On the Nilpotency Class of a Generalized 3-Abelian Group

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Abstract. A group G is called 3-abelian if the map $x \mapsto x^3$ is an endomorphism of G and it is called generalized 3-abelian, if there exist elements $c_1, c_2, c_3 \in G$ such that the map $\varphi : x \mapsto x^{c_1} x^{c_2} x^{c_3}$ is an endomorphism of G . Abdollahi, Daoud and Endimioni have proved that a generalized 3-abelian group G is nilpotent of class at most 10. Here, we improve the bound to 3 and we show that the exponent of its derived subgroup is finite and divides 9. We also prove that G is 3-Levi, 9-central, 9-abelian and 3-nilpotent of class at most 2.

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1. Introduction and Results

Let $n \geq 2$ be an integer. A group G is called n -abelian whenever $(xy)^n = x^n y^n$ for all $x, y \in G$; or equivalently the map $x \mapsto x^n$ is an endomorphism of G . Levi [7] has proved that a group G is 3-abelian if and only if it is 2-Engel and the exponent of its derived subgroup $[G, G]$ divides 3. Trotter [10] has proved that a 3-abelian group G is abelian whenever the map $x \mapsto x^3$ is an automorphism of G . A group G is called generalized n -abelian whenever there exist elements $c_1, \dots, c_n \in G$ such that the map $x \mapsto x^{c_1} \dots x^{c_n}$ is an endomorphism of G . The class of generalized n -abelian groups is closed under the formation images, and finite direct products. Obviously, every n -abelian group is generalized n -abelian and it is easy to see that every generalized 2-abelian group is abelian. It is clear that by conjugating we may assume one of $c_1, \dots, c_n$ to be the trivial element.

Abdollahi, Daoud and Endimioni [1, Theorem 3.1] have proved that a generalized 3-abelian group G is nilpotent of class at most 10, and abelian

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whenever the map $x \mapsto x^{c_1} x^{c_2} x^{c_3}$ is a monomorphism of G or the center $\zeta(G)$ has no element of order 3. Here we improve the previous results as following.

Theorem 1.1. *Let G be a generalized 3-abelian group. Then*

- i) G is nilpotent of class at most 3.*
- ii) The exponent of $[G, G]$ divides 9 and the exponent of $\gamma_3(G)$ divides 3.*

The free 4-generator group of exponent 3 is clearly a 3-abelian group and it is well known that it is nilpotent of class 3. Hence the bound 3 for the class of generalized 3-abelian groups is sharp.

Now for any non-zero integer n , a group G is called n -Levi if $[x, y^n] = [x, y]^n$ for all $x, y \in G$. It is called n -central if $n \geq 1$ and $[x, y^n] = 1$ for all $x, y \in G$. We show that there is a relation between generalized 3-abelian groups G and n -Levi or n -central groups.

Theorem 1.2. *Let G be a generalized 3-abelian group admitting an endomorphism of the form*

$$\phi : x \longmapsto x^a x x^b$$

for some $a, b \in G$. Then

- i) G is 3-Levi, 9-central and 9-abelian.*
- ii) The subgroup $\text{Im}\phi$ is abelian. In particular, if ϕ is injective or surjective, G is abelian.*

Let $m \neq 0$ be an integer. Baer [2] introduced the m -center of a group G as follows:

$$\mathcal{Z}(G, m) = \{a \in G \mid (ax)^m = a^m x^m \text{ and } (xa)^m = x^m a^m \text{ for all } x \in G\}.$$

The set $\mathcal{Z}(G, m)$ is a characteristic subgroup of G for any non-zero integer m . L.-C. Kappe and M. L. Newel [5] proved that

$$(ax)^m = a^m x^m \text{ for all } x \in G \iff (xa)^m = x^m a^m \text{ for all } x \in G.$$

Thus only one of the m -commutativity conditions suffices to define the m -center $\mathcal{Z}(G, m)$. If m is a positive integer, the upper m -central series $\mathcal{Z}_i(G, m)$ is defined inductively as the following: $\mathcal{Z}_0(G, m) = 1$, $\mathcal{Z}_1(G, m) = \mathcal{Z}(G, m)$ and $\mathcal{Z}_{i+1}(G, m) / \mathcal{Z}_i(G, m) = \mathcal{Z}(G / \mathcal{Z}_i(G, m), m)$ for $i \geq 1$.

We then get an ascending series.

$$1 = \mathcal{Z}_0(G, m) \leq \mathcal{Z}_1(G, m) \leq \dots \leq \mathcal{Z}_i(G, m) \leq \mathcal{Z}_{i+1}(G, m) \leq \dots$$

A group G is said to be m -nilpotent of class at most k , if $\mathcal{Z}_k(G, m) = G$. Karasev [6] introduced the m -derived subgroup of a group G as follows: For a given integer $m \geq 2$ and two elements x, y of G , the m -commutator of x and y is defined by $[x, y]_m = (xy)^m y^{-m} x^{-m}$. The m -derived subgroup of G is then the subgroup generated by the set of the m -commutators of G . It is a fully invariant subgroup of G and is denoted by $[G, G]_m$. If H is a normal subgroup of G , then G/H is m -abelian if and only if $[G, G]_m \leq H$. $G/[G, G]$ being abelian, $[G, G]_m \leq [G, G]$. We have the following theorem:

Theorem 1.3. *Every generalized 3-abelian group is 3-nilpotent of class at most 2. Moreover the exponent of its 3-derived subgroup divides 3.*

2. Proofs

Notations used in this paper are standard. For a group G and the elements $x_1, x_2, \dots, x_n, x, y \in G$, the commutators $[x_1, x_2, \dots, x_n]$ and $[x_n, y]$ are defined inductively by the rules:

$$[x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2, \quad [x_1, y] = [x, y] = x^{-1}xy$$

and for $n \geq 2$ $[x_1, x_2, \dots, x_{n+1}] = [[x_1, x_2, \dots, x_n], x_{n+1}]$, $[x_n, y] = [[x_{n-1}, y], y]$.

For a given integer $i \geq 1$, we denote by $[G, G]$, $\zeta_i(G)$ and $\gamma_3(G)$ respectively the derived subgroup, the i th-center and the third term of the lower central series of G . We denote by $H \leq G$ if H is a subgroup of G . To prove Theorems 1.1, 1.2 and 1.3 we need the following lemmas.

Lemma 2.1. *Let G be a generalized n -abelian group admitting an endomorphism of the form $\psi : x \mapsto x^{c_1} \cdots x^{c_n}$ where $c_1, \dots, c_n \in G$. Then, G is n -abelian whenever $c_1, \dots, c_n \in \zeta_2(G)$.*

Proof. Let $x \in G$. As $c_1, \dots, c_n \in \zeta_2(G)$ we have that $x, [x, c_1], \dots, [x, c_n]$ commute. Thus

$$x^\psi = \prod_{1 \leq i \leq n} x [x, c_i] = x^n \prod_{1 \leq i \leq n} [x, c_i].$$

Therefore

$$\begin{aligned} (xy)^\psi &= (xy)^n \prod_{1 \leq i \leq n} [xy, c_i] = (xy)^n \prod_{1 \leq i \leq n} [x, c_i][y, c_i] \\ x^\psi y^\psi &= x^n \prod_{1 \leq i \leq n} [x, c_i] y^n \prod_{1 \leq i \leq n} [y, c_i] = x^n y^n \prod_{1 \leq i \leq n} [x, c_i][y, c_i] \end{aligned}$$

whence $(xy)^n = x^n y^n$. □

Lemma 2.2. *Let G be a metabelian group, $x, y \in G$ and $u, v \in [G, G]$. Then, for any integer $n \geq 1$, the following assertions hold.*

- (a) $[uv, x] = [u, x][v, x]$.
- (b) $[x, y^n] = \prod_{1 \leq i \leq n} [x, y]^{(n \choose i)}$.
- (c) $(xy^{-1})^n = x^n \prod_{0 < i+j < n} [x, y]^{(n \choose i+j+1)} y^{-n}$.

Proof. (a) is easy to prove as $[G, G]$ is abelian. For the proofs of (b) and (c) see [4]. □

Proof of Theorem 1.1. i) Let G be a generalized 3-abelian group with the given endomorphism ϕ defined by $x^\phi = x^a x x^b$ for all $x \in G$, where $a, b \in G$ are fixed. To prove that G is nilpotent of class three, it is enough to show that every 4-generated subgroup $\langle g_1, g_2, g_3, g_4 \rangle$ of G is nilpotent of class at most 3. Now $H = \langle g_1, g_2, g_3, g_4, a, b \rangle$ is clearly invariant under the action of ϕ and is thus a generalized 3-abelian group as well. We can thus replace G by H and it suffices then to show that H is nilpotent of class at most 3. By [1, Théorème 3.1], H is nilpotent. Now one can use **ng** package of Werner Nickel [8] implemented in **GAP** [9] and **MAGMA** [3] to find the nilpotency class

of H . The package `nq` has the capability of computing the largest nilpotent quotient (if it exists) of a finitely generated group with finitely many identical relations and finitely many relations. For example, if we want to construct the largest nilpotent quotient of a group G with the following presentation

$$\langle x_1, \dots, x_n \mid r_1(x_1, \dots, x_n) = \dots = r_m(x_1, \dots, x_n) = 1, \\ w(x_1, \dots, x_n, y_1, \dots, y_k) = 1 \rangle,$$

where $r_1, \dots, r_m$ are relations on $x_1, \dots, x_n$ and $w(x_1, \dots, x_n, y_1, \dots, y_k) = 1$ is an identical relation in the group $\langle x_1, \dots, x_n \rangle$, one may apply the following code to use the package `nq` in GAP:

```
LoadPackage("nq"); #nq package of Werner Nickel
F:=FreeGroup(n+k);
L:=F/[r1(F.1,...,F.n),...,rm(F.1,...,F.n),
      w(F.1,...,F.n,F.(n+1),...,F.(n+k))];
H:=NilpotentQuotient(L,[F.(n+1),...,F.(n+k)]);
```

Note that we need to construct the free group of rank $n+k$ because as well as the n generators for G we also have an identical relation with k free variables. Note that the function `NilpotentQuotient(L)` attempts to compute the largest nilpotent quotient of L and it will terminate only if L has a largest nilpotent quotient. Note that our identical relation is $(xy)^\phi = x^\phi y^\phi$ for all $x, y \in G$, which can be written as follows:

$$(xy)^a (xy) (xy)^b = (x^a x x^b) (y^a y y^b)$$

or

$$y^a (xy) x^b (x x^b y^a y)^{-1} = 1.$$

So for our problem we need the following code:

```
LoadPackage("nq"); #nq package of Werner Nickel
F:=FreeGroup(8); x:=F.1; y:=F.2, a:=F.3; b:=F.4;
L:=F/[(F.2^F.3)*(F.1*F.2)*(F.1^F.4)*(F.1*(F.1^F.4)
      *(F.2^F.1)*F.2)^-1];
H:=NilpotentQuotient(L,[F.1,F.2]);
NilpotencyClassOfGroup(H);
```

This revealed that the class of H is 3.

ii) ϕ induces on the quotient group $G/\gamma_3(G)$ the endomorphism

$$\bar{x} \longmapsto \bar{x}^{\bar{a}} \bar{x} \bar{x}^{\bar{b}}.$$

$G/\gamma_3(G)$ is nilpotent of class at most 2; by lemma 2.1, it is 3-abelian. By Levi's Theorem, $\exp\left(\left[\frac{G}{\gamma_3(G)}, \frac{G}{\gamma_3(G)}\right]\right)$, which is equal to $\exp\left(\frac{[G,G]\gamma_3(G)}{\gamma_3(G)}\right)$, divides 3. Then $[G, G]^3 \leq \gamma_3(G)$ (*). It follows that $[G, G]^3 \leq \zeta(G)$ since $\gamma_3(G) \leq \zeta(G)$. Now let $x, y, z \in G$.

$$[x, y, z]^3 = [[x, y]^3, z] = 1$$

and so $\gamma_3(G)^3 = \{1\}$. From the relation (*), we get $[G, G]^9 \leq \gamma_3(G)^3$. Hence $[G, G]^9 = \{1\}$.

This completes the proof. $\square$

Proof of Theorem 1.2. i) Let $x, y \in G$.

$$[x, y^3] = \prod_{1 \leq i \leq 3} [x, {}_i y]^{(3)} = [x, y]^3 [x, y, y]^3 [x, y, y, y] = [x, y]^3.$$

$$[x, y^9] = \prod_{1 \leq i \leq 9} [x, {}_i y]^{(9)} = [x, y]^9 [x, y, y]^{36} \prod_{3 \leq i \leq 9} [x, {}_i y]^{(9)} = 1.$$

$$\begin{aligned} (xy^{-1})^9 &= x^9 \prod_{1 \leq i+j \leq 8} [x, {}_i y, {}_j x]^{(i+j+1)} y^{-9} \\ &= x^9 [x, y]^{36} [x, y, x]^{84} [x, y, y]^{84} \prod_{3 \leq i+j \leq 8} [x, {}_i y, {}_j x]^{(i+j+1)} y^{-9} = x^9 y^{-9}. \quad \square \end{aligned}$$

ii) Let $x, y \in G$. From part (i) and Theorem 1.1, we know that G is nilpotent of class at most 3 and $\gamma_3(G)^3 = \{1\}$. Therefore

$$x^\phi = x[x, a]x^2[x, b] = x^3[x, a][x, b][x, a, x]^2.$$

Using furthermore the facts from part (i) and Theorem 1.2 that $[G, G]^9 = \{1\}$ and that G is 3-Levi we get

$$[x^\phi, y^\phi] = [x^3[x, a][x, b], y^3[y, a][y, b]] = [x^3, y^3] = [x^3, y]^3 = [x, y]^9 = 1.$$

This completes the proof. $\square$

Proof of Theorem 1.3. We have seen that the quotient group $G/\gamma_3(G)$ is 3-abelian. So $[G, G]_3 \leq \gamma_3(G)^{(**)}$ and $\frac{G}{\mathcal{Z}(G, 3)}$ (which is isomorphic to $\frac{\frac{G}{\gamma_3(G)}}{\frac{\mathcal{Z}(G, 3)}{\gamma_3(G)}}$) also is 3-abelian. Therefore the series

$$1 \leq \mathcal{Z}(G, 3) \leq G$$

is 3-central and G is 3-nilpotent of class at most 2. The relation (**) yields $([G, G]_3)^3 \leq \gamma_3(G)^3 = 1$. This completes the proof. $\square$

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References

- [1] A. Abdollahi, B. Daoud and G. Endimioni, *Groupes n -abéliens généralisés*. Bull. Belg. Math. Soc. **13** (2006), 287-294.
- [2] R. Baer, *Factorization of n -soluble and n -nilpotent groups*. Proc. Amer. Math. Soc. **4** (1953), 15-26.
- [3] Wieb Bosma, John Cannon and Catherine Playoust, *The Magma algebra system I: The user language*. J. Symbolic Comput. **24** (1997), 235-265.

- [4] G. T. Hogan and W. P. Kappe, *On the H_p -problem for finite p -groups*. Proc. Amer. Math. Soc. **20** (1969), 450-454.
- [5] L. C. Kappe. and M. L. Newell, *On the n -centre of a group*. In: "Groups – St. Andrews 1989", Vol. 2, 339–352, London Math. Soc. Lecture Note Ser. **160**, Cambridge Univ. Press, Cambridge, 1991.
- [6] G. A. Karasev, *On the theory of n -nilpotent groups*. Mat. Zamretki **5** (1969), 653-654.
- [7] F. W. Levi, *Notes on group theory I, II*. J. Indian Math. Soc. (N.S.) **8** (1944), 1-9.
- [8] W. Nickel, NQ, 1998, *A refereed GAP 4 package*. see [9].
- [9] The GAP Group, *GAP-Groups, Algorithms, and Programming*. Version 4.4.12; 2008. [Http://www.gap-system.org](http://www.gap-system.org).
- [10] H. F. Trotter, *Groups in which raising to a power is an automorphism*. Canad. Math. Bull. **8** (1965), 825-827.

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