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Abstract. We fill in a gap in the proof of Theorem 5.1 of [1] on the boundedness of sublinear operators of singular type in variable exponent Herz type spaces $H^{p(\cdot),q(\cdot),\alpha(\cdot)}(\mathbb{R}^n)$. When q is constant, the formulation of Theorem 5.1 from [1] remains the same. In the case where q is variable, Theorem 5.1 needs a more precise formulation with respect to some auxiliary parameters of the space (not reflected in the notation $H^{p(\cdot),q(\cdot),\alpha(\cdot)}(\mathbb{R}^n)$ of the space).

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1. Introduction

We use all the definitions and notations from [1] and deal with sublinear operators T satisfying the well known size condition

$$|Tf(x)| \leq C \int_{\mathbb{R}^n} \frac{|f(y)| dy}{|x-y|^n}, \quad x \notin \text{supp } f. \quad (1.1)$$

Theorem 5.1 of [1] was formulated in [1] as follows.

Theorem A. *Let $p \in \mathcal{P}_{\infty}^{\log}(\mathbb{R}^n)$ with $1 < p_- < p_+ < \infty$, $q \in \mathcal{P}_{\infty}^{\log}(\nu, \infty)$ with $1 \leq q_- \leq q_+ < \infty$ and $\alpha \in \mathcal{P}_{\infty}(\nu, \infty)$. Then every sublinear operator T with the size condition (1.1), bounded in $L^{p(\cdot)}(\mathbb{R}^n)$, is also bounded in the Herz spaces $H_{\nu}^{p(\cdot),q(\cdot),\alpha(\cdot)}(\mathbb{R}^n)$, $\nu > 0$, if*

$$-\frac{n}{p_{\infty}} < \alpha_{\infty} < \frac{n}{p'_{\infty}}. \quad (1.2)$$

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In the corrected version of this theorem given below, the dependence, on the auxiliary parameters ν, γ, δ is already of importance when q is variable. For simplicity we choose $\nu = 2, \gamma = 1, \delta = 2$, but in the case of variable q we will also have to deal with values of γ and δ different from 1 and 2, respectively. Keeping this in mind, we slightly change the notation of the norm by redenoting the norm as follows

$$\|f\|_{H_{(\gamma,\delta)}^{p,q,\alpha}} := \|f\|_{L^{p(\cdot)}(B(0,2\gamma+\varepsilon))} + \left\| t^{\alpha(t)} \left\| f \chi_{R_{\gamma t, \delta t}} \right\|_{L^{p(\cdot)}} \right\|_{L^{q(\cdot)}((\gamma, \infty), \bar{d}t)}. \quad (1.3)$$

The corrected version of Theorem A runs as follows.

Theorem 1.1. *Let $p \in \mathcal{P}_{\infty}^{\log}(\mathbb{R}^n)$ with $1 < p_- < p_+ < \infty$, $q \in \mathcal{P}_{\infty}^{\log}(\nu, \infty)$ with $1 \leq q_- \leq q_+ < \infty$ and $\alpha \in \mathcal{P}_{\infty}(\nu, \infty)$. Let also*

$$-\frac{n}{p_{\infty}} < \alpha_{\infty} < \frac{n}{p'_{\infty}}. \quad (1.4)$$

Then every sublinear operator T with the size condition (1.1), bounded in $L^{p(\cdot)}(\mathbb{R}^n)$, is also bounded within the frameworks of the Herz spaces in the following sense:

$$\|Tf\|_{H_{(\gamma,\delta)}^{p,q,\alpha}} \leq C \|f\|_{H_{(\gamma',\delta')}^{p,q,\alpha}} \quad (1.5)$$

for any $0 < \gamma' < \gamma$ and $\delta < \delta' < \infty$. In the case where q is constant, (1.5) holds with $\gamma' = \gamma$ and $\delta' = \delta$.

Proof. We dwell only on the arguments where the changes should be made and skip the parts of the proof, where it remains unchanged.

The proof does not depend on the concrete choice of γ and δ , so we choose $\gamma = 1$ and $\delta = 2$ for simplicity, and work with the norms

$$\|f\|_{H_{(1,2)}^{p,q,\alpha}} = \|f\|_{L^{p(\cdot)}(B(0,2+\varepsilon))} + N_{(1,2)}(f)_{p,q,\alpha}, \quad (1.6)$$

and

$$\|f\|_{H_{(\gamma',\delta')}^{p,q,\alpha}} = \|f\|_{L^{p(\cdot)}(B(0,2+\varepsilon))} + N_{(\gamma',\delta')}(f)_{p,q,\alpha}, \quad \gamma' < 1, \delta' > 2, \quad (1.7)$$

where we denoted

$$N_{(\gamma',\delta')}(f)_{p,q,\alpha} := \left\| t^{\alpha_{\infty}} \left\| f \chi_{(\gamma' t, \delta' t)} \right\|_{L^{p(\cdot)}} \right\|_{L^{q((\gamma', \infty), \bar{d}t)}} \quad (1.8)$$

for brevity. It is obvious that $\|f\|_{H_{(1,2)}^{p,q,\alpha}} \leq \|f\|_{H_{(\gamma',\delta')}^{p,q,\alpha}}$.

The estimation of $\|Tf\|_{L^{p(\cdot)}(B(0,2+\varepsilon))}$ is the same as in [1].

In estimation of $N_{(1,2)}(Tf)_{p,q,\alpha}$ we use the splitting:

$$f(x) = f_0(x) + f_t(x) + g_t(x) + h_t(x),$$

where

$$\begin{aligned} f_0(x) &= f(x) \chi_{B(0,1/2)}(x), \quad f_t(x) = f(x) \chi_{B((0,\gamma' t) \setminus B(0,1/2))}(x), \\ g_t(x) &= f(x) \chi_{B(0,\delta' t) \setminus B(0,\gamma' t)}(x), \quad h_t(x) = f(x) \chi_{\mathbb{R}^n \setminus B(0,\delta' t)}(x). \end{aligned}$$

The term f_0 is the easiest one and needs only variable exponent Hölder inequality, the terms f_t and h_t are treated with application of the variable exponent Hardy inequality, while for g_t , which is in fact the term where changes should be made, we will use the fact that T is bounded on $L^{p(\cdot)}$.

Note that going out from the interval $(t, 2t)$ to a larger interval $(\gamma't, \delta't)$ with $\gamma' < 1$ and $\delta' > 2$ is due to estimations of the term Tg_t , but it does not appear in estimations of the terms Tf_0, Tf_t, Th_t .

Estimation of $Tf_0(x)$ and $Tf_t(x)$ is the same as in [1].

Estimation of Tg_t . By the boundedness of the operator T in the space $L^{p(\cdot)}(\mathbb{R}^n)$ we obtain $\|(Tg_t)\chi_{t,2t}\|_{L^{p(\cdot)}} \leq C \|g_t\|_{L^{p(\cdot)}} = C \|f\chi_{\gamma't,\delta't}\|_{L^{p(\cdot)}}$. Then

$$N_{(1,2)}(Tg_t)_{p,q,\alpha} \leq C \|f\|_{H_{(\gamma',\delta')}^{p,q,\alpha}}.$$

In the case where q is constant, we take γ' sufficiently close to 1 and δ' close to 2, so that $\frac{\delta'}{\gamma'} < 4$ and then

$$\|(Tg_t)\chi_{t,2t}\|_{L^{p(\cdot)}} \leq C \|f\chi_{\gamma't,\delta't}\|_{L^{p(\cdot)}} \leq C \sum_{j=-1}^1 \|f\chi_{2^j t, 2^{j+1} t}\|_{L^{p(\cdot)}}.$$

Consequently,

$$\begin{aligned} N_{(1,2)}(Tg_t)_{p,q,\alpha} &\leq C \|t^{\alpha\infty}\|_{L^{p(\cdot)}} \|f\chi_{\frac{t}{2},t}\|_{L^{p(\cdot)}} + C \|t^{\alpha\infty}\|_{L^q((2,\infty);\bar{d}t)} \|f\chi_{t,2t}\|_{L^{p(\cdot)}} \\ &\leq C \|t^{\alpha\infty}\|_{L^q((1,2);\bar{d}t)} \|f\chi_{t,2t}\|_{L^{p(\cdot)}} + C \|t^{\alpha\infty}\|_{L^q((2,\infty);\bar{d}t)} \|f\chi_{t,2t}\|_{L^{p(\cdot)}}. \end{aligned}$$

Since $\|f\chi_{t,2t}\|_{L^{p(\cdot)}} \leq \|f\chi_{0,4}\|_{L^{p(\cdot)}}$, we obtain

$$N_{(1,2)}(Tg_t)_{p,q,\alpha} \leq C \|f\|_{H_{(1,2)}^{p,q,\alpha}}$$

with Lemma 3.2 of [1] taken into account.

Estimation of Th_t . We take $x \in R_{t,2t}$ and have:

$$|Th_t(x)| \leq C \int_{|y|>\delta't} \frac{|f(y)| dy}{|x-y|^n}.$$

To proceed, we need the inequality

$$\int_{|y|>t} |\Phi(y)| dy \leq \frac{1}{|\ln \lambda|} \int_{\lambda t}^{\infty} \frac{d\tau}{\tau} \int_{\tau < |y| < 2\tau} |\Phi(y)| dy, \quad \frac{1}{2} \leq \lambda < 1, \quad (1.9)$$

the proof of which is straightforward:

$$\begin{aligned} \int_{\lambda t}^{\infty} \frac{d\tau}{\tau} \int_{\tau < |y| < 2\tau} |\Phi(y)| dy &= \int_{|y|>\lambda t} |\Phi(y)| dy \int_{\max(\lambda t, \frac{|y|}{2})}^{\frac{|y|}{2}} \frac{d\tau}{\tau} \\ &= \int_{\lambda t < |y| < 2\lambda t} |\Phi(y)| \ln \frac{|y|}{\lambda t} dy + \ln 2 \int_{|y|>2\lambda t} |\Phi(y)| dy \\ &\geq \ln \frac{1}{\lambda} \int_{t < |y| < 2\lambda t} |\Phi(y)| dy + \ln 2 \int_{|y|>2\lambda t} |\Phi(y)| dy \end{aligned}$$

$$\geq \ln \frac{1}{\lambda} \int_{|y|>t} |\Phi(y)| dy.$$

By means of (1.9) we then get

$$|Th_t(x)| \leq C \int_{\lambda\delta't}^{\infty} \left(\int_{R_{\tau,2\tau}} \frac{|f(y)|}{|x-y|^n} dy \right) \bar{d}\tau.$$

Here $|x-y| \geq |y|-|x| \geq \tau-2t \geq \tau - \frac{2}{\lambda\delta'}\tau = \frac{\lambda\delta'-2}{\lambda\delta'}\tau$. We may choose λ sufficiently close to 1 so that $\lambda\delta' > 2$ and then, with the property (4.17) of [1] taken into account, we obtain

$$\begin{aligned} t^{\alpha_\infty} \|\chi_{R_{t,2t}} Th_t(x)\|_{L^{p(\cdot)}} &\leq C t^{\alpha_\infty} \|\chi_{R_{t,2t}}\|_{p(\cdot)} \int_{2t}^{\infty} \tau^{-\frac{n}{p_\infty}} \|f\chi_{\tau,2\tau}\|_{L^{p(\cdot)}} \bar{d}\tau \\ &\leq C \int_t^{\infty} \left(\frac{t}{\tau}\right)^{\alpha_\infty + \frac{n}{p_\infty}} \psi(\tau) \bar{d}\tau, \end{aligned}$$

where $\psi(\tau) = \tau^{\alpha_\infty} \|f\chi_{R_{\tau,2\tau}}\|_{p(\cdot)} \chi_{(2,\infty)}(\tau)$. Thus we have arrived at a Hardy type inequality. Since $\alpha_\infty + \frac{n}{p_\infty} > 0$, the operator on the right-hand side is bounded in the space $L^{q(\cdot)}((2,\infty); \bar{d}t)$ by Corollary 4.5 of [1]. Consequently,

$$\left\| t^{\alpha_\infty} \|\chi_{R_{t,2t}} Th_t(x)\|_{L^{p(\cdot)}} \right\|_{L^{q(\cdot)}((2,\infty); \bar{d}t)} \leq C \|\psi\|_{L^{q(\cdot)}(\mathbb{R}_+; \bar{d}t)} \leq C \|f\|_{H_2^{p,q,\alpha}}.$$

□

A similar modification in the form (1.5) should be also made in the boundedness statements in applications to maximal and singular operators in Corollaries 5.2 and 5.3 of [1].

Finally, we use this opportunity to note a misprint in formula (3.4) of [1]: the inequality $|\alpha(t) - \alpha_{\pm\infty}| \leq \frac{A_{\pm}}{\ln(e+t)}$ must be replaced by a couple of inequalities $|\alpha(t) - \alpha(+\infty)| \leq \frac{A_{+\infty}}{\ln(e+t)}$ and $|\alpha(t) - \alpha(0)| \leq \frac{A_0}{\ln(e+t)}$.

References

- [1] Stefan Samko, Variable exponent Herz spaces. *Mediterr. J. Math.*, DOI: 10.1007/s00009-013-0285-x, 2013.

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