Sets of Univalence in Some Classes of Analytic Functions

Leopold Koczan and Katarzyna Trąbka-Więclaw

Abstract. This paper attempts to determine the “largest” sets of local univalence for a given class and the “largest” open sets in which all functions belonging to a given class are univalent. We establish general properties of sets of univalence for analytic functions with typical normalization. Moreover, we determine some examples of the sets of univalence for some particular subclasses of analytic functions.

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1. Introduction

Suppose that $\mathcal{A}$ is the family of all functions that are analytic in the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = 0$, $f'(0) = 1$.

By capital letters $A, B, \dots$ we denote compact subclasses of the class $\mathcal{A}$. By capital letters $\mathbb{A}, \mathbb{B}, \dots$ we denote sets of the complex plane. Let $\text{cl } \mathbb{A}$ be the closure of the set $\mathbb{A}$ and $\partial \mathbb{A}$ —the boundary of the set $\mathbb{A}$. Moreover, for the function $f \in \mathcal{A}$ and for the set $\mathbb{A}$ let $f_\theta(z) := e^{i\theta} f(e^{-i\theta} z)$ and $\mathbb{A}_\theta := \{e^{i\theta} a : a \in \mathbb{A}\}$ for $\theta \in \mathbb{R}$. Let $\mathbb{D}(p, r) := \{z \in \mathbb{C} : |z - p| < r\}$ (and so $\mathbb{D}(0, 1) = \mathbb{D}$).

In complex analysis, many authors investigated the problem of determining the “largest” discs at 0 in which all functions belonging to a given class are locally univalent or univalent. By the “largest” disc we mean the disc with the largest radius. For a compact class A these discs are unique and are called the disc of local univalence of the class A and the disc of univalence of the class A , respectively. The radii of these discs, we call the radius of local univalence of the class A and the radius of univalence of the class A , and are denoted by $r_{\text{LU}}(A)$ and $r_U(A)$, respectively.

The goal of this paper is to generalize the above problems, which means to determine the “largest” sets in which all functions belonging to a given class are locally univalent or univalent.

2. Main Results

Definition 2.1. A set $\mathbb{A} \subset \mathbb{D}$ is called the set of local univalence for the class $A \subset \mathcal{A}$ if:

1. all functions belonging to A are locally univalent in $\mathbb{A}$,
2. for every set $\mathbb{B}$ such that $\mathbb{A} \subset \mathbb{B} \subset \mathbb{D}$ and $\mathbb{A} \neq \mathbb{B}$ there exists a function in A that is not locally univalent in $\mathbb{B}$.

The set of local univalence for the class A is unique and we denote it by $\text{LU}(A)$. We have

$$\text{LU}(A) = \mathbb{D} \setminus \{z \in \mathbb{D} : \exists_{f \in A} f'(z) = 0\}.$$

For a compact class A , the set $\text{LU}(A)$ is open. Certainly, $0 \in \text{LU}(A)$, as a result of normalization (i.e., for all functions $f \in A$ we have $f'(0) = 1 \neq 0$). If a class is not compact, then the set of local univalence can be closed, for example $\text{LU}(\mathcal{A}) = \{0\}$. We consider only compact classes.

More problems appear while discussing the set of univalence for a given class. To avoid considerations about univalence at the boundary of a set, we use the following definition.

Definition 2.2. An open set $\mathbb{A} \subset \mathbb{D}$, $\mathbb{A} \neq \emptyset$, is called the set of univalence for the class $A \subset \mathcal{A}$ if:

1. all functions belonging to A are univalent in $\mathbb{A}$,
2. for every set $\mathbb{B}$ such that $\mathbb{A} \subset \mathbb{B} \subset \mathbb{D}$ and $\mathbb{A} \neq \mathbb{B}$ there exists a function in A that is not univalent in $\mathbb{B}$.

Let us notice that for the class of functions locally univalent in $\mathbb{D}$ with typical normalization, the set of univalence does not exist (is an empty set), although the set of local univalence is $\mathbb{D}$. Let $f(z) := (e^{az} - 1)/a$, $z \in \mathbb{D}$, $a \in \mathbb{C} \setminus \{0\}$. The function f is locally univalent in $\mathbb{D}$, because we have $f'(z) = e^{az}$ and $f'(z) \neq 0$ for all $z \in \mathbb{D}$. Furthermore, f is univalent in the disc $\mathbb{D}(0, \pi/|a|)$ and is not univalent in each disc with a larger radius. Since a is any complex number not equal to 0, in any disc at 0 there exist functions which are not univalent in this disc. Analogously, we can prove that this property holds for any disc at a point z_0 , $z_0 \in \mathbb{D}$, which is included in $\mathbb{D}$. Thus, the set of univalence for the class of functions locally univalent in $\mathbb{D}$ with typical normalization does not exist.

If we demand that the set $\mathbb{B}$ from Definition 2.2 is a domain, we get a generalization of Definition 2.2.

Definition 2.3. An open set $\mathbb{A} \subset \mathbb{D}$, $\mathbb{A} \neq \emptyset$, is called the set of domain univalence for the class $A \subset \mathcal{A}$ if:

1. all functions belonging to A are univalent in $\mathbb{A}$,
2. for every domain $\mathbb{B}$ such that $\mathbb{A} \subset \mathbb{B} \subset \mathbb{D}$ and $\mathbb{A} \neq \mathbb{B}$ there exists a function in A that is not univalent in $\mathbb{B}$.

Remark 2.4. 1. Each set of univalence for the class A is the set of domain univalence for this class.

2. The set of domain univalence for the class A does not have to be the set of univalence for this class, as will be shown in some examples (see Example 1).
3. The set of domain univalence for the class A , which is not the set of univalence for this class, is included in some set of univalence for this class (see Example 1).

From Remark 2.4(3) it follows that the set of domain univalence for the class A does not have to be the “largest” set of univalence for this class.

The questions arise as to whether sets of univalence are domains and whether sets of univalence are unique. The answer to the first question is negative, i.e., there exist sets of univalence which are not domains, as will be shown in Example 1. The answer to the second question leads to the following theorem.

Theorem 2.5. *If all functions belonging to a compact class A are univalent in the set $\text{LU}(A)$, then the set $\text{LU}(A)$ is the only set of univalence for this class. But if there are functions which are not univalent in the set $\text{LU}(A)$, then there are more than one set of univalence for the class A .*

Proof. Suppose that $\mathbb{A}$ is the set of univalence for the class A . We have $\mathbb{A} \subset \text{LU}(A)$ and $\mathbb{A} \neq \text{LU}(A)$. But $\mathbb{A} = \text{LU}(A)$. Contradiction. Thus, $\text{LU}(A)$ is the only set of univalence for the class A .

Assume now that $\mathbb{A}$ is the set of univalence for the class A not equal to $\text{LU}(A)$. We have $\mathbb{A} \subset \text{LU}(A)$ and $\mathbb{A} \neq \text{LU}(A)$. Therefore, there exists a point $z_0 \in \text{LU}(A)$ and $z_0 \notin \mathbb{A}$. Since the set $\text{LU}(A)$ is open, there exists a neighborhood of the point z_0 , $\mathbb{D}(z_0, r_1)$ included in $\text{LU}(A)$ and such that $\mathbb{D}(z_0, r_1) \cap \mathbb{A} = \emptyset$. From the facts that the class A is compact and all functions from the class A are locally univalent in this neighborhood, we conclude that there exists a neighborhood of the point z_0 , $\mathbb{D}(z_0, r_2)$, $r_2 \leq r_1$, in which all functions from the class A are univalent. This means that there exists the set of univalence containing $\mathbb{D}(z_0, r_2)$, in which all functions from the class A are univalent. Let us denote this set by $\mathbb{B}$. We have proved that there exist the sets $\mathbb{A}$ and $\mathbb{B}$, $\mathbb{A} \neq \mathbb{B}$, which are the sets of univalence for the class A , so the proof is complete. $\square$

Conjecture 2.6. *If the set $\text{LU}(A)$ is not the set of univalence for the class A , then there are infinitely many sets of univalence for this class.*

One can notice that the union of all sets of univalence for the class A is equal to $\text{LU}(A)$. If $\text{LU}(A)$ is the union of infinite number of different sets of univalence for the class A , then Conjecture 2.6 can be proved analogously as we have proved Theorem 2.5 (that sets of univalence are more than one). Therefore, the question arises as to whether $\text{LU}(A)$ can be the union of finite number of different sets of univalence for the class A . Example 1 shows that $\text{LU}(A)$ can be the union of three sets of univalence for the class A .

Another question arises as to whether the intersection of all sets of univalence for the class A can be an empty set. Example 6 shows that it can be an empty set.

Let us notice that even though all functions are univalent in the disc of univalence of a given class A , but it does not mean that the intersection of all sets of univalence for the class A contains the disc of univalence of this class. Inclusion occurs when the disc of univalence is equal to $\text{LU}(A)$.

Definition 2.7. A class A is called the class invariant under rotation if for each function $f \in A$ each function f_θ belongs to A , $\theta \in \mathbb{R}$.

Theorem 2.8. If a set $\mathbb{A}$ is the set of univalence for the class A invariant under rotation, then $\mathbb{A}_\theta$, $\theta \in \mathbb{R}$, are also the sets of univalence for this class.

Proof. Let $\mathbb{A}$ be the set of univalence for the class A . It follows that for any function $f \in A$ all functions f_α , $\alpha \in \mathbb{R}$, are univalent in $\mathbb{A}$. We will prove that all functions belonging to the class A are univalent in $\mathbb{A}_\theta$, $\theta \in \mathbb{R}$. Functions $z \mapsto e^{i\theta} f_\alpha(e^{-i\theta} z)$ are univalent in $\mathbb{A}_\theta$, $\theta \in \mathbb{R}$, because

$$e^{i\theta} f_\alpha(e^{-i\theta} z) = e^{i\theta} e^{i\alpha} f(e^{-i\alpha} e^{-i\theta} z) = e^{i(\theta+\alpha)} f(e^{-i(\theta+\alpha)} z) = f_{\theta+\alpha}(z).$$

Since functions $f_{\theta+\alpha}$ are univalent in $\mathbb{A}$ (because $\mathbb{A}$ is the set of univalence for the class A and $f_{\theta+\alpha} \in A$) and for $z \in \mathbb{A}_\theta$ we have $e^{-i\theta} z \in \mathbb{A}$, this implies that all functions belonging to the class A are univalent in $\mathbb{A}_\theta$. $\square$

Theorem 2.9. The disc of univalence of the class A invariant under rotation is the set of domain univalence for this class.

Proof. Two cases occur:

1. $r_{\text{LU}}(A) = r_U(A)$. Then there exists a point z_0 , $|z_0| = r_{\text{LU}}(A)$ and a function $f \in A$ such that $f'(z_0) = 0$. Let $\mathbb{B}$ be a domain containing the disc of univalence of the class A and not equal to this disc. This means that there exists a point z_0 , $|z_0| = r_U(A)$ such that a neighborhood of this point z_0 , $\mathbb{D}(z_0, \varepsilon)$ is included in the set $\mathbb{B}$ and $f'(z_0) = 0$. Thus, the set $\mathbb{D}(0, r_U(A)) \cup \mathbb{D}(z_0, \varepsilon) \subset \mathbb{B}$ and the function f is not univalent in $\mathbb{D}(0, r_U(A)) \cup \mathbb{D}(z_0, \varepsilon)$. So f is not also univalent in $\mathbb{B}$. Since the class A is invariant under rotation, f_θ , $\theta \in \mathbb{R}$ have the property $f'_\theta(z) = 0$ for $z = z_0 e^{i\theta}$. This means that there does not exist a domain $\mathbb{B}$ containing the disc of univalence of the class A , in which all functions from this class are univalent.
2. $r_{\text{LU}}(A) > r_U(A)$, so all functions belonging to the class A are locally univalent at the boundary of the disc of univalence of this class. There exist functions from the class A which are not univalent at the boundary of the disc of univalence of this class. Therefore, let f be such function that is not univalent at the boundary of the disc of univalence of the class A , i.e., $f(z_1) = f(z_2)$ for $z_1 \neq z_2$ and $|z_1| = |z_2| = r_U(A)$. For any ε we have that f is not univalent in $\mathbb{D}(0, r_U(A)) \cup \mathbb{D}(z_0, \varepsilon)$, where $\mathbb{D}(z_0, \varepsilon) \subset \mathbb{D}$. Analogously, as in the case of (1), we prove the conclusion of theorem. $\square$

Conjecture 2.10. The disc of univalence of the class A invariant under rotation is one of sets of univalence for this class.

Figure 1. Boundary of the set $f(\mathbb{D})$

3. Some Examples

Example 1. Let us consider a class consisting of one function $f(z) := z + z^2$, $z \in \mathbb{D}$, i.e., the class

$$A := \{z \mapsto z + z^2 : z \in \mathbb{D}\}.$$

The properties of the class A :

1. $\text{LU}(A) = \mathbb{D} \setminus \{-1/2\}$, $r_{\text{LU}}(A) = 1/2$.
2. $r_U(A) = 1/2$. Since we have $\text{Re}\{f'(z)\} > 0$ for $|z| < 1/2$, so according to the Noshiro–Warschawski Theorem (see for example [1]), the function f is univalent in the disc $\mathbb{D}(0, 1/2)$.
3. The function f maps the disc $\mathbb{D}$ onto the set shown in Fig. 1, which can be presented as a union of two disjoint sets $\mathbb{X}$ and $\mathbb{Y}$ such that all points of the set $\mathbb{X}$ are taken by f only once, and all points of the set $\mathbb{Y}$ are taken by f twice.
4. Let $\mathbb{E} := \mathbb{D} \cap \mathbb{D}(-1, 1)$ (see Fig. 2). The function f maps the set $\mathbb{E}$ onto the set shown in Fig. 4; the boundary of this set is the curve

$$\Gamma : \begin{cases} x(\theta) = \cos \theta + \cos 2\theta \\ y(\theta) = \sin \theta + \sin 2\theta \end{cases}, \quad \theta \in [2\pi/3, 4\pi/3].$$

All points of the set $f(\mathbb{E})$ are taken twice (at points symmetric with respect to the point $-1/2$).

5. Let $\mathbb{F} := \mathbb{D} \setminus \text{cl } \mathbb{E}$ (see Fig. 3). The function f maps the set $\mathbb{F}$ onto the set shown in Fig. 5.

Figure 2. Set $\mathbb{E}$ Figure 3. Set $\mathbb{F}$

6. The function f is univalent in an open set $\mathbb{B} \subset \mathbb{D}$ if and only if $\mathbb{B} \cap h(\mathbb{B}) = \emptyset$, where $h(z) := -z - 1$, $z \in \mathbb{D}$ (the set $h(\mathbb{B})$ is symmetric to the set $\mathbb{B}$ with respect to the point $-1/2$).

Proof. Assume that $\mathbb{B} \cap h(\mathbb{B}) = \emptyset$ and $f(z) = f(\zeta)$, $z, \zeta \in \mathbb{B}$, $z \neq \zeta$. We have $z + z^2 = \zeta + \zeta^2$ and thus $(z - \zeta)(1 + z + \zeta) = 0$, this gives $\zeta = -z - 1$. But

Figure 4. Set $f(\mathbb{B})$ Figure 5. Set $f(\mathbb{F})$

for $z \in \mathbb{B}$ we have $-z - 1 \notin \mathbb{B}$, because $\mathbb{B} \cap h(\mathbb{B}) = \emptyset$. Hence for $z, \zeta \in \mathbb{B}$, $z \neq \zeta$ we have $f(z) - f(\zeta) \neq 0$. This means that f is univalent in the set $\mathbb{B}$.

Figure 6. Set $\mathbb{G}(\gamma)$

Let us notice that $-1/2 \notin \mathbb{B}$, because if $-1/2 \in \mathbb{B}$, then $-1/2 \in h(\mathbb{B})$, and so $-1/2 \in \mathbb{B} \cap h(\mathbb{B})$. Contradiction. $\square$

7. The function f is univalent in the set $\mathbb{F}$. Since we have $\mathbb{F} \cap h(\mathbb{F}) = \emptyset$, so from the property (6) we obtain that f is univalent in $\mathbb{F}$.
8. For $\gamma \in [0, 2\pi)$ let

$$\mathbb{G}(\gamma) := \begin{cases} \mathbb{E} \cap \{z = x + iy : y \cos \gamma \\ \quad -x \sin \gamma - \frac{1}{2} \sin \gamma > 0\} & \text{for } \gamma \in (0, \pi) \cup (\pi, 2\pi), \\ \mathbb{E} \cap \mathbb{C}^+ & \text{for } \gamma = 0, \\ \mathbb{E} \cap \mathbb{C}^- & \text{for } \gamma = \pi, \end{cases}$$

where $\mathbb{C}^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, $\mathbb{C}^- = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$ (see Fig. 6).

From the property (6) it follows that f is univalent in $\mathbb{G}(\gamma)$ and $h(\mathbb{G}(\gamma))$. Since $h(\mathbb{G}(\gamma)) = \mathbb{G}(\pi + \gamma)$ for $\gamma \in [0, \pi)$, we can restrict ourselves to $\mathbb{G}(\gamma)$. Among the sets $\mathbb{G}(\gamma)$, only the set $\mathbb{G}(\pi/2)$ (see Fig. 8) is the set of domain univalence for the class A , but it is not the set of univalence for the class A (Remark 2.4(2)). While for $\gamma \neq \pi/2$, the sets $\mathbb{G}(\gamma)$ are not the sets of domain univalence for the class A (so they are not the sets of univalence for the class A , either).

9. For $\gamma \in [0, 2\pi)$ let

$$\mathbb{H}(\gamma) := \mathbb{B}(\gamma) \setminus \partial \mathbb{B}(\gamma), \text{ where } \mathbb{B}(\gamma) := \mathbb{G}(\gamma) \cup \operatorname{cl} \mathbb{F}$$

(see Fig. 7). The sets $\mathbb{H}(\gamma)$ are the sets of univalence for the class A (so they are also the sets of domain univalence for the class A —Remark 2.4(1)). Let us notice that $\mathbb{H}(\pi/2)$ is the set of univalence for the class A containing $\mathbb{G}(\pi/2)$ —the set of domain univalence for this class (Remark

Figure 7. Set $\mathbb{H}(\gamma)$ Figure 8. Set $\mathbb{G}(\pi/2)$

2.4(3)). Furthermore, $\mathbb{H}(\pi/2)$ is the set of univalence for the class A , which is not a domain (see Fig. 9).

10. The union of three sets of univalence for the class A is equal to $\text{LU}(A)$. For $\gamma \in (0, \pi/2)$ we have $\text{LU}(A) = \mathbb{H}(\gamma) \cup \mathbb{H}(\pi - \gamma) \cup \mathbb{H}(3\pi/2)$.

Figure 9. Set $\mathbb{H}(\pi/2)$

Example 2. Let us consider a class which is the "smallest" class invariant under rotation, containing the class A from Example 1, i.e., the class

$$B := \{z \mapsto z + e^{-i\theta} z^2 : z \in \mathbb{D}, \theta \in \mathbb{R}\}.$$

Certainly, $A \subset B$ and $A \neq B$. The properties of the class B :

1. $\text{LU}(B) = \mathbb{D} \setminus \{z \in \mathbb{C} : |z| = 1/2\}$, $r_{\text{LU}}(B) = 1/2$.
2. $r_U(B) = 1/2$. Since we have $\text{Re}\{f'(z)\} > 0$ for $|z| < 1/2$, so according to the Noshiro–Warschawski Theorem (see for example [1]), the function f is univalent in the disc $\mathbb{D}(0, 1/2)$.
3. The set $\mathbb{D}(0, 1/2)$ is the set of univalence for the class B .

Proof. From Theorem 2.9 it follows that the disc of univalence $\mathbb{D}(0, 1/2)$ is the set of domain univalence for the class B . We will prove now that $\mathbb{D}(0, 1/2)$ is the set of univalence for this class. Assume that $\mathbb{D}(0, 1/2)$ is not the set of univalence for the class B . This means that there exists an open set $\mathbb{B}$ such that $\mathbb{D}(0, 1/2) \subset \mathbb{B} \subset \mathbb{D}$ and $\mathbb{D}(0, 1/2) \neq \mathbb{B}$, in which all functions belonging to the class B are univalent. For the function $f(z) := z + z^2$ each function f_θ , $\theta \in \mathbb{R}$, belongs to B . It is enough to show that f is not univalent in $\mathbb{B}_\theta$ for some $\theta \in \mathbb{R}$ (Theorem 2.8). Since $\mathbb{D}(0, 1/2) \subset \mathbb{B}$ and $\mathbb{D}(0, 1/2) \neq \mathbb{B}$, so there exists a point z_0 such that $1/2 < |z_0| < 1$ and $z_0 \in \mathbb{B}$. Choosing $\theta_0 = \pi - \arg z_0$ we have $z_0 e^{i\theta_0} \in (-1, -1/2)$. The set $\mathbb{B}_{\theta_0}$ is open, thus there exists a neighborhood $\mathbb{D}(z_0 e^{i\theta_0}, \varepsilon) \subset \mathbb{B}_{\theta_0}$. The function f is not univalent in $\mathbb{D}(0, 1/2) \cup \mathbb{D}(z_0 e^{i\theta_0}, \varepsilon)$, because $(\mathbb{D}(0, 1/2) \cup \mathbb{D}(z_0 e^{i\theta_0}, \varepsilon)) \cup h(\mathbb{D}(0, 1/2) \cup \mathbb{D}(z_0 e^{i\theta_0}, \varepsilon)) \neq \emptyset$, where $h(z) := -z - 1$, $z \in \mathbb{D}$. Contradiction (see property (6) from Example 1). $\square$

4. The set $\mathbb{G}(\pi/2)$ is the set of domain univalence for the class B , but it is not the set of univalence for this class.

Proof. Suppose that $\mathbb{E}$, $\mathbb{F}$ and $\mathbb{G}(\gamma)$ are defined as in Example 1. From Example 1 we know that the function $f(z) := z + z^2$ is univalent in $\mathbb{G}(\gamma)$, $\gamma \in [0, 2\pi)$ and that $\mathbb{G}(\pi/2)$ is the set of domain univalence for the class A . Furthermore, f is univalent in $\mathbb{G}(\pi/2)_\theta$, $\theta \in [0, 2\pi)$ (see Theorem 2.8). For $\theta \in [2\pi/3, 4\pi/3]$ we have $\mathbb{G}(\pi/2)_\theta \cap \mathbb{E} = \emptyset$; it means that $\mathbb{G}(\pi/2)_\theta$ is included in $\mathbb{F}$, in which f is univalent. For $\theta \in (0, 2\pi/3) \cup (4\pi/3, 2\pi)$ we have $\mathbb{G}(\pi/2)_\theta \cap \mathbb{E} \neq \emptyset$; then it is easy to check that $\mathbb{G}(\pi/2)_\theta \cap \mathbb{E}$ is included in $\mathbb{G}(\pi/2 + \theta)$, in which f is univalent. Hence, f is univalent in $\mathbb{G}(\pi/2)_\theta$. So we have proved that all functions belonging to the class B are univalent in $\mathbb{G}(\pi/2)$. Thus $\mathbb{G}(\pi/2)$ is the set of domain univalence for the class B , because it is the set of domain univalence for the class A defined in Example 1.

To show that $\mathbb{G}(\pi/2)$ is not the set of univalence for the class B , it is enough to find some open set $\mathbb{B}$ such that $\mathbb{G}(\pi/2) \subset \mathbb{B} \subset \mathbb{D}$ and $\mathbb{G}(\pi/2) \neq \mathbb{B}$, in which each function f_θ , $\theta \in [0, 2\pi)$ is univalent. Let $\mathbb{B} = \mathbb{G}(\pi/2) \cup \mathbb{X}$, where $\mathbb{X} = \mathbb{D}^+(0, 1/2) \setminus (\text{cl } \mathbb{D}(i, 1) \cup \text{cl } \mathbb{D}(-i, 1))$ and $\mathbb{D}^+(0, 1/2)$ is the right half-disc of the disc $\mathbb{D}(0, 1/2)$. The function f is univalent in $\mathbb{B}_\theta$, $\theta \in [0, 2\pi)$, because $\mathbb{B}_\theta \cap h(\mathbb{B}_\theta) = \emptyset$, where $h(z) := -z - 1$, $z \in \mathbb{D}$ (see property (6) from Example 1).

The set $\mathbb{G}(\pi/2)$ is included in some set of univalence for the class B , which is not the disc of univalence of this class. $\square$

Example 3. Let us consider a class which is the convex hull of the class B from Example 2, i.e., the class

$$C := \{z \mapsto z + az^2 : z \in \mathbb{D}, |a| \leq 1\}.$$

Certainly, $B \subset C$ and $B \neq C$. The properties of the class C :

1. $\text{LU}(C) = \mathbb{D}(0, 1/2)$, $r_{\text{LU}}(C) = 1/2$. Because for $f(z) = z + az^2$ we have $f'(z) = 0$ for $z_a = -1/(2a)$, $|z_a| \geq 1/2$.
2. $r_U(C) = 1/2$ and the set $\mathbb{D}(0, 1/2)$ is the set of univalence for the class C . Since for $|z| < 1/2$ we have $\text{Re } \{f'(z)\} > 0$ for $f \in C$, according to the Noshiro–Warschawski Theorem (see for instance [1]), the function f is univalent in the disc $\mathbb{D}(0, 1/2)$.
3. The set $\mathbb{D}(0, 1/2)$ is the only one set of univalence for the class C , because it is equal to the set of local univalence for this class (see Theorem 2.5).

Example 4. The set of univalence and the set of local univalence for the class of typically real functions

$$T := \{f \in \mathcal{A} : \text{Im } z \text{ Im } f(z) \geq 0, z \in \mathbb{D}\}$$

coincide (see for instance [2]). It is the Goluzin lens presented in Fig. 10, i.e.,

$$\text{LU}(T) = \{z \in \mathbb{D} : |z^2 + 1| > 2|z|\} = \mathbb{D}(i, \sqrt{2}) \cap \mathbb{D}(-i, \sqrt{2}).$$

Certainly, $r_{\text{LU}}(T) = r_U(T) = \sqrt{2} - 1$.

Example 5. The set of local univalence for the class of odd typically real functions

$$T^{(2)} := \{f \in T : f(z) = -f(-z), z \in \mathbb{D}\}$$

Figure 10. Boundary of the set $\text{LU}(T)$ Figure 11. Boundary of the set $\text{LU}(T^{(2)})$

is the set presented in Fig. 11, i.e.,

$$\begin{aligned} \text{LU}(T^{(2)}) &= \{z \in \mathbb{D} : |3z^4 + 2z^2 + 3| > 8|z|^2\} \setminus \{\pm ir : r \geq \sqrt{2} - 1\} \\ &= \{z \in \mathbb{D} : |3z^4 + 2z^2 + 3| > 8|z|^2\} \setminus \{z \in \mathbb{C} : z^2 \leq -(\sqrt{2} - 1)^2\}, \end{aligned}$$

which was shown in [6]. Certainly, $\text{LU}(T) \subset \text{LU}(T^{(2)})$.

Figure 12. Boundary of the set $\mathbb{K}$

Koczan and Zaprawa [4, 5] proved that there can be infinitely many sets of univalence for the class $T^{(2)}$, and they determined some of these sets. The Goluzin lens is one of the sets of univalence for the class $T^{(2)}$ (see [6]).

Example 6. Let us consider a class consisting of one function

$$g(z) := \frac{1}{\pi} \tan \frac{\pi z}{1 + z^2}, \quad z \in \mathbb{D},$$

i.e., the class

$$D := \left\{ z \mapsto \frac{1}{\pi} \tan \frac{\pi z}{1 + z^2} : z \in \mathbb{D} \right\}.$$

The function g is locally univalent ($g'(z) \neq 0$, $z \in \mathbb{D}$), i.e., $\text{LU}(D) = \mathbb{D}$ and typically real (it is called the universal function; it was introduced by Goodman [3]). The sets of univalence for the class D are the following sets (see [7]):

1. the set $\mathbb{K}$ presented in Fig. 12, which is bounded, symmetric with respect to both axes of the complex plane, and which has the boundary in the first quadrant of the coordinate system given by polar equation $\rho = r(\theta) e^{i\theta}$, where

$$r(\theta) = (\alpha - \sqrt{\alpha^2 - 4})/2, \quad \theta \in [0, \pi/2],$$

where $\alpha = \cos \theta + \sqrt{\cos^2 \theta + 4 \sin^2 \theta}$,

2. the set $\mathbb{L}$ presented in Fig. 13, which is bounded, symmetric with respect to the real axis, and which has the boundary consisting of the segment $[-i, i]$ and the arc given by polar equation $\rho = r(\theta) e^{i\theta}$, where

$$r(\theta) = \begin{cases} 1 & \text{for } \theta \in (-\pi/3, \pi/3), \\ (\beta - \sqrt{\beta^2 - 16})/4 & \text{for } \theta \in [-\pi/2, -\pi/3] \cup [\pi/3, \pi/2], \end{cases}$$

Figure 13. Boundary of the set $\mathbb{L}$ Figure 14. Boundary of the set $\mathbb{M}$

where $\beta = \cos \theta + \sqrt{\cos^2 \theta + 16 \sin^2 \theta}$,

3. the set $\mathbb{M}$ presented in Fig. 14, which is symmetric to the set $\mathbb{L}$ with respect to the imaginary axis .

The disc of univalence of the class D is the set $\mathbb{D}(0, 1/\sqrt{3})$ and $r_U(D) = 1/\sqrt{3}$ (see [7]).

Let us notice that $\mathbb{L} \cap \mathbb{M} = \emptyset$, so the intersection of all sets of univalence for the class D is an empty set.

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Leopold Koczan
Fundamentals of Technology Faculty
Lublin University of Technology
ul. Nadbystrzycka 38
20-618 Lublin
Poland
e-mail: l.koczan@pollub.pl

Katarzyna Trąbka-Więclaw
Department of Mathematics
Mechanical Engineering Faculty
Lublin University of Technology
ul. Nadbystrzycka 36
20-618 Lublin
Poland
e-mail: k.trabka@pollub.pl

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