mathbb{R}$ which are either continuous at zero or monotonic (see [1, Theorem 11] and [1, Theorem 13], respectively). One step to derive a general solution (with no regularity assumptions) has been done in his paper as well (see

[1, Theorem 7]). Namely, in the case where G is an Abelian group uniquely divisible by 2, every solution $f: G \rightarrow G$ of $(*)$ with $f(G) \subset -f(G)$ has to be an idempotent additive self-mapping on G . Our goal is to provide a wider (but still partial) answer to the question about the general form of solutions of Eq. $(*)$ in the case $G = \mathbb{R}$.

For any function $f: \mathbb{R} \rightarrow \mathbb{R}$ we define

$$S = f(\mathbb{R}), \quad Z = S \cap (-S), \quad P = S \setminus Z.$$

If f is a solution of $(*)$ then, in view of [1, Lemma 1], $f|_S$ is the identity function and

$$f(nx) = nf(x) \quad \text{for all } x \in \mathbb{R}, n \in \mathbb{N}. \quad (1.2)$$

It is also easily verified that $(S, +)$ forms a subsemigroup of the group $(\mathbb{R}, +)$ and hence $(Z, +)$ is a subgroup of $(\mathbb{R}, +)$. Moreover, $(P, +)$ is a semigroup. Indeed, if $t, u \in P$ then clearly $t + u \in S$ and $f(t + u) = t + u$. If $t + u \in Z$ then we would have $t + u = -v$ for some $v \in S$. Therefore, $-t = u + v \in S$ which yields $t \in Z$; a contradiction. Consequently, $t + u \in S \setminus Z = P$.

By virtue of (1.2), we have

$$S = \mathbb{Q}^+ \cdot S, \quad Z = \mathbb{Q}^+ \cdot Z, \quad P = \mathbb{Q}^+ \cdot P, \quad (1.3)$$

where $\mathbb{Q}^+ := \mathbb{Q} \cap (0, \infty)$.

Note that the assumption of [1, Theorem 7] quoted above says that S is a group, i.e. $P = \emptyset$. Our aim is to solve Eq. $(*)$ in the more general case where $S \cup (-S)$ forms a group (which is equivalent to $S \cup (-S) = S - S$). Because of Fechner's result, we may (and we actually do) assume that $P \neq \emptyset$.

Our main result reads as follows.

Theorem 1. *Assume $f: \mathbb{R} \rightarrow \mathbb{R}$ is a function satisfying the following conditions:*

- (H₁) $S \cup (-S) = S - S$,
- (H₂) *there exists $\xi \in P$ such that*

$$P \subset \bigcup_{n \in \mathbb{N}} \left(\frac{1}{n} \xi + P \right) \quad (1.4)$$

and

$$\bigcap_{n \in \mathbb{N}} (n\xi + P) = \emptyset. \quad (1.5)$$

Then f is a solution of Eq. $()$ if and only if there exist: an additive function $A: \mathbb{R} \rightarrow \mathbb{R}$, a subgroup R of $\mathbb{R}$ such that $A|_R: R \rightarrow A(\mathbb{R})$ is an isomorphism, and a function $z: \mathbb{R} \rightarrow Z$, where $Z \subset \ker A$ is a group, with the properties:*

- (i) $z|_{-D}$ is additive, where $D = A^{-1}([0, \infty))$,
- (ii) $z(x) = -z(-x) + z(-A|_R^{-1}(|A(x)|))$ for all $x \in D$,
- (iii) $z(x) = x$ for all $x \in Z$,

such that

$$f(x) = A|_R^{-1}(|A(x)|) + z(x) \quad \text{for all } x \in \mathbb{R}. \quad (1.6)$$

Conditions (i)–(iii) give a clear description of any function z which comes into play. Namely, when we ensure that z is additive on the semigroup $-D$ and z is identity on $Z \subset \ker A \subset -D$, then the values of z on the set D are well-defined by formula (ii). As we see, there is a lot of freedom for the choice of function z ; in a very particular case it may be an arbitrary additive function being identity on Z and vanishing on R .

We also have a variety of additive functions A (and subgroups R) which yield solutions of Eq. (*) with the aid of (1.6). In a particular case A may be an arbitrary idempotent additive function (i.e. $A \circ A = A$) and $R = A(\mathbb{R})$. Then $A|_R$ is simply the identity function on R . In such a case the term $A|_R^{-1}(|A(x)|)$ reduces to $|A(x)|$.

Remark 1. If A and z are as above, then the following conditions hold:

- (iv) $z|_D$ is additive,
- (v) $z(x) = -z(-x) + z(-A|_R^{-1}(|A(x)|))$ for all $x \in \mathbb{R}$,
- (vi) $z(x) = 0$ for all $x \in D \cap R$.

Proof. For (iv) note that in view of (ii) and (i), for $x, y \in D$ we have

$$\begin{aligned} z(x) + z(y) &= z(-A|_R^{-1}(|A(x)|)) + z(-A|_R^{-1}(|A(y)|)) - z(-x) - z(-y) \\ &= z(-A|_R^{-1}(A(x) + A(y))) - z(-x - y) = z(x + y). \end{aligned}$$

To get (v) it is enough to fix $x \in -D$ and put $-x$ in the place of x in formula (ii). Finally, for an arbitrary $x \in D \cap R$ we have $x = A|_R^{-1}(|A(y)|)$ for a certain $y \in \mathbb{R}$ and putting this into equation (ii) we get $z(x) = -z(-x) + z(-x) = 0$. $\square$

Remark 2. If (H_1) holds, $P \neq \emptyset$ and S is bounded from one side, then (H_2) holds.

Proof. If S (and hence also P) is bounded from one side, then condition (1.4) is trivially fulfilled with an arbitrary $\xi \in P$. Now, we show that (1.5) also holds with any $\xi \in P$. Suppose, on the contrary, that there is $p \in P$ such that $p - \frac{1}{n}\xi \notin P$ for $n \in \mathbb{N}$. In view of (1.3) and (H_1) , we have

$$p - \frac{1}{n}\xi \in P - P \subset S - S = S \cup (-S). \quad (1.7)$$

Since $p - \frac{1}{n}\xi \notin P$, we have either $p - \frac{1}{n}\xi \notin S$, or $p - \frac{1}{n}\xi \in Z$. By (1.7), in both cases $p - \frac{1}{n}\xi \in -S$, which implies $\xi - np \in S$ for all $n \in \mathbb{N}$ and contradicts the fact that S is bounded from one side. $\square$

In the last section we will show that the situation where (H_2) is not valid is possible and produces new solutions of Eq. (*). They “pretend” to behave partially like an additive function and partially like the absolute value function.

2. The proof

One of the crucial parts of the proof is a construction of an additive mapping which is positive on a given semigroup (and nowhere else). To this end we apply a method used by Sablik in the proof of [3, Proposition 3.1]. Our strategy is also based on a theorem of Simon and Volkmann [4] which says that a function $g: \mathbb{R} \rightarrow \mathbb{R}$ satisfies the equation

$$g(x) + g(y) = \max\{g(x+y), g(x-y)\} \quad \text{for all } x, y \in \mathbb{R} \quad (2.1)$$

if and only if there is an additive function $A: \mathbb{R} \rightarrow \mathbb{R}$ such that

$$g(x) = |A(x)| \quad \text{for all } x \in \mathbb{R}. \quad (2.2)$$

A new proof of this statement, quite different from the original one, may be found in the next section.

Proof of Theorem 1. Suppose $f: \mathbb{R} \rightarrow \mathbb{R}$ is a solution of $(*)$ which fulfils (H_1) and (H_2) . Denote by X the group $S \cup (-S)$ and by $\tilde{X}$ the quotient group X/Z . Let also $\tilde{S} = \{x+Z : x \in S\}$. Plainly, $\tilde{S} \cup (-\tilde{S}) = \tilde{X}$. Moreover, $\tilde{S} \cap (-\tilde{S}) = \{0\}$ (0 stands here for the coset Z). Indeed, suppose $t \in S$ and $t+Z \in -\tilde{S}$. Then there is $u \in S$ such that $t+Z = -u+Z$, i.e. $t+u \in Z$. Thus $-t-u \in S$ which gives $f(-t-u) = -t-u$ and $-t = f(-t-u) + u \in S$. Consequently, $t \in -S$ and therefore $t \in Z$, which yields $t+Z = Z$.

Let $\tilde{S}^* = \tilde{S} \setminus \{0\}$. Then, obviously, $\tilde{S}^* \cup (-\tilde{S}^*) = \tilde{X} \setminus \{0\}$ and $\tilde{S}^* \cap (-\tilde{S}^*) = \emptyset$. Moreover, $\tilde{S}^* + \tilde{S}^* \subset \tilde{S}^*$. Indeed, for every $t, u \in P$ we have $(t+Z) + (u+Z) = (t+u) + Z \in \tilde{S}$. If $(t+u) + Z = 0$, then we would have $t+u \in Z$ and the argumentation used in the preceding paragraph shows that $t \in Z$, which is impossible.

Now, we are going to construct an additive function $a: \tilde{X} \rightarrow \mathbb{R}$ such that

$$a^{-1}((0, \infty)) = \tilde{S}^*. \quad (2.3)$$

To this end pick any $\xi \in P$ satisfying conditions (1.4) and (1.5). For every $x \in P$ we define

$$A_x = \left\{ \frac{m}{n} \in \mathbb{Q} : m \in \mathbb{Z}, n \in \mathbb{N} \text{ and } nx + m\xi \in P \right\}.$$

By means of (1.4), there is $n \in \mathbb{N}$ such that $x \in \frac{1}{n}\xi + P$. Hence, in view of (1.3), we have $nx - \xi \in P$, which implies $\inf A_x < 0$. In what follows we show that $\inf A_x > -\infty$.

Suppose $\frac{m}{n} \in A_x$ and $\frac{m'}{n'} > \frac{m}{n}$ for some $m, m' \in \mathbb{Z}$ and $n, n' \in \mathbb{N}$. Then $m'n - mn' > 0$ and since P is a semigroup, we obtain

$$n(n'x + m'\xi) = n'(nx + m\xi) + (m'n - n'm)\xi \in P.$$

Thus, (1.3) implies $\frac{m'}{n'} \in A_x$. We have just shown the following condition:

$$\text{if } q \in A_x \text{ and } \mathbb{Q} \ni q' > q, \text{ then } q' \in A_x. \quad (2.4)$$

Consequently, to obtain $\inf A_x > -\infty$ it is enough to show that $-n \notin A_x$ for at least one $n \in \mathbb{N}$. This fact, however, follows from condition (1.5), since otherwise we would have $x - n\xi \in P$ for every $n \in P$, i.e. $x \in \bigcap_{n \in \mathbb{N}} (n\xi + P)$.

Let $x, y \in P$ and suppose $x + Z = y + Z$. Then $y = x + z$ for a certain $z \in Z$. Now, if $nx + m\xi \in P$ for some $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, then

$$ny + m\xi = (nx + m\xi) + nz \in P + Z = P,$$

which proves $A_x \subset A_y$. Interchanging the roles of x and y we obtain $A_x = A_y$. Therefore, we may define a function $a: \tilde{X} \rightarrow \mathbb{R}$ by the formula

$$a(x + Z) = \begin{cases} -\inf A_x, & \text{if } x \in P, \\ 0, & \text{if } x \in Z, \\ \inf A_{-x}, & \text{if } x \in -P. \end{cases}$$

Condition (2.3) is then straightforward. We shall prove the additivity of a . By the definition, if a is additive on the semigroup $\tilde{S}^*$, then it is additive on the whole of $\tilde{X}$. In other words, it is enough to show that for all $x, y \in P$ we have

$$\inf A_{x+y} = \inf A_x + \inf A_y. \quad (2.5)$$

Fix $x, y \in P$. Let $r = \frac{m}{n} > \inf A_x$ and $s = \frac{m'}{n'} > \inf A_y$, where $m, m' \in \mathbb{Z}$ and $n, n' \in \mathbb{N}$. Then condition (2.4) implies $nx + m\xi \in P$ and $n'y + m'\xi \in P$. Hence

$$nn'(x + y) + (mn' + m'n)\xi = n'(nx + m\xi) + n(n'y + m'\xi) \in P,$$

which means that $r + s \in A_{x+y}$ and, in view of (2.4), $r + s \geq \inf A_{x+y}$. This proves that

$$\inf A_{x+y} \leq \inf A_x + \inf A_y.$$

To get the reverse inequality let $r = \frac{m}{n} < \inf A_x$ and $s = \frac{m'}{n'} < \inf A_y$. Then $r \notin A_x$, thus $nx + m\xi \notin P$ and hence, by (1.3), $kx + km\xi \notin P$ for every $k \in \mathbb{N}$. If we had $nx + m\xi \in Z$, then

$$knx + (km + 1)\xi \in Z + \xi \subset P$$

whence

$$A_x \ni \frac{km + 1}{kn} \xrightarrow{k \rightarrow \infty} \frac{m}{n} = r;$$

a contradiction. Therefore, $nx + m\xi \in -P$ and, analogously, $n'y + m'\xi \in -P$. Hence

$$nn'(x + y) + (mn' + m'n)\xi = n'(nx + m\xi) + n(n'y + m'\xi) \in -P,$$

which means that $r + s \notin A_{x+y}$ and, in view of (2.4), $r + s \leq \inf A_{x+y}$. Consequently, equality (2.5) (and the additivity of a) has been proved.

Define a function $g: \mathbb{R} \rightarrow \mathbb{R}$ by the formula

$$g(x) = a(f(x) + Z).$$

In what follows we will show that g fulfils Eq. (2.1). By (1.2), we have $g(2x) = 2g(x)$ for every $x \in \mathbb{R}$. Thus substitution $x \mapsto \frac{x+y}{2}$, $y \mapsto \frac{x-y}{2}$ shows that Eq. (2.1) is equivalent to the following one:

$$g(x+y) + g(x-y) = 2 \max\{g(x), g(y)\}. \quad (2.6)$$

Putting $y = -x$ in Eq. (*) and applying (1.2) we infer that $f(f(x) - f(-x)) = f(x) - f(-x)$, i.e. $f(x) - f(-x) \in S$ for every $x \in \mathbb{R}$. Interchanging the roles of x and $-x$ we obtain $f(-x) - f(x) \in S$, therefore $f(x) - f(-x) \in Z$. This means that g is an even function. Consequently, the roles of x and y in Eq. (2.6) are symmetric. Fix arbitrarily $x, y \in \mathbb{R}$; for the proof of equality (2.6) it is enough to distinguish the two following cases.

Case 1. If $g(x) = g(y)$, then $a(f(x) + Z) = a(f(y) + Z)$ hence $a((f(x) - f(y)) + Z) = 0$. By virtue of (2.3), it is possible exclusively when $f(x) - f(y) \in Z$. Thus $f(x) - f(y) \in S$ and $f(y) - f(x) \in S$. Equation (*) then yields

$$2f(x) = f(x+y) + f(x-y) \quad \text{and} \quad 2f(y) = f(x+y) + f(y-x). \quad (2.7)$$

Summing up Eqs. (2.7), taking values of a on both sides, and making use of the evenness of g we obtain

$$2g(x) + 2g(y) = 2g(x+y) + g(x-y) + g(y-x) = 2g(x+y) + 2g(x-y),$$

which is nothing else but (2.6), since $g(x) = g(y)$.

Case 2. If $g(x) > g(y)$, then $a((f(x) - f(y)) + Z) > 0$ and, by virtue of (2.3), $f(x) - f(y) \in P \subset S$. Equation (*) yields $2f(x) = f(x+y) + f(x-y)$, thus $2g(x) = g(x+y) + g(x-y)$, which gives (2.6), since $g(x) > g(y)$.

In the light of the mentioned result of Simon and Volkmann, there is an additive mapping $A: \mathbb{R} \rightarrow \mathbb{R}$ fulfilling (2.2) or, equivalently,

$$a(f(x) + Z) = |A(x)| \quad \text{for all } x \in \mathbb{R}. \quad (2.8)$$

As it follows from the definition, a is a monomorphism on the group $\tilde{X}$, hence equality (2.8) gives

$$f(x) + Z = a^{-1}(|A(x)|) \quad \text{for all } x \in \mathbb{R}.$$

Let $\kappa: \tilde{X} \rightarrow X$ be an arbitrary selection (i.e. $\kappa(\omega) \in \omega$ for $\omega \in \tilde{X}$). Let also $\tilde{A}: \mathbb{R} \supset a(\tilde{X}) \rightarrow X$ be given as $\tilde{A} = \kappa \circ a^{-1}$. Then we may rewrite the above equality in the form

$$f(x) = \tilde{A}(|A(x)|) + z(x) \quad \text{for all } x \in \mathbb{R}, \quad (2.9)$$

with a suitable function $z: \mathbb{R} \rightarrow Z$.

For an arbitrary $x \in Z$, since $f(x) = x$, we obtain $a^{-1}(|A(x)|) = 0$, so it has to be $A(x) = 0$. Therefore, $A|_Z = 0$. Since $f(f(x)) = f(x)$ for $x \in \mathbb{R}$, equality (2.9) implies

$$\tilde{A}(|A(x)|) + z(x) = \tilde{A} \left\{ \left| A[\tilde{A}(|A(x)|) + z(x)] \right| \right\} + z(f(x)).$$

By the additivity of A and the fact that $A(z(x)) = 0$, we thus obtain

$$\tilde{A}(|A(x)|) - \tilde{A} \left\{ \left| A[\tilde{A}(|A(x)|)] \right| \right\} = z(f(x)) - z(x) \in Z.$$

Hence

$$a^{-1}(|A(x)|) = a^{-1} \left\{ \left| A[\tilde{A}(|A(x)|)] \right| \right\},$$

therefore

$$|A(x)| = \left| A[\tilde{A}(|A(x)|)] \right| \quad \text{for all } x \in \mathbb{R},$$

i.e.

$$A(\tilde{A}(t)) = \pm t \quad \text{for all } t \in A(\mathbb{R}). \quad (2.10)$$

Observe that for all $x, y \in a(\tilde{X})$ we have

$$\tilde{A}(x+y) - \tilde{A}(x) - \tilde{A}(y) \in Z$$

and since $A|_Z = 0$, we infer that the composition $A \circ \tilde{A}: a(\tilde{X}) \rightarrow X$ is additive. Consequently, Eq. (2.10) gives

$$(A \circ \tilde{A})|_{A(\mathbb{R})} = \pm \text{id}_{A(\mathbb{R})}.$$

Replacing A by $-A$ if needed we may assume that there is $+\text{id}_{A(\mathbb{R})}$ on the right-hand side. Denoting $R = \tilde{A}(A(\mathbb{R}))$ we infer that the mappings $\tilde{A}|_{A(\mathbb{R})}: A(\mathbb{R}) \rightarrow R$ and $A|_R: R \rightarrow A(\mathbb{R})$ are bijections, and

$$\tilde{A}|_{A(\mathbb{R})} = A|_R^{-1}. \quad (2.11)$$

Because there is no algebraic structure in the set R (since the selection κ has been chosen arbitrarily), we shall now replace it by some *subgroup* R' of $\mathbb{R}$ such that $A|_{R'}: R' \rightarrow A(\mathbb{R})$ is still a bijection. To this end it is enough to pick any Hamel basis H of the linear space $\mathbb{R}$ over $\mathbb{Q}$, and define R' as the linear space spanned by a maximal set $H' \subset H$ for which the sequence $(A(h) : h \in H')$ is linearly independent. Then, since both $A|_R: R \rightarrow A(\mathbb{R})$ and $A|_{R'}: R' \rightarrow A(\mathbb{R})$ are bijections, there is a one-to-one correspondence $R \leftrightarrow R'$ such that for all $r \leftrightarrow r'$ we have $A(r) = A(r')$. This allows us to replace the selection $\kappa: \tilde{X} \rightarrow R$ by a new selection $\kappa': \tilde{X} \rightarrow R'$ such that $\kappa(\omega) \leftrightarrow \kappa'(\omega)$ for $\omega \in \tilde{X}$. Defining the function $\tilde{A}$ in the new way as $\tilde{A} = \kappa' \circ a^{-1}$ we obtain formula (2.9) again (with a new function $z: \mathbb{R} \rightarrow Z$) and all the arguments of the preceding paragraph remain valid. Therefore, with no loss of generality we may assume that equality (2.11) holds true where R is a group. In particular, $A|_R^{-1}$ is additive.

Comparing (2.11) to representation (2.9) we infer that

$$f(x) = A|_R^{-1}(|A(x)|) + z(x) \quad \text{for all } x \in \mathbb{R}, \quad (2.12)$$

and using Eq. (*) we obtain

$$\begin{aligned} & A|_R^{-1} \{ |A[A|_R^{-1}(|A(x)|) - A|_R^{-1}(|A(y)|) + z(x) - z(y)]| \} + z(f(x) - f(y)) \\ &= A|_R^{-1}(|A(x+y)|) + A|_R^{-1}(|A(x-y)|) - A|_R^{-1}(|A(x)|) - A|_R^{-1}(|A(y)|) \\ & \quad + z(x+y) + z(x-y) - z(x) - z(y). \end{aligned} \quad (2.13)$$

Now, applying the additivity of A and $A|_R^{-1}$, identity (1.1), and the fact $A(z(x) - z(y)) = 0$, we conclude that

$$z(f(x) - f(y)) = z(x+y) + z(x-y) - z(x) - z(y) \quad \text{for all } x, y \in \mathbb{R}. \quad (2.14)$$

Putting $y = 0$ in equality (2.14) we get

$$z(f(x)) = z(x) \quad \text{for all } x \in \mathbb{R}, \quad (2.15)$$

whereas putting $x = 0$ yields

$$z(-f(x)) = z(-x) \quad \text{for all } x \in \mathbb{R}. \quad (2.16)$$

For an arbitrary $x \in Z$, in the light of (2.12) and the fact that $A|_Z = 0$, we have $x = f(x) = z(x)$. Therefore

$$z|_Z = \text{id}_Z, \quad (2.17)$$

which proves condition (iii).

Observe that $z|_S$ is additive. In fact, for all $x, y \in \mathbb{R}$ we have $f(f(x) + f(y)) = f(x) + f(y)$ hence formula (2.12), jointly with (2.15), yields $z(f(x) + f(y)) = z(x) + z(y) = z(f(x)) + z(f(y))$.

Assuming $|A(x)| \geq |A(y)|$ formula (2.12) gives

$$\begin{aligned} f(x) - f(y) &= A|_R^{-1}(|A(x)| - |A(y)|) + z(x) - z(y) \\ &= A|_R^{-1}(|A(\pm x \pm y)|) + z(x) - z(y) \\ &= f(x \pm y) - z(x \pm y) + z(x) - z(y) \in S + Z = S \end{aligned}$$

(with an appropriate choice of signs). Therefore, the additivity of $z|_S$ and conditions (2.15), (2.17) imply $z(f(x) - f(y)) = z(x) - z(y)$. By virtue of equality (2.14), $2z(x) = z(x+y) + z(x-y)$. Representation (2.12) obviously implies that $z(2t) = 2z(t)$ for every $t \in \mathbb{R}$, hence the foregoing equality gives

$$z(x) = z\left(\frac{x+y}{2}\right) + z\left(\frac{x-y}{2}\right). \quad (2.18)$$

We are going to show that Eq. (2.18) (which holds if $|A(x)| \geq |A(y)|$) forces the functions $z|_D$ and $z|_{-D}$ to be additive.

Fix t, u which both belong to D or both to $-D$. Putting $x = t + u$ and $y = t - u$ we see that $|A(x)| \geq |A(y)|$, hence Eq. (2.18) yields $z(t + u) = z(t) + z(u)$. This means that $z|_D$ and $z|_{-D}$ are additive and particularly proves condition (i).

According to (2.16), (2.12), (2.17) and the additivity of $z|_{-D}$ we get that for every $x \in \mathbb{R}$

$$\begin{aligned} z(-x) &= z(-f(x)) = z(-A|_R^{-1}(|A(x)|) - z(x)) \\ &= z(-A|_R^{-1}(|A(x)|)) - z(x), \end{aligned}$$

which proves assertion (ii). This completes the proof of the “only if” part.

For the “if” part assume f is given by formula (1.6) with: an additive mapping $A: \mathbb{R} \rightarrow \mathbb{R}$, a group R such that $A|_R: R \rightarrow A(\mathbb{R})$ is an isomorphism, and a function $z: \mathbb{R} \rightarrow Z$, where $Z \subset \ker A$ is a group, which fulfill (i)-(iii). Calculation (2.13) shows that f satisfies Eq. (*) if and only if (2.14) holds true. In order to prove (2.14) fix $x, y \in \mathbb{R}$ and let us distinguish two possible cases.

Case 1. If $|A(x)| \geq |A(y)|$ then

$$\begin{aligned} f(x) - f(y) &= A|_R^{-1}(|A(x)|) - A|_R^{-1}(|A(y)|) + z(x) - z(y) \\ &= A|_R^{-1}(|A(\pm x \pm y)|) + z(x) - z(y). \end{aligned}$$

Therefore, in the light of Remark 1(iv), (vi) and condition (iii),

$$z(f(x) - f(y)) = z(A|_R^{-1}(|A(\pm x \pm y)|)) + z(z(x) - z(y)) = z(x) - z(y).$$

Thus, equality (2.14) reduces to $2z(x) = z(x + y) + z(x - y)$ or, equivalently,

$$z(x) = z\left(\frac{x+y}{2}\right) + z\left(\frac{x-y}{2}\right).$$

However, $\frac{x+y}{2}$ and $\frac{x-y}{2}$ either both belong to D or both to $-D$, hence the desired equality follows from condition (i) and Remark 1(iv).

Case 2. If $|A(x)| < |A(y)|$ then

$$A|_R^{-1}(|A(x)|) - A|_R^{-1}(|A(y)|) = A|_R^{-1}(-|A(\varepsilon_1 x - \varepsilon_2 y)|),$$

where $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$ are such that $\varepsilon_1 x \in D$ and $\varepsilon_2 y \in D$. Arguing as above and using Remark 1(v) we obtain

$$\begin{aligned} z(f(x) - f(y)) &= z(A|_R^{-1}(-|A(\varepsilon_1 x - \varepsilon_2 y)|)) + z(z(x) - z(y)) \\ &= z(\varepsilon_1 x - \varepsilon_2 y) + z(-\varepsilon_1 x + \varepsilon_2 y) + z(x) - z(y). \end{aligned}$$

If $\varepsilon_1 \varepsilon_2 = 1$, then equality (2.14) reduces to $z(y - x) + 2z(x) = z(x + y)$ or, equivalently,

$$z\left(\frac{y-x}{2}\right) + z(x) = z\left(\frac{x+y}{2}\right).$$

If $\varepsilon_1 \varepsilon_2 = -1$ then equality (2.14) reduces to $z(-x - y) + 2z(x) = z(x - y)$ or, equivalently,

$$z\left(\frac{-x-y}{2}\right) + z(x) = z\left(\frac{x-y}{2}\right).$$

It is easily verified that in the first case $\frac{y-x}{2}$ and x either both belong to D or both to $-D$, whereas in the second case the same is true for $\frac{-x-y}{2}$ and x . Consequently, the desired equality follows again from condition (i) and Remark 1(iv). □

3. A new proof of the theorem of Simon and Volkmann

We give here a new proof of the theorem of Simon and Volkmann [4] quoted above. In fact, instead of Eq. (2.1) we will be considering the following one (which was also introduced in [4]):

$$|f(x) - f(y)| = \min\{f(x+y), f(x-y)\} \quad (3.1)$$

under the additional assumption

$$f(2x) = 2f(x) \quad \text{for all } x \in G, \quad (3.2)$$

where G is a group. As it was communicated to the author by Professor Volkmann, such a system of equations is equivalent to the equation

$$f(x) + f(y) = \max\{f(x+y), f(x-y)\}. \quad (3.3)$$

Indeed, if f satisfies (3.3) then obviously (3.2) is valid and, moreover,

$$\begin{aligned} \min\{f(x+y), f(x-y)\} &= f(x+y) + f(x-y) - \max\{f(x+y), f(x-y)\} \\ &= \max\{f(2x), f(2y)\} - f(x) - f(y) \\ &= \max\{f(x), f(y)\} - \min\{f(x), f(y)\} \\ &= |f(x) - f(y)| \quad \text{for all } x, y \in G. \end{aligned}$$

In a similar manner we may prove that (3.1) and (3.2) imply (3.3).

Observe that Eq. (3.1) immediately forces f to be subadditive. Therefore, (3.2) automatically implies $f(nx) = nf(x)$ for all $x \in G$ and $n \in \mathbb{N}$. Moreover, $f^{-1}(\{0\})$ is equal to the set of periods of f . We will be using all these facts without explicit mentioning.

Theorem 2. *Let $(G, +)$ be an Abelian group. A function $f: G \rightarrow \mathbb{R}$ is a solution of Eq. (3.1) satisfying (3.2) if and only if there exists an additive function $a: G \rightarrow \mathbb{R}$ such that $f(x) = |a(x)|$ for all $x \in G$.*

Proof. The sufficiency of our assertion is easily verified, so assume that $f: G \rightarrow \mathbb{R}$ is a solution of Eq. (3.1) such that $f(nx) = nf(x)$ for all $x \in G$ and $n \in \mathbb{N}$. For transparency we divide the proof into several parts. In parts I–IV we additionally assume that G is an Abelian free group. In the last part we will show how to reduce the general case to that special one.

I. Let $\{h_t\}_{t \in T}$ be any basis of the free group G and denote

$$v_t = f(h_t) \quad \text{for } t \in T.$$

Let $\mathcal{C}$ be the family consisting of all sequences with the domain T and taking values in the set $\{-1, 1\}$ and let

$$U = \{t \in T : v_t \neq 0\}.$$

We introduce an equivalence relation $\sim$ in $\mathcal{C}$ given by the formula

$$(\delta_t)_{t \in T} \sim (\varepsilon_t)_{t \in T} \Leftrightarrow \begin{cases} \delta_t = \varepsilon_t & \text{for all } t \in U, \text{ or} \\ \text{the sequences } \delta|_U \text{ and } \varepsilon|_U \text{ are constant.} \end{cases}$$

The last condition is equivalent to the following one: if a_δ, a_ε are additive functions satisfying $a_\delta(h_t) = \delta_t v_t$ and $a_\varepsilon(h_t) = \varepsilon_t v_t$ for $t \in T$, then $|a_\delta| = |a_\varepsilon|$.

Take arbitrarily $t, u \in U$, $t \neq u$ (if any such pair (t, u) exists). Then

$$f(h_t + h_u) \neq f(h_t - h_u).$$

Indeed, suppose that it is not the case. Putting $x = h_t + h_u$, $y = h_t - h_u$ in Eq. (3.1) we then obtain either $v_t = 0$ or $v_u = 0$, which is a contradiction. Consequently, exactly one of the two values $f(h_t + h_u)$, $f(h_t - h_u)$ equals $|v_t - v_u|$. In the case where the first one does, we define $\sigma_{t,u} = -1$, if the latter one does, we define $\sigma_{t,u} = 1$. There exists exactly one (up to the equivalence $\sim$) sequence $(\varepsilon_t)_{t \in T} \in \mathcal{C}$ fulfilling

$$\varepsilon_t \varepsilon_u = \sigma_{t,u} \quad \text{for all } t, u \in U, t \neq u.$$

To see this it is enough to check the following transitivity condition: if $t, u, w \in U$ are pairwise different and $\sigma_{t,u} = 1 = \sigma_{u,w}$, then $\sigma_{t,w} = 1$. Indeed, by the definition of $\sigma_{t,u}$ and $\sigma_{u,w}$, we have

$$f(h_t - h_u) = |v_t - v_u|$$

and

$$f(h_u - h_w) = |v_u - v_w|.$$

Observe that independently of the values $|v_t - v_u|$ and $|v_u - v_w|$ we can either add or subtract these two equations reducing the term v_u on the right-hand side. As a result we get one of the two equalities:

$$f(h_t - h_u) + f(h_u - h_w) = |v_t - v_w|, \quad (3.4)$$

or

$$|f(h_t - h_u) - f(h_u - h_w)| = |v_t - v_w|. \quad (3.5)$$

Formula (3.4), jointly with the subadditivity of f , yields

$$f(h_t - h_w) \leq |f(h_t) - f(h_w)|.$$

However, Eq. (3.1) implies then that the above inequality is in fact an equality, i.e. $\sigma_{t,w} = 1$. In the case where formula (3.5) is valid, suppose $\sigma_{t,w} = -1$, which means that $f(h_t + h_w) = |v_t - v_w|$ and (3.5) yields

$$|f(h_t - h_u) - f(h_u - h_w)| = f(h_t + h_w).$$

Therefore, there are two possibilities:

$$f(h_t + h_w) = \pm(f(h_t - h_u) - f(h_u - h_w)).$$

The first one implies

$$f(h_t - h_u) = f(h_t + h_w) + f(h_u - h_w) \geq f(h_t + h_u)$$

and hence, by Eq. (3.1),

$$f(h_t + h_u) = |v_t - v_u|,$$

which contradicts $\sigma_{t,u} = 1$. The latter possibility similarly leads to contradiction with $\sigma_{u,w} = 1$.

We have assigned to our solution f the two sequences: $(v_t)_{t \in T}$ and $(\varepsilon_t)_{t \in T}$. Our present aim is to show that these two sequences completely determine all the values of the function f .

II. In the first step we will show that the values of f are determined for all arguments being linear combinations of exactly two elements from the basis H . Fix $t, u \in T$, $t \neq u$. We may assume that $t, u \in U$; otherwise at least one of the numbers h_t, h_u is a period of f , which obviously implies that all values $f(mh_t + nh_u)$ are determined for $m, n \in \mathbb{Z}$. Denote by $\mathcal{S}$ the family of all 2-homogeneous solutions of Eq. (3.1) to which we have assigned the same sequences $(v_t)_{t \in T}$, $(\varepsilon_t)_{t \in T}$ as those assigned to the solution f , and let

$$\begin{aligned} \mathcal{N}_{t,u} = \{ & (m, n) \in \mathbb{Z} \times \mathbb{Z} : \text{for every } g \in \mathcal{S} \text{ we have} \\ & f(mh_t + nh_u) = g(mh_t + nh_u)\}. \end{aligned}$$

We shall prove that

$$\mathbb{Z} \times \mathbb{Z} = \mathcal{N}_{t,u}. \quad (3.6)$$

First, observe that both pairs $(1, 1)$, $(1, -1)$ belong to $\mathcal{N}_{t,u}$. Indeed, in the case where $\sigma_{t,u} = 1$ we have

$$f(h_t - h_u) = |v_t - v_u|. \quad (3.7)$$

If, moreover, $v_t \geq v_u$ then

$$f(2h_t + h_u) \geq 2v_t - v_u \geq v_u,$$

thus the substitution $x \mapsto h_t + h_u$, $y \mapsto h_t$ into Eq. (3.1) yields

$$|f(h_t + h_u) - v_t| = v_u,$$

which implies either

$$f(h_t + h_u) = v_t + v_u, \quad (3.8)$$

or

$$f(h_t + h_u) = v_t - v_u.$$

However, the last equality and equality (3.7) give $f(h_t + h_u) = f(h_t - h_u)$, contrary to the assumption $t, u \in U$. Consequently, Eqs. (3.7) and (3.8) determine the values $f(h_t - h_u)$ and $f(h_t + h_u)$ uniquely in the case $v_t \geq v_u$. The opposite case may be treated similarly: we have

$$f(h_t + h_u) = |v_t - v_u| \quad (3.9)$$

and

$$f(h_t - 2h_u) \geq 2v_u - v_t \geq v_t,$$

which, in view of the substitution $x \mapsto h_t - h_u$, $y \mapsto h_u$, yields

$$|f(h_t - h_u) - v_u| = v_t.$$

Like in the previous case we exclude one of the possibilities and we obtain

$$f(h_t - h_u) = v_u - v_t.$$

Analogous arguments work in the case where $\sigma_{t,u} = -1$. Consequently,

$$(1, 1), (1, -1) \in \mathcal{N}_{t,u}. \quad (3.10)$$

Before proceeding to the next part of the proof of our claim (3.6), we shall distinguish two possibilities: $v_t \geq v_u$ and $v_u \geq v_t$.

Case 1 ($v_t \geq v_u$). We are going to prove that

$$\mathcal{N}^+ := \{(m, n) \in \mathbb{Z} \times \mathbb{Z} : |m| \geq |n|\} \subset \mathcal{N}_{t,u} \quad (3.11)$$

using an induction on $|m| - |n|$. By virtue of (3.10) and the homogeneity of f , every pair $(m, n) \in \mathcal{N}^+$ with $|m| = |n|$ belongs to $\mathcal{N}_{t,u}$. Now, fix $r \in \mathbb{N}$ and $(m, n) \in \mathcal{N}^+$ with $|m| - |n| = r$, and assume that

$$\{(m, n) \in \mathbb{Z} \times \mathbb{Z} : 0 \leq |m| - |n| < r\} \subset \mathcal{N}_{t,u}.$$

With no loss of generality we may suppose that $m \in \mathbb{N}$, $m \geq 2$.

First, consider the case where $n \in \mathbb{N}$ (and hence $1 \leq n < m = n + r$). For arbitrary $k, l \in \mathbb{Z}$, $k > l$, putting $x \mapsto kh_t + lh_u$, $y \mapsto h_t$ and $x \mapsto kh_t + lh_u$, $y \mapsto h_u$ in Eq. (3.1), we obtain

$$|f(kh_t + lh_u) - v_t| = \min\{f((k+1)h_t + lh_u), f((k-1)h_t + lh_u)\}$$

and

$$|f(kh_t + lh_u) - v_u| = \min\{f(kh_t + (l+1)h_u), f(kh_t + (l-1)h_u)\};$$

moreover,

$$f(kh_t + lh_u) \geq kv_t - lh_u \geq \max\{v_t, v_u\}.$$

Hence

$$\bigwedge_{\substack{k, l \in \mathbb{Z} \\ k > l}} \left\{ \begin{array}{l} (k+1, l), (k-1, l) \in \mathcal{N}_{t,u}, \text{ or} \\ (k, l+1), (k, l-1) \in \mathcal{N}_{t,u} \end{array} \right\} \Rightarrow (k, l) \in \mathcal{N}_{t,u}.$$

Applying this implication, the inductive hypothesis and 2-homogeneity we get:

(a) if m, n are even then

$$\left(\frac{m}{2}, \frac{n}{2}\right) \in \mathcal{N}_{t,u} \Rightarrow (m, n) \in \mathcal{N}_{t,u};$$

(b) if m is even, n is odd then

$$\begin{aligned} \left(\frac{m}{2}, \frac{n-1}{2}\right) \in \mathcal{N}_{t,u} &\Rightarrow (m, n-1) \in \mathcal{N}_{t,u}, \\ (m, n-1), (m, n+1) \in \mathcal{N}_{t,u} &\Rightarrow (m, n) \in \mathcal{N}_{t,u}; \end{aligned}$$

(c) if m is odd, n is even then

$$\begin{aligned} \left(\frac{m+1}{2}, \frac{n}{2}\right) \in \mathcal{N}_{t,u} &\Rightarrow (m+1, n) \in \mathcal{N}_{t,u}, \\ (m-1, n), (m+1, n) \in \mathcal{N}_{t,u} &\Rightarrow (m, n) \in \mathcal{N}_{t,u}; \end{aligned}$$

(d) if m, n are odd then

$$\begin{aligned} \left(\frac{m+1}{2}, \frac{n+1}{2}\right), \left(\frac{m+1}{2}, \frac{n-1}{2}\right) &\in \mathcal{N}_{t,u} \\ \Rightarrow (m+1, n-1), (m+1, n+1) &\in \mathcal{N}_{t,u} \\ \Rightarrow (m+1, n) \in \mathcal{N}_{t,u}, \\ (m-1, n), (m+1, n) \in \mathcal{N}_{t,u} &\Rightarrow (m, n) \in \mathcal{N}_{t,u}. \end{aligned}$$

The case where $n < 0$ may be treated as follows. We put $x \mapsto mh_t + nh_u$, $y \mapsto mh_t - nh_u$ and $x \mapsto mh_t$, $y \mapsto nh_u$ in Eq. (3.1) obtaining

$$|f(mh_t + nh_u) - f(mh_t - nh_u)| = \min\{2mv_t, 2nv_u\}$$

and

$$\min\{f(mh_t + nh_u), f(mh_t - nh_u)\} = |mv_t - nv_u|.$$

These two equations determine the value $f(mh_t + nh_u)$, since we already know the values: $f(mh_t - nh_u)$, v_t , v_u . This completes the proof of (3.11).

It remains to show that

$$\mathcal{N}^- := \{(m, n) \in \mathbb{Z} \times \mathbb{Z} : |m| \leq |n|\} \subset \mathcal{N}_{t,u}. \quad (3.12)$$

Fix any pair $(m, n) \in \mathcal{N}^-$; we may assume that $m \neq 0$. Choose $k \in \mathbb{N}$ such that $(k-1)|m| > |n|$ and substitute $x \mapsto mh_t + nh_u$, $y \mapsto kmh_t$ in Eq. (3.1). We get

$$|f(mh_t + nh_u) - k|m|v_t| = \min\{f((1+k)mh_t + nh_u), f((1-k)mh_t + nh_u)\}. \quad (3.13)$$

The two values on the right-hand side are already determined. Moreover,

$$f(mh_t + nh_u) \leq |m|v_t + |n|v_u \leq k|m|v_t$$

and, consequently, Eq. (3.13) determines the value $f(mh_t + nh_u)$, i.e. $(m, n) \in \mathcal{N}_{t,u}$.

Case 2 ($v_u \geq v_t$). We argue analogously to the previous case; we start with showing (3.12) and then deduce (3.11) from (3.12).

Summarizing this part of the proof we may write

$$\mathbb{Z} \times \mathbb{Z} = \mathcal{N}^+ \cup \mathcal{N}^- \subset \mathcal{N}_{t,u},$$

which gives (3.6).

III. Now we will show that the value $f(x+y+z)$ is determined provided we know the values of f on every sum of not more than two terms from $\{x, y, z\}$. In the light of the preceding part of the proof, it would imply that all the values of f are completely determined by the sequences $(v_t)_{t \in T}$ and $(\varepsilon_t)_{t \in T}$.

Applying Eq. (3.1) repeatedly we get

$$\begin{aligned} \min\{f(x+y+z), f(x+y-z)\} &= |f(x+y) - f(z)|, \\ \min\{f(x+y+z), f(x-y+z)\} &= |f(x+z) - f(y)|, \\ \min\{f(x+y+z), f(x-y-z)\} &= |f(y+z) - f(x)|. \end{aligned} \quad (3.14)$$

Let $a \geq b \geq c \geq d$ be all the values: $f(x+y+z)$, $f(x+y-z)$, $f(x-y+z)$, $f(x-y-z)$. We may assume that they are pairwise different, since otherwise one of the sums of not more than two terms from $\{x, y, z\}$ would be a period of f , which automatically determines $f(x+y+z)$.

If the three expressions on the right-hand sides of Eqs. (3.14) occur not to be pairwise different, then the repeating one has to be equal to $f(x+y+z)$. In the opposite case we have

$$f(x+y+z) \in \{a, b\}$$

and at least one of the numbers c, d appears in one of the right-hand sides. Then, taking the equation with the least right-hand side, we get

$$\min\{f(x+y+z), f(x \pm y \pm z)\} \in \{c, d\},$$

which determines the value $e := f(x \pm y \pm z)$, where the sequence of signs is one of: $(+, -)$, $(-, +)$, $(-, -)$. Applying Eq. (3.1) once more we obtain

$$|f(x+y+z) - e| = \min\{2f(x'), 2f(x'')\}, \quad (3.15)$$

where each of x' , x'' is a sum of not more than two terms from $\{x, y, z\}$. However,

$$\{a, b\} \ni f(x+y+z) \geq e \in \{c, d\},$$

thus formula (3.15) determines $f(x+y+z)$.

IV. Let $a: G \rightarrow \mathbb{R}$ be the additive function satisfying $a(h_t) = \varepsilon_t v_t$ for $t \in T$. Then $|a|$ is a solution of our equation to which we assign the same sequence $(v_t)_{t \in T}$, $(\varepsilon_t)_{t \in T}$ as those assigned to f . By virtue of parts II and III, it has to be $f = |a|$.

V. Now, assume that G is an arbitrary Abelian group (not necessarily free). Then G is an image of a certain Abelian free group H through a homomorphism $\varphi: H \rightarrow G$. By the isomorphism theorem, there is an isomorphism

$\Phi: H/\ker\varphi \rightarrow G$, such that $\Phi \circ \kappa = \varphi$, where $\kappa: H \rightarrow H/\ker\varphi$ is the canonical projection. Let $\tilde{f}: H \rightarrow \mathbb{R}$ be given as $\tilde{f} = f \circ \varphi$. The function $\tilde{f}$, being a 2-homogenous solution of the equation considered, which acts on an Abelian free group, admits the representation $\tilde{f} = |\tilde{a}|$ for some additive function $\tilde{a}: H \rightarrow \mathbb{R}$. Define a mapping $a: G \rightarrow \mathbb{R}$ by the formula

$$a(\Phi(x + \ker\varphi)) = \tilde{a}(x).$$

It is easy to see that $\tilde{a}(x)$ does not depend on the specific member of the coset $x + \ker\varphi$. Moreover, a is additive and for every $y \in G$, $y = \Phi(x + \ker\varphi)$ for some $x \in H$, we have

$$f(y) = f(\Phi \circ \kappa)(x) = (f \circ \varphi)(x) = \tilde{f}(x) = |\tilde{a}(x)| = |a(y)|,$$

which completes the proof. $\square$

4. Concluding remarks

Now, we are going to deduce some corollaries from our main result. The first one requires the following theorem due to Maksa and Rätz [2] and extended by Sablik (cf. [3, Proposition 3.2]). Namely, if G is a group and $a, b: G \rightarrow \mathbb{R}$ are additive mappings satisfying $a^{-1}((0, \infty)) = b^{-1}((0, \infty))$, then there is a positive number γ such that $a = \gamma b$.

Corollary 1. *Assume $f: \mathbb{R} \rightarrow [0, \infty)$ is a function satisfying (H_1) . Then f is a solution of Eq. (*) if and only if there exists an additive function $A: \mathbb{R} \rightarrow \mathbb{R}$ such that $A \circ A = A$ and*

$$f(x) = |A(x)| \quad \text{for all } x \in \mathbb{R}. \quad (4.1)$$

Proof. It is easily verified that every function of the form (4.1) fulfills (*). Assuming now that f is a non-negative solution of (*) fulfilling (H_1) we infer, in view of Remark 2, that (H_2) holds true. Thus, Theorem 1 gives (1.6) with appropriate functions A and $z: \mathbb{R} \rightarrow Z$. However, since $S \subset [0, \infty)$, condition (1.3) implies $Z = \{0\}$ and consequently z vanishes. Therefore

$$f(x) = A_R^{-1}(|A(x)|) \quad \text{for all } x \in \mathbb{R}. \quad (4.2)$$

Now, consider the group $A(\mathbb{R})$ and the function $A|_R^{-1}: A(\mathbb{R}) \rightarrow \mathbb{R}$. This is an additive mapping which takes positive values exactly on the set $A(\mathbb{R}) \cap (0, \infty)$. The same is trivially true for the identity function on $A(\mathbb{R})$. Thus, by the above mentioned result of Maksa, Rätz and Sablik, there is a positive number γ such that $A|_R^{-1}(A(x)) = \gamma A(x)$ for all $x \in \mathbb{R}$. Hence, obviously $A|_R^{-1}(|A(x)|) = \gamma|A(x)|$ and, by formula (4.2), $f(x) = \lambda|A(x)|$ for all $x \in \mathbb{R}$. Putting this into Eq. (*) immediately gives $\lambda = 1$, thus (4.1) has been proved. $\square$

Corollary 2. Assume $f: \mathbb{R} \rightarrow (-\infty, 0]$ is a function satisfying (H_1) . Then f is a solution of Eq. $(*)$ if and only if there exists an additive function $A: \mathbb{R} \rightarrow \mathbb{R}$, such that $A \circ A = A$ and

$$f(x) = -|A(x)| \quad \text{for all } x \in \mathbb{R}.$$

Proof. It is enough to consider, instead of $f(x)$, the function $-f(-x)$ and to apply Corollary 1. $\square$

Corollary 3. Assume $f: \mathbb{R} \rightarrow \mathbb{R}$ is an even function satisfying (H_1) and (H_2) . Then f is a solution of Eq. $(*)$ if and only if there exist: an additive function $A: \mathbb{R} \rightarrow \mathbb{R}$ and a subgroup R of $\mathbb{R}$, such that $A|_R: R \rightarrow A(\mathbb{R})$ is an isomorphism, such that

$$f(x) = A|_R^{-1}(|A(x)|) \quad \text{for all } x \in \mathbb{R}. \quad (4.3)$$

Proof. If f is an even function, then (1.6) implies that z is even as well. Making use of condition (iii) for every $x \in Z$ we obtain $x = z(x) = z(-x) = -x$, which proves that $Z = \{0\}$. Consequently, z vanishes and formula (1.6) reduces to (4.3). $\square$

Even under additional assumptions (H_1) and (H_2) the description of solutions of Eq. $(*)$ is rather complicated and we have a variety of them. What is more, the general situation is even worse (or better—depends on the point of view). Namely, if $T \subset \mathbb{R}$ is an arbitrary semigroup satisfying:

$$(T_1) \quad T \cup (-T) = T - T,$$

$$(T_2) \quad T \cap (-T) = \{0\},$$

then the function $\chi_T: T \cup (-T) \rightarrow T \cup (-T)$ defined by

$$\chi_T(x) = \begin{cases} x, & \text{if } x \in T, \\ -x, & \text{if } x \in -T, \end{cases}$$

satisfies the equation considered, i.e.

$$\begin{aligned} \chi_T(\chi_T(x) - \chi_T(y)) &= \chi_T(x + y) + \chi_T(x - y) - \chi_T(x) - \chi_T(y) \\ &\text{for all } x, y \in T \cup (-T). \end{aligned}$$

In particular, taking a Hamel basis $\{h_\alpha\}_{\alpha \in A}$ with a well-ordered set $(A, \leq)$, we may define

$$T = \left\{ x \in \mathbb{R} : x = \sum_{i=1}^n \lambda_i h_{\alpha_i}, \lambda_i \in \mathbb{Q}, \alpha_1 \leq \dots \leq \alpha_n \text{ and } \lambda_n > 0 \right\}.$$

Then χ_T yields a solution of the above equation, whereas condition (H_2) does not hold true.

We can make the above example even a bit more complicated by choosing an arbitrary additive function $A: \mathbb{R} \rightarrow \mathbb{R}$ and a subgroup R of $\mathbb{R}$ such that $A|_R: R \rightarrow A(\mathbb{R})$ is an isomorphism. If a semigroup $T \subset \mathbb{R}$ satisfies conditions

(T_1) , (T_2) and $A(\mathbb{R}) \subset T \cup (-T)$, then the composition $f = A|_R^{-1} \circ \chi_T \circ A$ fulfills Eq. (*).

Taught by the examples above, the author does not attempt to give any conjecture about the general form of solutions $f: \mathbb{R} \rightarrow \mathbb{R}$ of Eq. (*) with no additional assumptions. Nevertheless, one conjecture seems to be reasonable. Namely, for every solution which is known to the author, assumption (H_1) occurs to be true. It seems probable that this is a general rule.

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