On uniformly continuous Nemytskij operators generated by set-valued functions

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Abstract. Let $I = [0, 1]$, let Y be a real normed linear space, C a convex cone in Y and Z a real Banach space. Denote by $clb(Z)$ the set of all nonempty, convex, closed and bounded subsets of Z . If a superposition operator N generated by a set-valued function $F: I \times C \rightarrow clb(Z)$ maps the set $H_\alpha(I, C)$ of all Hölder functions $\varphi: I \rightarrow C$ into the set $H_\beta(I, clb(Z))$ of all Hölder set-valued functions $\phi: I \rightarrow clb(Z)$ and is uniformly continuous, then

$$F(x, y) = A(x, y) \overset{*}{+} B(x), \quad x \in I, y \in C$$

for some set-valued functions A, B such that $A(\cdot, y), B \in H_\beta(I, clb(Z)), y \in C$ and $A(x, \cdot), x \in I$ are $\overset{*}{+}$ -additive and continuous on C into $clb(Z)$. A converse result is also investigated.

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1. Introduction

Let $I, J \subset \mathbb{R}$ be intervals, $\alpha \in (0, 1]$, $x_0 \in I$ and let $Lip^\alpha(I, J)$ denote the set of all functions $\varphi: I \rightarrow J$ for which the set

$$\left\{ \frac{|\varphi(x) - \varphi(y)|}{|x - y|^\alpha} : x, y \in I, x \neq y \right\}$$

is bounded with the functional

$$\|\varphi\|_{Lip^\alpha} := |\varphi(x_0)| + \sup_{x, y \in I, x \neq y} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^\alpha}. \quad (1)$$

Each function $f: I \times J \rightarrow \mathbb{R}$ generates an operator of substitution (or Nemytskij operator) N defined on $Lip^\alpha(I, J)$ by the formula

$$(N\varphi)(x) = f(x, \varphi(x)), \quad \varphi \in Lip(I, J), x \in I.$$

Recently Matkowski has shown (cf. [6]) that if a superposition operator N of a generator f maps the set $Lip^\alpha(I, J)$ into the Banach space $Lip^\alpha(I, \mathbb{R})$ and is uniformly continuous with respect to the norm (1), then

$$f(x, y) = a(x)y + b(x), \quad x \in I, y \in J$$

for some $a, b \in Lip^\alpha(I, \mathbb{R})$. In this paper we are going to prove a counterpart of Matkowski's result for Nemytskij operators generated by set-valued functions with values in a set $clb(Z)$ of all nonempty, bounded, closed, convex subsets of a normed linear space Z .

The problem of characterizing Lipschitzian operators of a substitution generated by set-valued functions with nonempty, compact, convex values, mapping the set of functions satisfying the Hölder condition into the set of such set-valued functions was investigated by Ludew (cf. [4]). Lipschitzian Nemytskij operators generated by set-valued functions with values in $clb(Z)$ acting in the space of functions satisfying the Lipschitz condition were studied in [9].

Let $(Z, \|\cdot\|)$ be a real, normed linear space. For a bounded $A \subset Z$ one can define a number $\|A\|$ as follows $\|A\| := \sup\{\|z\| : z \in A\}$.

We introduce a binary operation $+$ in $clb(Z)$ by the formula

$$A +^* B = \text{cl}(A + B),$$

where $A + B$ is an algebraic sum of A and B and $\text{cl}A$ is the closure of A . Observe, that for arbitrary $A, B \in clb(Z)$ the set $A + B$ does not have to be closed. A corresponding example can be found e.g. in [10]. The pair $(clb(Z), +^*)$ is an Abelian semigroup with the set $\{0\}$ as the zero element. We can multiply elements of $clb(Z)$ by nonnegative numbers and the conditions

$$1 \cdot A = A, \quad \lambda(\mu A) = (\lambda\mu)A, \quad \lambda(A +^* B) = \lambda A +^* \lambda B, \quad (\lambda + \mu)A = \lambda A +^* \mu A$$

hold for all $A, B \in clb(Z)$ and $\lambda, \mu \geq 0$. This means that the set $clb(Z)$ with operations $+$ and $\cdot$ is an *abstract convex cone*. The cancellation law, i.e.

$$A +^* B = C +^* B \implies A = C$$

in $clb(Z)$ follows e.g. from Theorem II-17 in [2, p. 48].

For two nonempty subsets A, B of Z and $z \in Z$ define

$$\rho(z, B) = \inf\{\|z - b\| : b \in B\}, \quad e(A, B) = \sup\{\rho(a, B) : a \in A\}.$$

The Hausdorff distance $d(A, B)$ between A and B is defined by

$$d(A, B) = \max\{e(A, B), e(B, A)\}.$$

If S denotes the closed unit ball in Z , then

$$d(A, B) = \inf\{t > 0 : A \subset B + tS, B \subset A + tS\}$$

for all nonempty and bounded sets A and B . It is easily seen that $(clb(Z), d)$ is a metric space. It is complete, provided Z is a Banach space (cf. e.g. [2, p. 40]).

It is easy to check that the Hausdorff distance is invariant with respect to translation, i.e.,

$$d(A \overset{*}{+} B, C \overset{*}{+} B) = d(A + B, C + B) = d(A, C)$$

(cf. e.g. [3]) and

$$d(\lambda A, \lambda B) = |\lambda|d(A, B)$$

for all $\lambda \geq 0$ and $A, B, C \in clb(Z)$.

A subset C of a real linear space Y is said to be a *convex cone* if $\lambda C \subset C$ for all $\lambda \geq 0$ and $C + C \subset C$.

A set-valued function $F: C \rightarrow clb(Z)$ defined on a convex cone C is **additive* (**Jensen*) if

$$F(x + y) = F(x) \overset{*}{+} F(y) \quad \left(F\left(\frac{x + y}{2}\right) = \frac{1}{2}(F(x) \overset{*}{+} F(y)) \right)$$

for all $x, y \in C$. A function F is $\mathbb{Q}_+$ -homogeneous if $F(\lambda y) = \lambda F(y)$ for all $\lambda \in \mathbb{Q} \cap [0, \infty)$ and $y \in C$.

The following lemmas will be needed in the paper.

Lemma 1. (Corollary 4 in [9]) *Let C be a convex cone in a real linear space Y and let Z be a Banach space. A set-valued function $F: C \rightarrow clb(Z)$ is *Jensen if and only if there exist an *additive set-valued function $A: C \rightarrow clb(Z)$ and a set $B \in clb(Z)$ such that*

$$F(x) = A(x) \overset{*}{+} B$$

for $x \in C$.

Lemma 2. (Lemma 2 in [7]) *Let Y, Z be two real, normed linear spaces and let C be a convex cone in Y . Suppose F is a $\mathbb{Q}_+$ -homogeneous set-valued function defined on C with nonempty values in Z . The equality*

$$\lim_{y \rightarrow 0, y \in C} \|F(y)\| = 0 \tag{2}$$

holds if and only if there exists a positive constant M such that

$$\|F(y)\| \leq M\|y\| \quad \text{for } y \in C.$$

In the set of all $\mathbb{Q}_+$ -homogeneous set-valued functions in C with nonempty values in Z , satisfying (2) we can introduce the functional

$$\|F\| = \sup_{x \in C, x \neq 0} \frac{\|F(x)\|}{\|x\|}. \tag{3}$$

By Lemma 2, $\|F\|$ is finite. This functional will be called a norm.

Lemma 3. (Lemma 4 and Remark 2 in [7]) *Let Y be a Banach space, Z a real, normed linear space and let C be a convex cone in Y . Suppose that $(F_j : j \in J)$ is a family of $\ast$ -additive, continuous set-valued functions $F_j : C \rightarrow clb(Z)$. If $\text{int}C \neq \emptyset$ and for each $y \in C$ the set $\bigcup_{j \in J} F_j(y)$ is bounded in Z , then there exists a constant $M \in (0, +\infty)$ such that*

$$\sup_{j \in J} \|F_j\| \leq M.$$

A function $\alpha : [0, \infty) \rightarrow [0, \infty)$ is said to be an α -function, if $\alpha(t) > 0$ for $t \in (0, \infty)$, $\alpha(0) = 0 = \lim_{t \rightarrow 0} \alpha(t)$, $\alpha(1) = 1$ and both α and α^* , where

$$\alpha^*(t) = \begin{cases} \frac{t}{\alpha(t)} & \text{for } t \in (0, \infty), \\ 0 & \text{for } t = 0, \end{cases}$$

are increasing (cf. [1, p. 182]).

Note that the function $\alpha(t) = t^p$, where $p \in (0, 1]$ is an α -function.

For two α -functions α and β we write

$$\alpha \prec \beta \iff \alpha(t) = O(\beta(t)) \quad \text{as } t \rightarrow 0.$$

Let α be an α -function, $I = [0, 1]$ and let C be a convex cone in a real, normed linear space Y . By definition, the set $H_\alpha(I, C)$ consists of all functions $\varphi : I \rightarrow C$ such that

$$h_\alpha(\varphi) := \sup_{s \in (0, 1]} \frac{\omega(\varphi, s)}{\alpha(s)} < +\infty, \quad (4)$$

where

$$\omega(\varphi, s) := \sup\{\|\varphi(x_1) - \varphi(x_2)\| : x_1, x_2 \in I, |x_1 - x_2| \leq s\}$$

(cf. [5]). By $H_\alpha(I, clb(Z))$ we denote the set of all set-valued functions $\phi : I \rightarrow clb(Z)$ such that $h_\alpha(\phi) < +\infty$, where

$$\omega(\phi, s) := \sup\{d(\phi(x_1), \phi(x_2)) : x_1, x_2 \in I, |x_1 - x_2| \leq s\}.$$

Observe that all functions from $H_\alpha(I, C)$ and from $H_\alpha(I, clb(Z))$ are continuous. In fact, let us fix $x_1, x_2 \in I$ and let $\varphi \in H_\alpha(I, C)$. We have

$$\|\varphi(x_1) - \varphi(x_2)\| \leq \omega(\varphi, |x_1 - x_2|) \leq h_\alpha(\varphi)\alpha(|x_1 - x_2|). \quad (5)$$

Since α is continuous at 0, (5) shows that φ is uniformly continuous, by (4). The same reasoning applies to $\phi \in H_\alpha(I, clb(Z))$.

We introduce a metric ρ_α in the set $H_\alpha(I, C)$ putting $\rho_\alpha(\varphi, \bar{\varphi}) = \|\varphi - \bar{\varphi}\|_\alpha$, where

$$\|\varphi\|_\alpha := \|\varphi(0)\| + h_\alpha(\varphi).$$

In the set $H_\alpha(I, clb(Z))$ one can define a metric setting

$$d_\alpha(\phi, \bar{\phi}) := d(\phi(0), \bar{\phi}(0)) + \sup_{s \in (0, 1]} \frac{\omega(\phi, \bar{\phi}, s)}{\alpha(s)}, \quad \phi, \bar{\phi} \in H_\alpha(I, clb(Z)),$$

where

$$\omega(\phi, \bar{\phi}, s) := \sup\{d(\phi(x_1) + \bar{\phi}(x_2), \phi(x_2) + \bar{\phi}(x_1)) : x_1, x_2 \in I, |x_1 - x_2| \leq s\}$$

(cf. [8]). It may be proved that $d_\alpha(\phi, \bar{\phi}) < \infty$ and that d_α is a metric in $H_\alpha(I, clb(Z))$ (cf. [4]).

Consider the set

$$\mathcal{L}(C, clb(Z)) := \{A: C \rightarrow clb(Z) : A \text{ is } * \text{ additive and continuous}\}.$$

Since every $*$ additive set-valued function $A: C \rightarrow clb(Z)$ is $\mathbb{Q}_+$ -homogeneous, for each $A \in \mathcal{L}(C, clb(Z))$ we have

$$\|A(y)\| \leq \|A\| \|y\|, \quad y \in C,$$

where $\|A\|$ is defined by (3). Thus, for $A, B \in \mathcal{L}(C, clb(Z))$ we have $d(A(y), B(y)) \leq \|A(y)\| + \|B(y)\| \leq (\|A\| + \|B\|)\|y\|$ and

$$d_{\mathcal{L}}(A, B) := \sup_{y \in C, y \neq 0} \frac{d(A(y), B(y))}{\|y\|}$$

is finite. It is easily seen, that $d_{\mathcal{L}}$ yields a metric in $\mathcal{L}(C, clb(Z))$.

2. Main results

Let α, β be α -functions. We shall prove (in Theorem 2) that a uniformly continuous operator of a substitution N mapping $H_\alpha(I, C)$ into $H_\beta(I, clb(Z))$ has to be generated by a function $F: I \times C \rightarrow clb(Z)$ of the form

$$F(x, y) = A(x, y) \overset{*}{+} B(x),$$

where $A(x, \cdot)$ is a $*$ additive continuous set-valued function and $A(\cdot, y), B(\cdot)$ belong to $H_\beta(I, clb(Z))$.

Theorem 1. *Let $I = [0, 1]$ and Y be a real normed linear space, Z a Banach space and let C be a convex cone in Y . Assume that $\gamma: [0, \infty) \rightarrow [0, \infty)$ is continuous at 0, $\gamma(0) = 0$, and the superposition operator N is generated by a set-valued function $F: I \times C \rightarrow clb(Z)$.*

(a) *Suppose that N maps $H_\alpha(I, C)$ into $H_\beta(I, clb(Z))$ and*

$$d_\beta(N\varphi, N\bar{\varphi}) \leq \gamma(\|\varphi - \bar{\varphi}\|_\alpha), \quad \varphi, \bar{\varphi} \in H_\alpha(I, C). \quad (6)$$

Then there exist functions $A: I \times C \rightarrow clb(Z)$ and $B: I \rightarrow clb(Z)$ such that $A(\cdot, y), B \in H_\beta(I, clb(Z))$ for every $y \in C$, $A(x, \cdot) \in \mathcal{L}(C, clb(Z))$ for every $x \in I$ and

$$F(x, y) = A(x, y) \overset{*}{+} B(x), \quad x \in I, y \in C.$$

Moreover, the inequality

$$d(A(x, y_1) + A(\bar{x}, y_2), A(\bar{x}, y_1) + A(x, y_2)) \leq \gamma(\|y_1 - y_2\|)\beta(\|\bar{x} - x\|) \quad (7)$$

holds for all $x, \bar{x} \in I$ and $y_1, y_2 \in C$.

- (b) Assume that γ is increasing and the condition $\frac{1}{\beta} \prec \gamma(\frac{1}{\alpha})$ does not hold. Then the operator N maps $H_\alpha(I, C)$ into $H_\beta(I, \text{clb}(Z))$ and satisfies inequality (6) if and only if the function F is of the form

$$F(x, y) = B(x), \quad x \in I, \quad y \in C,$$

where $B \in H_\beta(I, \text{clb}(Z))$. In this case N is a constant operator.

Proof. (a) Note, that for a given $y \in C$ a constant function $\varphi(t) = y, t \in I$, belongs to the space $H_\alpha(I, C)$. Since N maps $H_\alpha(I, C)$ into $H_\beta(I, \text{clb}(Z))$, we have $N\varphi = F(\cdot, y) \in H_\beta(I, \text{clb}(Z))$. Consequently, $F(\cdot, y)$ is continuous.

For arbitrarily fixed $y, \bar{y} \in C$, take $\varphi, \bar{\varphi}: I \rightarrow C$ defined by

$$\varphi(t) = y, \quad \bar{\varphi}(t) = \bar{y}, \quad t \in I.$$

Then $\varphi, \bar{\varphi} \in H_\alpha(I, C)$ and, by the assumption, functions $N\varphi = F(\cdot, y)$, $N\bar{\varphi} = F(\cdot, \bar{y})$ belong to $H_\beta(I, \text{clb}(Z))$ and

$$\|\varphi - \bar{\varphi}\|_\alpha = \|y - \bar{y}\|.$$

From the definition of the metric d_β we get

$$d(N\varphi(0), N\bar{\varphi}(0)) + \frac{\omega(N\varphi, N\bar{\varphi}, 1)}{\beta(1)} \leq d_\beta(N\varphi, N\bar{\varphi}).$$

Hence, by (6), for all $x \in I$

$$d(F(0, y), F(0, \bar{y})) + d(F(x, y) + F(0, \bar{y}), F(x, \bar{y}) + F(0, y)) \leq \gamma(\|y - \bar{y}\|). \quad (8)$$

Since

$$\begin{aligned} d(F(x, y), F(x, \bar{y})) &= d(F(x, y) + F(0, \bar{y}), F(x, \bar{y}) + F(0, \bar{y})) \\ &\leq d(F(x, y) + F(0, \bar{y}), F(x, \bar{y}) + F(0, y)) \\ &\quad + d(F(x, \bar{y}) + F(0, y), F(x, \bar{y}) + F(0, \bar{y})) \\ &= d(F(0, y), F(0, \bar{y})) + d(F(x, y) + F(0, \bar{y}), F(x, \bar{y}) + F(0, y)), \end{aligned}$$

(8) implies that

$$d(F(x, y), F(x, \bar{y})) \leq \gamma(\|y - \bar{y}\|) \quad \text{for } x \in I.$$

This inequality, the continuity of γ at 0 and the equality $\gamma(0) = 0$ show that F is continuous with respect to the second variable.

Let us fix $x, \bar{x} \in I, x < \bar{x}, y_1, y_2, \bar{y}_1, \bar{y}_2 \in C$ and define functions

$$\varphi_i(t) := \begin{cases} y_i & \text{for } 0 \leq t \leq x, \\ \frac{\bar{y}_i - y_i}{\bar{x} - x}(t - x) + y_i & \text{for } x < t < \bar{x}, \\ \bar{y}_i & \text{for } \bar{x} \leq t \leq 1 \end{cases}$$

for $i = 1, 2$. Obviously, $\varphi_i(I) \subseteq C$. We shall prove that $\varphi_i \in H_\alpha(I, C)$. It is easy to see that

$$\omega(\varphi_i, s) = \|\bar{y}_i - y_i\| \text{ for } \bar{x} - x \leq s \leq 1,$$

$$\omega(\varphi_i, s) = \frac{s}{\bar{x} - x} \|\bar{y}_i - y_i\| \text{ for } 0 \leq s \leq \bar{x} - x.$$

Since the function $t \mapsto \frac{t}{\alpha(t)}$ is increasing

$$\sup_{s \in (0, 1]} \frac{\omega(\varphi_i, s)}{\alpha(s)} = \frac{\|\bar{y}_i - y_i\|}{\alpha(\bar{x} - x)}.$$

Therefore, $\varphi_i \in H_\alpha(I, C)$ and $\|\varphi_i\|_\alpha = \|y_i\| + \frac{\|\bar{y}_i - y_i\|}{\alpha(\bar{x} - x)}$. In particular

$$\|\varphi_1 - \varphi_2\|_\alpha = \|y_1 - y_2\| + \frac{\|\bar{y}_1 - \bar{y}_2 - y_1 + y_2\|}{\alpha(\bar{x} - x)}. \quad (9)$$

From (6) and the definition of d_β

$$\frac{\omega(N\varphi_1, N\varphi_2, \bar{x} - x)}{\beta(\bar{x} - x)} \leq d_\beta(N\varphi_1, N\varphi_2) \leq \gamma(\|\varphi_1 - \varphi_2\|_\alpha)$$

and since $\varphi_i(x) = y_i$ and $\varphi_i(\bar{x}) = \bar{y}_i$,

$$d(F(x, y_1) + F(\bar{x}, \bar{y}_2), F(\bar{x}, \bar{y}_1) + F(x, y_2)) \leq \gamma(\|\varphi_1 - \varphi_2\|_\alpha) \beta(\bar{x} - x). \quad (10)$$

Taking arbitrary $u, v \in C$ and setting $y_1 = \bar{y}_2 = \frac{u+v}{2}, \bar{y}_1 = u, y_2 = v$ we obtain

$$\|\varphi_1 - \varphi_2\|_\alpha = \frac{\|u - v\|}{2}$$

and

$$d\left(F\left(x, \frac{u+v}{2}\right) + F\left(\bar{x}, \frac{u+v}{2}\right), F(\bar{x}, u) + F(x, v)\right) \leq \gamma\left(\frac{\|u - v\|}{2}\right) \beta(\bar{x} - x).$$

Letting $\bar{x}$ tend to x , since $\lim_{t \rightarrow 0} \beta(t) = 0$, from the continuity of F with respect to the first variable we get

$$d\left(2F\left(x, \frac{u+v}{2}\right), F(x, u) + F(x, v)\right) = 0,$$

i.e.

$$F\left(x, \frac{u+v}{2}\right) = \frac{1}{2}[F(x, u) + F(x, v)]$$

for all $x \in I$. This shows, that $F(x, \cdot)$ is * Jensen, which implies that there exist functions $A: I \times C \rightarrow clb(Z)$ and $B: I \rightarrow clb(Z)$ such that $A(x, \cdot)$ is * additive for $x \in I$ and

$$F(x, y) = A(x, y) \overset{*}{+} B(x), \quad x \in I, y \in C \quad (11)$$

(cf. Lemma 1).

To prove that $A(x, \cdot)(x \in I)$ is continuous let us fix $y, \bar{y} \in C$. We have

$$\begin{aligned} d(A(x, y), A(x, \bar{y})) &= d(A(x, y) \overset{*}{+} B(x), A(x, \bar{y}) \overset{*}{+} B(x)) \\ &= d(F(x, y), F(x, \bar{y})), \end{aligned}$$

therefore, the continuity of $F(x, \cdot)$ implies the continuity of $A(x, \cdot)$.

From the * additivity of $A(x, \cdot)$ we have $A(x, 0) = \{0\}$, whence

$$F(x, 0) = A(x, 0) \overset{*}{+} B(x) = B(x). \quad (12)$$

Since $F(\cdot, y) \in H_\beta(I, clb(Z))$ for all $y \in C$, (12) shows that $B \in H_\beta(I, clb(Z))$.

Now we shall prove that $A(\cdot, y) \in H_\beta(I, clb(Z))$ for every $y \in C$. Let us fix $s \in (0, 1]$, $x, \bar{x} \in I$ such that $|x - \bar{x}| \leq s$ and $y \in C$. Of course

$$\begin{aligned} d(A(x, y), A(\bar{x}, y)) &= d(A(x, y) + B(x), A(\bar{x}, y) + B(x)) \\ &\leq d(A(x, y) + B(x), A(\bar{x}, y) + B(\bar{x})) \\ &\quad + d(A(\bar{x}, y) + B(\bar{x}), A(\bar{x}, y) + B(x)) \\ &= d(F(x, y), F(\bar{x}, y)) + d(B(x), B(\bar{x})), \end{aligned}$$

whence

$$d(A(x, y), A(\bar{x}, y)) \leq \omega(F(\cdot, y), s) + \omega(B, s)$$

and

$$\frac{\omega(A(\cdot, y), s)}{\beta(s)} \leq h_\beta(F(\cdot, y)) + h_\beta(B).$$

The fact that $A(\cdot, y) \in H_\beta(I, clb(Z))$ for every $y \in C$ follows now from the inequality above.

Finally, to show (7) take $x, \bar{x} \in I$ such that $x < \bar{x}$ and $y_1, y_2 \in C$. Putting $\bar{y}_1 = y_1, \bar{y}_2 = y_2$ in (9) and (10) we obtain

$$d(F(x, y_1) + F(\bar{x}, y_2), F(\bar{x}, y_1) + F(x, y_2)) \leq \gamma(\|y_1 - y_2\|)\beta(\bar{x} - x).$$

Therefore

$$\begin{aligned} d(A(x, y_1) + B(x) + A(\bar{x}, y_2) + B(\bar{x}), A(\bar{x}, y_1) + B(\bar{x}) + A(x, y_2) + B(x)) \\ = d(A(x, y_1) + A(\bar{x}, y_2), A(\bar{x}, y_1) + A(x, y_2)) \leq \gamma(\|y_1 - y_2\|)\beta(\bar{x} - x). \end{aligned}$$

The obtained inequality

$$d(A(x, y_1) + A(\bar{x}, y_2), A(\bar{x}, y_1) + A(x, y_2)) \leq \gamma(\|y_1 - y_2\|)\beta(|\bar{x} - x|)$$

for all $y_1, y_2 \in C$ and $x, \bar{x} \in I$, $x < \bar{x}$ is also true in the case of $x \geq \bar{x}$, which completes the proof of part (a).

(b) It suffices to show necessity. Setting $y_1 = y_2$ in (9) and (10) we obtain

$$d(F(\bar{x}, \bar{y}_1), F(\bar{x}, \bar{y}_2)) \leq \gamma \left(\frac{\|\bar{y}_1 - \bar{y}_2\|}{\alpha(\bar{x} - x)} \right) \beta(\bar{x} - x)$$

for all $x, \bar{x} \in I$ such that $x < \bar{x}$ and for all $\bar{y}_1, \bar{y}_2 \in C$. In the case $\|\bar{y}_1 - \bar{y}_2\| \leq 1$ by the monotonicity of γ we have

$$d(F(\bar{x}, \bar{y}_1), F(\bar{x}, \bar{y}_2)) \leq \gamma \left(\frac{1}{\alpha(\bar{x} - x)} \right) \beta(\bar{x} - x). \quad (13)$$

Since the condition $\frac{1}{\gamma(\frac{1}{\alpha})} \prec \beta$ does not hold, we can find a sequence $(t_n), t_n \in (0, 1], t_n \rightarrow 0$, such that

$$\beta(t_n) \gamma \left(\frac{1}{\alpha(t_n)} \right) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (14)$$

Take $x \in [0, 1)$ and $\bar{x}_n := x + t_n$. Then $\bar{x}_n \in [0, 1]$ for a large enough n and $\bar{x}_n \rightarrow x$. Since $F(\cdot, y)$, $y \in C$ is continuous, from (13) and (14) we deduce that $F(x, \bar{y}_1) = F(x, \bar{y}_2)$, $x \in [0, 1]$ and $\bar{y}_1, \bar{y}_2 \in C$.

In the case $\|\bar{y}_1 - \bar{y}_2\| > 1$, fix n large enough to have $\frac{1}{n} \|\bar{y}_1 - \bar{y}_2\| \leq 1$. Setting $y^i = \bar{y}_1 + \frac{i}{n}(\bar{y}_2 - \bar{y}_1)$, $i = 0, 1, \dots, n-1$, we obtain $\|y^{i+1} - y^i\| \leq 1$. By the above, we get $F(x, y^i) = F(x, y^{i+1})$ for all $x \in I$ and $i = 0, 1, \dots, n-1$, whence $F(x, \bar{y}_1) = F(x, \bar{y}_2)$ for all $x \in I$ and $\bar{y}_1, \bar{y}_2 \in C$. Consequently, $F(x, y) = F(x, 0) =: B(x)$ for $x \in I$ and $y \in C$, which completes the proof. $\square$

Remark 1. Denote by $\mathcal{A}$ the set of all functions $\varphi \in H_\alpha(I, C)$ of the form

$$\varphi(t) := \begin{cases} y & \text{for } 0 \leq t \leq x, \\ \frac{\bar{y}-y}{\bar{x}-x}(t-x) + y & \text{for } x < t < \bar{x}, \\ \bar{y} & \text{for } \bar{x} \leq t \leq 1 \end{cases}$$

for some $x, \bar{x} \in I, x < \bar{x}, y, \bar{y} \in C$. From the argument used in the proof follows that Theorem 1 remains valid if inequality (6) is postulated only for all $\varphi, \bar{\varphi} \in \mathcal{A}$.

Remark 2. Assuming, that γ in Theorem 1 is increasing does not cause any loss of generality.

Indeed, for a given $\gamma: [0, \infty) \rightarrow [0, \infty)$ we can take $\gamma^*: [0, \infty) \rightarrow [0, \infty)$ defined by $\gamma^*(t) = \sup_{s \in [0, t]} \gamma(s)$.

Condition (6) in Theorem 1 can be replaced by the uniform continuity of N .

Theorem 2. Let Y be a real normed linear space, Z a Banach space and C a convex cone in Y . Suppose that the superposition operator N of the generator $F: I \times C \rightarrow \text{clb}(Z)$ maps $H_\alpha(I, C)$ into $H_\beta(I, \text{clb}(Z))$ and that N is uniformly continuous. Then there exist functions $A: I \times C \rightarrow \text{clb}(Z)$ and $B: I \rightarrow \text{clb}(Z)$

such that $A(\cdot, y), B \in H_\beta(I, clb(Z))$ for every $y \in C, A(x, \cdot) \in \mathcal{L}(C, clb(Z))$ for every $x \in I$ and

$$F(x, y) = A(x, y) \overset{*}{+} B(x), \quad x \in I, \quad y \in C.$$

Proof. Suppose that N is uniformly continuous. Then for every $\varepsilon > 0$ there is $\delta > 0$ such that for all $\varphi, \bar{\varphi} \in H_\alpha(I, C)$

$$\|\varphi - \bar{\varphi}\|_\alpha \leq \delta \implies d_\beta(N\varphi, N\bar{\varphi}) \leq \varepsilon.$$

Let $\gamma: [0, \infty) \rightarrow [0, \infty)$ be defined by the formula

$$\gamma(t) := \sup\{d_\beta(N\varphi, N\bar{\varphi}) : \|\varphi - \bar{\varphi}\|_\alpha \leq t\}, \quad t \geq 0.$$

The function γ is well defined. Indeed, fix $\delta > 0$ such that for all $\varphi, \bar{\varphi} \in H_\alpha(I, C)$

$$\|\varphi - \bar{\varphi}\|_\alpha \leq \delta \implies d_\beta(N\varphi, N\bar{\varphi}) \leq 1. \quad (15)$$

Hence, we have $\gamma(t) \leq 1$ for all $t \in [0, \delta]$. Take $t \geq 0, s \geq 0, t + s > 0$ and $\varphi, \bar{\varphi} \in H_\alpha(I, C)$ such that $\|\varphi - \bar{\varphi}\|_\alpha \leq t + s$. The function $\psi = \frac{t}{t+s}\varphi + \frac{s}{t+s}\bar{\varphi}$ also belongs to $H_\alpha(I, C)$ and

$$\|\varphi - \psi\|_\alpha = \frac{s}{t+s}\|\varphi - \bar{\varphi}\|_\alpha \leq s, \quad \|\psi - \bar{\varphi}\|_\alpha = \frac{t}{t+s}\|\varphi - \bar{\varphi}\|_\alpha \leq t.$$

Thus, by the definition of γ it follows that

$$d_\beta(N\varphi, N\bar{\varphi}) \leq d_\beta(N\varphi, N\psi) + d_\beta(N\psi, N\bar{\varphi}) \leq \gamma(s) + \gamma(t)$$

and consequently

$$\gamma(s+t) \leq \gamma(s) + \gamma(t).$$

In particular, $\gamma(2t) \leq 2\gamma(t)$, whence by induction we obtain

$$\gamma(nt) \leq n\gamma(t) \quad (16)$$

for all $n \in \mathbb{N}$ and $t \geq 0$. For a given $t \geq 0$ there exists a positive integer n such that $\frac{t}{n} < \delta$. From (15) and (16) it follows that

$$\gamma(t) = \gamma\left(n\frac{t}{n}\right) \leq n\gamma\left(\frac{t}{n}\right) \leq n < \infty.$$

Since N is uniformly continuous, γ is continuous at 0, $\gamma(0) = 0$ and obviously

$$d_\beta(N\varphi, N\bar{\varphi}) \leq \gamma(\|\varphi - \bar{\varphi}\|_\alpha), \quad \varphi, \bar{\varphi} \in H_\alpha(I, C),$$

thus the result is a consequence of Theorem 1. $\square$

Remark 3. If in Theorem 1, $\gamma(t) = Lt$ (for some constant $L \geq 0$) and the function F maps $I \times C$ into the space $cc(Z)$ of all nonempty, convex and compact subsets of Z , we can replace $\overset{*}{+}$ by the usual algebraic sum of two sets and we get the result obtained by Ludew in [4].

3. A converse result

The following result may be proved in the same way as Lemma 5 in [7].

Lemma 4. *Let Y and Z be two real, normed linear spaces and C a convex cone in Y with nonempty interior. Then there exists a positive constant M_0 such that for every continuous, $*$ additive, set-valued function $F: C \rightarrow clb(Z)$ the inequality*

$$d(F(x), F(y)) \leq M_0 \|F\| \|x - y\|, \quad x, y \in C$$

holds.

The following theorem is a converse of part (a) of Theorem 1.

Theorem 3. *Let Y be a Banach space, Z a real normed linear space, C a convex cone in Y with nonempty interior and let α, β be two α -functions such that $\alpha \prec \beta$. Assume that $A(\cdot, y), B \in H_\beta(I, clb(Z))$ for $y \in C$ and $A(x, \cdot) \in \mathcal{L}(C, clb(Z))$ for $x \in I$. Moreover, assume that for some increasing, continuous at 0 function $\gamma: [0, \infty) \rightarrow [0, \infty)$, such that $\gamma(0) = 0$, the inequality*

$$d(A(x, y_1) + A(\bar{x}, y_2), A(\bar{x}, y_1) + A(x, y_2)) \leq \gamma(\|y_1 - y_2\|) \beta(|\bar{x} - x|) \quad (17)$$

holds for all $x, \bar{x} \in I$ and $y_1, y_2 \in C$. If a set-valued function $F: I \times C \rightarrow clb(Z)$ is of the form

$$F(x, y) = A(x, y) + B(x), \quad x \in I, \quad y \in C,$$

then the operator of the substitution N generated by F maps the set $H_\alpha(I, C)$ into $H_\beta(I, clb(Z))$ and satisfies inequality (6) with a function γ_1 , where $\gamma_1(t) = c(t + \gamma(t))$, $t \in [0, \infty)$ and c is a constant.

Proof. First, we shall prove that for an arbitrary $y \in C$ the set $\bigcup_{x \in I} A(x, y)$ is bounded. Let $x \in I, y \in C$. We have

$$\begin{aligned} \|A(x, y)\| &= d(A(x, y), \{0\}) \leq d(A(x, y), A(0, y)) + d(A(0, y), \{0\}) \\ &= d(A(x, y), A(0, y)) + \|A(0, y)\|. \end{aligned}$$

Moreover, since $A(\cdot, y) \in H_\beta(I, clb(Z))$,

$$d(A(x, y), A(0, y)) \leq \omega(A(\cdot, y), 1) = \frac{\omega(A(\cdot, y), 1)}{\beta(1)} \leq h_\beta(A(\cdot, y)) < \infty.$$

Therefore

$$\|A(x, y)\| \leq h_\beta(A(\cdot, y)) + \|A(0, y)\|, \quad x \in I.$$

Since $\{A(x, \cdot)\}_{x \in I}$ is a family of $*$ additive and continuous functions, by Lemma 3 there exists a constant $M > 0$ such that

$$\sup_{x \in I} \|A(x, y)\| \leq M \|y\|, \quad y \in C.$$

Hence, and by Lemma 4, we deduce that

$$d(A(x, y), A(x, \bar{y})) \leq M_0 M \|y - \bar{y}\| \quad (18)$$

for all $x \in I$ and $y, \bar{y} \in C$.

We shall prove now that N maps $H_\alpha(I, C)$ into $H_\beta(I, clb(Z))$. Let $\varphi \in H_\alpha(I, C)$ and $x, \bar{x} \in I$. The inequality

$$d(A(x, y), A(\bar{x}, y)) \leq \gamma(\|y\|)\beta(|\bar{x} - x|) \quad (19)$$

is a consequence of (17). From (18) and (19) we get

$$\begin{aligned} d(N\varphi(x), N\varphi(\bar{x})) &= d(A(x, \varphi(x)) + B(x), A(\bar{x}, \varphi(\bar{x})) + B(\bar{x})) \\ &\leq d(A(x, \varphi(x)), A(\bar{x}, \varphi(\bar{x}))) + d(B(x), B(\bar{x})) \\ &\leq d(A(x, \varphi(x)), A(\bar{x}, \varphi(x))) \\ &\quad + d(A(\bar{x}, \varphi(x)), A(\bar{x}, \varphi(\bar{x}))) + d(B(x), B(\bar{x})) \\ &\leq \gamma(\|\varphi(x)\|)\beta(|\bar{x} - x|) + M_0 M \|\varphi(x) - \varphi(\bar{x})\| + d(B(x), B(\bar{x})) \end{aligned}$$

for all $x, \bar{x} \in I$. Since

$$\|\varphi(x)\| \leq \|\varphi(0)\| + \frac{\|\varphi(x) - \varphi(0)\|}{\alpha(x - 0)} \alpha(x - 0), \quad x \in (0, 1]$$

we have $\|\varphi(x)\| \leq \|\varphi\|_\alpha$ for every $x \in I$. Now let us take $s \in (0, 1]$ and $x, \bar{x} \in I$ such that $|x - \bar{x}| \leq s$. Due to the monotonicity of γ and β

$$d(N\varphi(x), N\varphi(\bar{x})) \leq \gamma(\|\varphi\|_\alpha)\beta(s) + M_0 M \omega(\varphi, s) + \omega(B, s).$$

Thus, for every $s \in (0, 1]$ we obtain

$$\begin{aligned} \frac{\omega(N\varphi, s)}{\beta(s)} &\leq \gamma(\|\varphi\|_\alpha) + M_0 M \frac{\omega(\varphi, s)}{\alpha(s)} \frac{\alpha(s)}{\beta(s)} + \frac{\omega(B, s)}{\beta(s)} \\ &\leq \gamma(\|\varphi\|_\alpha) + L M_0 M h_\alpha(\varphi) + h_\beta(B), \end{aligned}$$

where $L > 1$ is some constant such that $\frac{\alpha(s)}{\beta(s)} \leq L, s \in (0, 1]$ (by the assumption $\alpha \prec \beta$). Therefore, $h_\beta(N\varphi) < \infty$ and $N\varphi \in H_\beta(I, clb(Z))$.

It remains to show that N satisfies (6). Let $\varphi, \bar{\varphi} \in H_\alpha(I, C), s \in (0, 1]$ and take $x, \bar{x} \in I$ such that $|\bar{x} - x| \leq s$. Inequalities (18) and (17) imply that

$$\begin{aligned} &d(N\varphi(x) + N\bar{\varphi}(\bar{x}), N\varphi(\bar{x}) + N\bar{\varphi}(x)) \\ &= d(A(x, \varphi(x)) + A(\bar{x}, \bar{\varphi}(\bar{x})), A(\bar{x}, \varphi(\bar{x})) + A(x, \bar{\varphi}(x))) \\ &= d(A(x, \varphi(x) + \bar{\varphi}(\bar{x})) + A(\bar{x}, \bar{\varphi}(\bar{x})), A(x, \varphi(\bar{x})) + A(\bar{x}, \bar{\varphi}(x))) \\ &\leq d(A(x, \varphi(x) + \bar{\varphi}(\bar{x})), A(x, \varphi(\bar{x}) + \bar{\varphi}(x))) \\ &\quad + d(A(\bar{x}, \bar{\varphi}(\bar{x})) + A(x, \varphi(\bar{x})), A(x, \bar{\varphi}(\bar{x})) + A(\bar{x}, \varphi(\bar{x}))) \\ &\leq M_0 M \|(\varphi - \bar{\varphi})(x) - (\varphi - \bar{\varphi})(\bar{x})\| + \gamma(\|(\varphi - \bar{\varphi})(\bar{x})\|)\beta(|\bar{x} - x|) \\ &\leq M_0 M \omega(\varphi - \bar{\varphi}, s) + \gamma(\|\varphi - \bar{\varphi}\|_\alpha)\beta(s). \end{aligned}$$

Hence,

$$\frac{\omega(N\varphi, N\bar{\varphi}, s)}{\beta(s)} \leq M_0 M \frac{\omega(\varphi - \bar{\varphi}, s)}{\alpha(s)} \frac{\alpha(s)}{\beta(s)} + \gamma(\|\varphi - \bar{\varphi}\|_\alpha)$$

and therefore,

$$\sup_{s \in (0,1]} \frac{\omega(N\varphi, N\bar{\varphi}, s)}{\beta(s)} \leq M_0 M L h_\alpha(\varphi - \bar{\varphi}) + \gamma(\|\varphi - \bar{\varphi}\|_\alpha).$$

Consequently,

$$\begin{aligned} d_\beta(N\varphi, N\bar{\varphi}) &= d(N\varphi(0), N\bar{\varphi}(0)) + \sup_{s \in (0,1]} \frac{\omega(N\varphi, N\bar{\varphi}, s)}{\beta(s)} \\ &\leq M_0 M \|(\varphi - \bar{\varphi})(0)\| + M_0 M L h_\alpha(\varphi - \bar{\varphi}) + \gamma(\|\varphi - \bar{\varphi}\|_\alpha) \\ &\leq M_0 M L \|\varphi - \bar{\varphi}\|_\alpha + \gamma(\|\varphi - \bar{\varphi}\|_\alpha). \end{aligned}$$

Setting $c = \max\{1, M_0 M L\}$ and $\gamma_1(t) = c(t + \gamma(t))$ we get

$$d_\beta(N\varphi, N\bar{\varphi}) \leq \gamma_1(\|\varphi - \bar{\varphi}\|_\alpha).$$

This finishes the proof. $\square$

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