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On Monotonic Integrable Solutions for Quadratic Functional Integral Equations

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Abstract. We study the solvability of functional quadratic integral equations in the space of integrable functions on the interval $I = [0, 1]$. We concentrate on a.e. monotonic solutions for considered problems. The existence result is obtained under the assumption that the functions involved in the investigated equation satisfy Carathéodory conditions. As a solution space we consider both $L_1(I)$ and $L_p(I)$ spaces for $p > 1$.

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1. Introduction

Linear and nonlinear integral equations are considered as a branch of the applications of functional analysis. This branch is a great importance not only for the specialist in this field but also for those whose interest lies in other branch of mathematics with especial reference to mathematical physics, engineering and biology.

The object of this paper is to study the solvability of a nonlinear Urysohn functional integral equation

$$x(t) = f_1(t, x(\phi_1(t))) + f_2(t, x(t)) \int_0^1 u(t, s, x(\phi_2(s))) ds, \quad t \in I. \quad (1.1)$$

Special cases for considered equation (quadratic integral equations) were investigated in connection with some applications of such a kind of problems in the theories of radiative transfer, neutron transport and in the kinetic theory of gases (cf. [5, 15, 22, 23]). More general problem (motivated by some practical interests in plasma physics) was investigated in [39]. The existence of continuous solutions for particular cases of the considered problem was

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investigated since many years (see [18, 35] or a very recent paper [2]). On the other hand, different kind of integral equations (including quadratic integral equations) should be investigated in different function spaces. This was remarked, for instance, in [39, Theorem 3.14] for the case of $L_p(I)$ -solutions, for the Hammerstein integral equation see also [35, 42] for L_p -solutions or [6, 31, 45] for integrable solutions. A very interesting survey about different classes of solutions (not only in $C(I)$ or $L_p(I)$, but also in Orlicz spaces $L_\varphi(I)$ or even in ideal spaces) for a class of integral equations related to our equation can be found in [4].

Next, let us recall that the equations involving the functional dependence have still growing number of applications (cf. [33]). We try to cover the results of this type. Let us mention, for example, the results from [7, 14].

We are interested in monotonic solutions of the above problem. The considered problem can cover, for instance, as particular cases:

1. $f_1(t, x) = g(t)$, $f_2(t, x) = \lambda$ the functional Urysohn integral equation ([7, 9, 14]),
2. $f_1(t, x) = g(t)$, $f_2(t, x) = x$, $\phi_2(t) = t$ the functional-integral equation ([41]),
3. $f_2(t, x) = 0$ the abstract functional equation ([9], for instance),
4. for continuous solutions with $\phi_1(t) = \phi_2(t) = t$ and $u(t, s, x) = \frac{u_1(t, s, x)}{\Gamma(\alpha) \cdot (t - \tau)^{1-\alpha}}$ see [20, 30],
5. $f_2(t, x) = \lambda$ the functional integral equation (for continuous solutions see [1, 12, 28]),
6. $f_2(t, x) = x$ the quadratic (functional) Urysohn integral equation ([16, 15], for instance).

Our problem, as well as, the particular cases was investigated mainly in cases when the solutions are elements of the space of continuous functions. Thus the proofs are based on very special properties of this space (the compactness criterion, in particular), cf. [20, 40].

On the other hand, by the practical interest it is worthwhile to consider discontinuous solutions. Here we are looking for integrable solutions. Thus the operators F_1 , F_2 and U should take their values in the space $L_1(I)$. Let us recall that we are interested in finding monotonic solutions (a.e. monotonic in the case of integrable solutions). In such a case discontinuous solutions are expected even in a simplest case i.e. when

$$f_1(t, x) = h(t) = \begin{cases} 0 & t \text{ is rational,} \\ t & t \text{ is irrational} \end{cases}$$

An interesting example of discontinuous solutions for integral equations is taken from [39, Example 3.5]:

$$\chi_{[1/2, 1]}(t) \cdot (2t - 1) \cdot x(t) + \chi_{[0, 1/2]}(t) \cdot (1 - 2t) \cdot (x(t) - 1) \int_0^1 (1 - x(s)) ds = 0.$$

In the paper [24], we study the particular case of the above problem on \mathbb{R}^+ when $f_1(t, x) = g(t)$ and $f_2(t, x) = x$. Here we extend the earlier result by

considering functional integral equation in a more general form. Moreover, we prove the existence of solutions in some subspaces of $L_1(0, 1)$.

Let us add a few comments about functional dependence, i.e. functions ψ_1 and ψ_2 . Our set of assumptions is based on the paper [14]. Functions of the form $\psi_i(t) = t^\alpha$ ($\alpha > 0$) or $\psi_i(t) = t - \tau(t)$ with some set of assumptions for τ are most important cases covered in our paper. Let us note that functional equations with state dependent delay are very useful in many mathematical models including the population dynamics, the position control or the cell biology. A very interesting survey about such a theory and their applications can be found in [33].

The last aspect of our results is to investigate the monotonicity property of solutions. This is important property and there are many papers devoted to its study. Let us note some recent ones [16, 17, 24, 30], for instance.

The results obtained in the current paper create some extensions for several known ones i.e. in addition to those mentioned previously also for the results from earlier papers or books ([3, 9, 21, 27, 34, 43, 44, 48], for example).

2. Notation and Auxiliary Facts

Let \mathbb{R} be the field of real numbers, \mathbb{R}^+ be the interval $[0, \infty)$ and $L_1(I)$ be the space of Lebesgue integrable functions (equivalence classes of functions) on a measurable subset I of \mathbb{R} , with the standard norm

$$\|x\| = \int_I |x(t)| dt.$$

Recall that by L_p we will denote the space of (equivalence classes of) functions x satisfying $\int_0^1 |x(s)|^p ds < \infty$. In this paper we will denote by I an interval $[0, 1]$. By $\|\cdot\|_p$ we will denote the norm in $L_p(I)$.

One of the most important operator studied in nonlinear functional analysis is the so-called superposition operator [3].

Definition 2.1. Assume that a function $f : I \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the Carathéodory conditions i.e. it is measurable in t for any $x \in \mathbb{R}$ and continuous in x for almost all $t \in I$. Then to every function $x(t)$ being measurable on I we may assign the function

$$F_f(x)(t) = f(t, x(t)), \quad t \in I.$$

The operator F_f defined in such a way is called the superposition (Nemytskii) operator generated by the function f .

Furthermore, for every $f \in L_1$ and every $\phi : I \rightarrow I$ we define the superposition operator generated by the functions f and ϕ , $F_{\phi, f} : L_1(I) \rightarrow L_1(I)$ as

$$F_{\phi, f}(t) = f(t, x(\phi(t))) \quad , \quad t \in I$$

In $L_p(I)$ we have the “automatic” continuity of the Nemytskii operator ([3, 36]):

Theorem 2.2. *Let f satisfies the Carathéodory conditions. The superposition operator F generated by the function f maps continuously the space $L_p(I)$ into $L_q(I)$ ($p, q \geq 1$) if and only if*

$$|f(t, x)| \leq a(t) + b \cdot |x|^{\frac{p}{q}}, \tag{2.1}$$

for all $t \in I$ and $x \in \mathbb{R}$, where $a \in L_q(I)$ and $b \geq 0$.

It should be also noted that the superposition operator F takes its values in $L_\infty(I)$ iff the generating function f is independent on x (cf. [3, Theorem 3.17]). This remark allows us to reduce the number of the considered cases.

Let $S = S(I)$ denotes the set of measurable (in Lebesgue sense) functions on I and let $meas$ stand for the Lebesgue measure in \mathbb{R} . Identifying the functions equal almost everywhere the set S furnished with the metric

$$d(x, y) = \inf_{a>0} [a + meas\{s : |x(s) - y(s)| \geq a\}],$$

we obtain a complete metric space. Moreover, the convergence in measure on I is equivalent to the convergence with respect the metric to d (Proposition 2.14 in [46]). The compactness in such a space is called a “compactness in measure” and such sets have very nice properties when considered as subsets of L_p -spaces of integrable functions ($p \geq 1$).

The following theorems give different sufficient conditions for compactness in measure that will be more convenient for our discussion ([8, 37]).

Theorem 2.3. *Let X be a bounded subset of $L_1(I)$ and suppose that there is a family of measurable subsets $(\Omega_c)_{0 \leq c \leq 1}$ of the interval I such that $meas\Omega_c = c$ for every $c \in I$ and for $x \in X$*

$$x(t_1) \geq x(t_2), \quad (t_1 \in \Omega_c, t_2 \notin \Omega_c).$$

Then this family is equimeasurable and the set X is compact in measure in $L_1(I)$.

It is clear that by putting $\Omega_c = [0, c) \cup E$ or $\Omega_c = [0, c) \setminus E$, where E is a set with measure zero, this family contains nonincreasing functions (possibly except for a set E). We will call the functions from this family “a.e. nonincreasing” functions. This is the case, when we choose an integrable and nonincreasing function y and the all the functions equal a.e. to y satisfies the above condition. Thus we can write that elements from $L_1(I)$ belong to this class of functions. Due to the compactness criterion in the space of measurable functions (with the topology convergence in measure) (see Lemma 4.1 in [8]) we have a desired theorem concerning the compactness in measure of a subset X of $L_1(I)$ (cf. [8, Corollary 4.1] or [29, Section III.2]). Let us recall, in metric spaces the set U_0 is compact if and only if each sequence from U_0 has a subsequence that converges in U_0 (i.e. sequentially compact).

Lemma 2.4. *Let X be a bounded subset of $L_p(I)$ consisting of functions which are a.e. nondecreasing (or a.e. nonincreasing) on the interval I . Then X is compact in measure in $L_p(I)$.*

Proof. Let $R > 0$ be such that $X \subset B_R \subset L_p(I)$. It is known that X is compact in measure as a subset of S . Since the compactness in measure is equivalent to sequential compactness, we are interested in studying the properties of the latter on. By taking an arbitrary sequence (x_n) in X we obtain that there exists a subsequence (x_{n_k}) convergent in measure to some x in the space S . Since the balls in $L_p(I)$ spaces ($p \geq 1$) are closed in the topology of convergence in measure, we obtain $x \in B_R \subset L_p(I)$ and finally $x \in X$. □

In the paper we will need to distinguish between two different cases: when an operator take their values in Lebesgue spaces $L_p(I)$ or in a space of essentially bounded functions $L_\infty(I)$ (for Nemytskii operators see Theorem 2.2). For Urysohn operators the continuity is not “automatic” as in the case of superposition operators. Let us recall an important sufficient condition:

Theorem 2.5. [37, Theorem 10.1.10] *Let $u : I \times I \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies Carathéodory conditions i.e. it is measurable in (t, s) for any $x \in \mathbb{R}$ and continuous in x for almost all $(t, s) \in I \times I$. Assume that $U(x)(t) = \int_I u(t, s, x(s))ds$ maps $L_p(I)$ into $L_q(I)$ ($q < \infty$) and for each $h > 0$ the function*

$$R_h(t, s) = \max_{|x| \leq h} |u(t, s, x)|$$

is integrable on s for a.e. $t \in I$. If moreover for each $h > 0$ this operator satisfies

$$\lim_{meas D \rightarrow 0} \sup_{|x| \leq h} \left\| \int_D u(t, s, x(s))ds \right\|_{L_q(I)} = 0$$

and for arbitrary non-negative $z(t) \in L_p(I)$

$$\lim_{\delta \rightarrow 0} \left\| \int_D \sup_{|x| \leq z} \left\| \int_D u(t, s, x(s)) ds \right\|_{L_q(I)} = 0,$$

then U is a continuous operator.

The first two conditions are satisfied when $\int_I R_h(t, s)ds \in L_q(I)$, for instance.

We will use also the majorant principle for Urysohn operators (cf. [37, Theorem 10.1.11]). The following theorem which is a particular case of much more general result ([37, Theorem 10.1.16]), will be very useful in the proof of the main result for operators in $L_\infty(I)$:

Theorem 2.6. [37] *Let $u : I \times I \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies Carathéodory conditions i.e. it is measurable in (t, s) for any $x \in \mathbb{R}$ and continuous in x for almost all (t, s) . Assume that*

$$|u(t, s, x)| \leq k(t, s) \cdot (a(s) + b \cdot |x|),$$

where the nonnegative function k is measurable in (t, s) , a is a positive integrable function, $b > 0$ and such that the linear integral operator with the

kernel $k(t, s)$ maps $L_1(I)$ into $L_\infty(I)$. Then the operator U maps $L_1(I)$ into $L_\infty(I)$. Moreover, if for arbitrary $h > 0$

$$\lim_{\delta \rightarrow 0} \left\| \int_D \max_{|x_1| \leq h, |x_1 - x_2| \leq \delta} |u(t, s, x_1) - u(t, s, x_2)| ds \right\|_{L_\infty(I)} = 0,$$

then U is a continuous operator.

We mention also that some particular conditions guaranteeing the continuity of the operator U may be found in [47, 48].

Let us recall some properties of operators preserving monotonicity properties of functions.

Lemma 2.7. [9, Lemma 4.2] *Suppose the function $t \rightarrow f(t, x)$ is a.e. nonincreasing on a finite interval I for each $x \in \mathbb{R}$ and the function $x \rightarrow f(t, x)$ is a.e. nonincreasing on \mathbb{R} for any $t \in I$. Then the superposition operator F generated by f transforms functions being a.e. nonincreasing on I into functions having the same property.*

We will use the fact that the superposition operator takes the bounded sets compact in measure into the sets with the same property.

Lemma 2.8. [37, Lemma 17.5] *Assume that a function $f : I \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies Carathéodory conditions. The superposition operator F maps a sequence of functions convergent in measure into a sequence of functions convergent in measure.*

Thus we can prove the following (cf. [24, Proposition 4.1]):

Proposition 2.9. *Assume that a function $f : I \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies Carathéodory conditions and the function $t \rightarrow f(t, x)$ is a.e. nonincreasing on a finite interval I for each $x \in \mathbb{R}$ and the function $x \rightarrow f(t, x)$ is a.e. nonincreasing on \mathbb{R} for any $t \in I$. Assume that $F : L_p(I) \rightarrow L_q(I)$. Then $F(V)$ is compact in measure for arbitrary bounded and compact in measure subset V of $L_p(I)$.*

Proof. Let V be a bounded and compact in measure subset of $L_p(I)$. By our assumption $F(V) \subset L_q(I)$. It is known that as a subset of S the set $F(V)$ is compact in measure (cf. [8]). It was noted that the topology of convergence in measure is metrizable, so the compactness of the set is equivalent with the sequential compactness. By taking an arbitrary sequence $(y_n) \subset F(V)$ we get a sequence (x_n) in V such that $y_n = F(x_n)$. Since $(x_n) \subset V$, as follows from Lemma 2.8 F transforms this sequence into the sequence convergent in measure. Thus (y_n) is compact in measure, so is $F(V)$. \square

For the integral operator of the form $K_0(x)(t) = \int_I k(t, s)x(s) ds$ we have the following theorem due to Krzyż ([38, Theorem 6.2]):

Theorem 2.10. *The operator K_0 preserves the monotonicity of functions iff*

$$\int_0^b k(t_1, s) ds \geq \int_0^b k(t_2, s) ds$$

for $t_1 < t_2$, $t_1, t_2 \in I$ and for any $b \in I$.

Next, we give some definitions and results which will be needed further on. Assume that $(E, \|\cdot\|)$ is an arbitrary Banach space with zero element θ . Denote by $B(x, R)$ the closed ball centered at x and with radius R . The symbol B_R stands for the ball $B(\theta, R)$.

If X is a subset of E , then \bar{X} and $\text{conv}X$ denote the closure and convex closure of X , respectively. We denote the standard algebraic operations on sets by the symbols λX and $X + Y$. Moreover, we denote by M_E the family of all nonempty and bounded subsets of E and N_E its subfamily consisting of all relatively compact subsets.

Now we present the concept of a regular measure of noncompactness:

Definition 2.11. [13] A mapping $\mu : M_E \rightarrow [0, \infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

- (i) $\mu(X) = 0 \Leftrightarrow X \in N_E$.
- (ii) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.
- (iii) $\mu(\bar{X}) = \mu(\text{conv}X) = \mu(X)$.
- (iv) $\mu(\lambda X) = |\lambda| \mu(X)$, for $\lambda \in \mathbb{R}$.
- (v) $\mu(X + Y) \leq \mu(X) + \mu(Y)$.
- (vi) $\mu(X \cup Y) = \max\{\mu(X), \mu(Y)\}$.
- (vii) If X_n is a sequence of nonempty, bounded, closed subsets of E such that $X_{n+1} \subset X_n$, $n = 1, 2, 3, \dots$, and $\lim_{n \rightarrow \infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

An example of such a mapping is the following:

Definition 2.12. [13] Let X be a nonempty and bounded subset of E . The Hausdorff measure of noncompactness $\chi(X)$ is defined as

$$\chi(X) = \inf\{r > 0 : \text{there exists a finite subset } Y \text{ of } E \text{ such that } X \subset Y + B_r\}.$$

Another regular measure was defined in the space $L_1(I)$ (cf. [17]). For any $\varepsilon > 0$, let c be a measure of equiintegrability of the set X (the so-called Sadovskii functional [3, p. 39]) i.e.

$$c(X) = \lim_{\varepsilon \rightarrow 0} \left\{ \sup_{x \in X} \left\{ \sup_{D \subset I, \text{meas}D \leq \varepsilon} \int_D |x(t)| dt \right\} \right\}.$$

Restricted to the family compact in measure subsets of this space it forms a regular measure of noncompactness (cf. [32]).

An importance of such a kind of functions can be clarified by using the contraction property with respect to this measure instead of compactness in the Schauder fixed point theorem. Namely, we have the theorem due to Darbo ([13, 26]):

Theorem 2.13. *Let Q be a nonempty, bounded, closed, and convex subset of E and let $H : Q \rightarrow Q$ be a continuous transformation which is a contraction with respect to the measure of noncompactness μ , i.e. there exists $k \in [0, 1)$ such that*

$$\mu(H(X)) \leq k\mu(X),$$

for any nonempty subset X of E . Then H has at least one fixed point in the set Q .

3. Main Result

Denote by H the operator associated with the right hand side of equation (1.1) which takes the form

$$x = H(x),$$

where

$$H(x)(t) = f_1(t, x(\phi_1(t))) + f_2(t, x(t)) \cdot \int_0^1 u(t, s, x(\phi_2(s))) ds. \tag{3.1}$$

This operator will be written as $H(x) = F_{\phi_1, f_1}(x) + A(x)$,

$$A(x)(t) = F_{f_2}(x)(t) \cdot U(x)(t) = F_{f_2}(x)(t) \cdot \int_0^1 u(t, s, x(\phi_2(s))),$$

and the superposition operator F as in Definition 2.1. Thus equation (1.1) becomes

$$x(t) = F_{\phi_1, f_1}(x)(t) + A(x)(t).$$

We shall treat the equation (1.1) under the following assumptions listed below

- (i) $f_i : I \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies Carathéodory conditions and there are positive integrable on I functions a_i and constants $b_i \geq 0$ such that

$$|f_i(t, x)| \leq a_i(t) + b_i|x| \quad , \quad i = 1, 2,$$

for all $t \in [0, 1]$ and $x \in \mathbb{R}$. Moreover, $f_i(t, x) \geq 0$ for $x \geq 0$ and f_i is assumed to be nonincreasing with respect to both variable t and x separately for $i = 1, 2$.

- (ii) $u : I \times I \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies Carathéodory conditions i.e. it is measurable in (t, s) for any $x \in \mathbb{R}$ and continuous in x for almost all (t, s) . The function u is nonincreasing with respect to each variable, separately.

- (iii) Assume that

$$|u(t, s, x)| \leq k(t, s)(a_3(s) + b_3|x|), \text{ for all } t, s \geq 0 \text{ and } x \in \mathbb{R},$$

where the function k is measurable in (t, s) , $a_3 \in L_1(I)$ and a constant $b_3 > 0$. Assume that the linear integral operator K_0 with the kernel $k(t, s)$ maps $L_1(I)$ into $L_\infty(I)$. Moreover, assume that for arbitrary $h > 0$ ($i = 1, 2$)

$$\lim_{\delta \rightarrow 0} \left\| \int_D \max_{|x_i| \leq h, |x_1 - x_2| \leq \delta} |u(t, s, x_1) - u(t, s, x_2)| ds \right\|_{L_\infty(I)} = 0.$$

- (iv) $\phi_i : I \rightarrow I$ are increasing, absolutely continuous functions (for $i = 1, 2$). Moreover, there are constants $M_i > 0$ such that $\phi'_i \geq M_i$ a.e on $(0, 1)$ (for $i = 1, 2$).

- (v) $\int_0^b k(t_1, s) ds \geq \int_0^b k(t_2, s) ds$ for $t_1, t_2 \in I$ with $t_1 < t_2$ and for any $b \in [0, 1]$.

(vi) let $W > \sqrt{\frac{4b_2b_3\|K_0\|_{L_\infty(I)}}{M_2} (\|a_1\|_1 + \|K_0\|_{L_\infty(I)}\|a_2\|_1\|a_3\|_1)}$, where

$$W = \left(\frac{b_1}{M_1} + \frac{b_3}{M_2} \|K_0\|_{L_\infty(I)} \|a_2\|_1 + b_2 \|K_0\|_{L_\infty(I)} \|a_3\|_1 \right) - 1$$

and let R denotes a positive solution of the quadratic equation

$$\begin{aligned} & \frac{b_2b_3\|K_0\|_{L_\infty(I)}}{M_2} \cdot t^2 \\ & - \left[1 - \left(\frac{b_1}{M_1} + \frac{b_3}{M_2} \|K_0\|_{L_\infty(I)} \|a_2\|_1 + b_2 \|K_0\|_{L_\infty(I)} \|a_3\|_1 \right) \right] \cdot t \\ & + (\|a_1\|_1 + \|K_0\|_{L_\infty(I)} \|a_2\|_1 \|a_3\|_1) = 0. \end{aligned}$$

Then we can prove the following theorem.

Theorem 3.1. *Let the assumptions (i)–(vi) be satisfied. Put*

$$L = \left[\frac{b_1}{M_1} + b_2 \|K_0\|_{L_\infty(I)} [\|a_3\|_1 + \frac{b_3}{M_2} R] \right].$$

If $L < 1$, then the equation (1.1) has at least one integrable solution a.e. nonincreasing on I .

Proof. First of all observe that by assumption (i) and Theorem (2.2) we have that F_{ϕ_1, f_1} and F_{f_2} are continuous mappings from $L_1(I)$ into itself. By assumption (iii) and Theorem 2.6 we can deduce that U maps $L_1(I)$ into $L_\infty(I)$. From the Hölder inequality the operator A maps $L_1(I)$ into itself continuously. Finally, for a given $x \in L_1(I)$ the function $H(x)$ belongs to $L_1(I)$ and is continuous. Thus

$$\begin{aligned} \|H(x)\|_1 & \leq \|F_{\phi_1, f_1} x\|_1 + \|Ax\|_1 \\ & \leq \int_0^1 [a_1(t) + b_1|x(\phi_1(t))|] dt \\ & \quad + \int_0^1 [a_2(t) + b_2|x(t)|] \int_0^1 |u(t, s, x(\phi_2(s)))| ds dt \\ & \leq \int_0^1 [a_1(t) + b_1|x(\phi_1(t))|] dt \\ & \quad + \int_0^1 [a_2(t) + b_2|x(t)|] \int_0^1 k(t, s)[a_3(s) + b_3|x(\phi_2(s))|] ds dt \\ & \leq \|a_1\|_1 + \frac{b_1}{M_1} \int_0^1 |x(\phi_1(t))| \phi_1'(t) dt \\ & \quad + \int_0^1 \int_0^1 k(t, s) a_2(t) [a_3(s) + b_3|x(\phi_2(s))|] ds dt \\ & \quad + b_2 \int_0^1 \int_0^1 k(t, s) |x(t)| [a_3(s) + b_3|x(\phi_2(s))|] ds dt \end{aligned}$$

$$\begin{aligned}
 &\leq \|a_1\|_1 + \frac{b_1}{M_1} \int_{\phi_1(0)}^{\phi_1(1)} |x(u)| du \\
 &\quad + \int_0^1 [a_3(s) + b_3|x(\phi_2(s))|] \int_0^1 k(t, s)a_2(t) dt ds \\
 &\quad + b_2 \int_0^1 [a_3(s) + b_3|x(\phi_2(s))|] \int_0^1 k(t, s)|x(t)| dt ds \\
 &\leq \|a_1\|_1 + \frac{b_1}{M_1} \int_0^1 |x(t)| dt \\
 &\quad + \|K_0\|_{L_\infty(I)} \|a_2\|_1 \int_0^1 [a_3(s) + b_3|x(\phi_2(s))|] ds \\
 &\quad + b_2 \|K_0\|_{L_\infty(I)} \|x\|_1 \int_0^1 [a_3(s) + b_3|x(\phi_2(s))|] ds \\
 &\leq \|a_1\|_1 + \frac{b_1}{M_1} \|x\|_1 \\
 &\quad + \|K_0\|_{L_\infty(I)} \|a_2\|_1 \int_0^1 [a_3(s) + \frac{b_3}{M_2} |x(\phi_2(s))| \phi_2'(s)] ds \\
 &\quad + b_2 \|K_0\|_{L_\infty(I)} \|x\|_1 \int_0^1 [a_3(s) + \frac{b_3}{M_2} |x(\phi_2(s))| \phi_2'(s)] ds \\
 &\leq \|a_1\|_1 + \frac{b_1}{M_1} \|x\|_1 + \|K_0\|_{L_\infty(I)} \|a_2\|_1 [\|a_3\|_1 + \frac{b_3}{M_2} \|x\|_1] \\
 &\quad + b_2 \|K_0\|_{L_\infty(I)} \|x\|_1 [\|a_3\|_1 + \frac{b_3}{M_2} \|x\|_1] \\
 &= \|a_1\|_1 + \|K_0\|_{L_\infty(I)} \|a_2\|_1 \|a_3\|_1 + [\frac{b_1}{M_1} + \frac{b_3}{M_2} \|K_0\|_{L_\infty(I)} \|a_2\|_1 \\
 &\quad + b_2 \|K_0\|_{L_\infty(I)} \|a_3\|_1] \cdot \|x\|_1 + \left[\frac{b_2 b_3 \|K_0\|_{L_\infty(I)}}{M_2} \right] \cdot (\|x\|_1)^2.
 \end{aligned}$$

By our assumption (vi) , it follows that there exists a positive constant R being the positive solution of the equation from assumption (vi) and such that H maps the ball B_R into itself.

Further, let Q_R stand for the subset of B_R consisting of all functions which are a.e. nonincreasing on I . Similarly as claimed in [10] we are able to show that this set is nonempty, bounded (by R), convex and closed in $L_1(I)$. Only the last property needs some comments. Let (y_n) be a sequence of elements in Q_R convergent in $L_1(I)$ to y . Then the sequence is convergent in measure and as a consequence of the Vitali convergence theorem and of the characterization of convergence in measure (the Riesz theorem) we obtain the existence of a subsequence (y_{n_k}) of (y_n) which converges to y almost uniformly on I . Moreover, y is nonincreasing a.e. on I which means that $y \in Q_R$ and so the set Q_R is closed. Now, in view of Theorem 2.4 the set Q_R is compact in measure. To see this it suffices to put $\Omega_c = [0, c] \setminus P$ for any $c \geq 0$, where P denotes a suitable set with $meas P = 0$.

Now, we will show that H preserve the monotonicity of functions. Take $x \in Q_R$, then $x(t)$ and $x(\phi_i(t))$ are a.e. nonincreasing on I and consequently each f_i is also of the same type by virtue of the assumption (i) and Theorem 2.7. Further, $Ux(t)$ is a.e. nonincreasing on I due to assumption (ii). Moreover, F_{ϕ_1, f_1} , $A(x)(t)$ are also of the same type. Thus we can deduce that $H(x) = F_{\phi_1, f_1} + A(x)$ is also a.e. nonincreasing on I . This fact, together with the assertion $H : B_R \rightarrow B_R$ gives that H is also a self-mapping of the set Q_R . From the above considerations it follows that H maps continuously Q_R into Q_R .

From now we will assume that X is a nonempty subset of Q_R and the constant $\varepsilon > 0$ is arbitrary, but fixed. Then for an arbitrary $x \in X$ and for a set $D \subset I$, $meas D \leq \varepsilon$ we obtain

$$\begin{aligned} \int_D |(H(x))(t)| dt &\leq \int_D [a_1(t) + b_1|x(\phi_1(t))|] dt \\ &\quad + \int_D [a_2(t) + b_2|x(t)|] \int_0^1 |u(t, s, x(\phi_2(s)))| ds dt \\ &\leq \|a_1\chi_D\|_1 + \frac{b_1}{M_1} \int_D |x(\phi_1(t))|\phi_1'(t) dt \\ &\quad + \int_D \int_0^1 k(t, s)a_2(t)[a_3(s) + b_3|x(\phi_2(s))|] ds dt \\ &\quad + b_2 \int_D \int_0^1 k(t, s)|x(t)|[a_3(s) + b_3|x(\phi_2(s))|] ds dt \\ &\leq \|a_1\chi_D\|_1 + \frac{b_1}{M_1} \|x\chi_D\|_1 \\ &\quad + \|K_0\|_{L_\infty(I)} \|a_2\chi_D\|_1 [\|a_3\|_1 + \frac{b_3}{M_2} R] \\ &\quad + b_2 \|K_0\|_{L_\infty(I)} \|x\chi_D\|_1 [\|a_3\|_1 + \frac{b_3}{M_2} R]. \end{aligned}$$

Hence, taking into account the equalities

$$\lim_{\varepsilon \rightarrow 0} \{ \sup [\int_D a_i(t) dt : D \subset I, meas D \leq \varepsilon] \} = 0, \quad i = 1, 2,$$

and by the definition of $c(X)$ (cf. Section 2) we get

$$c(H(X)) \leq [\frac{b_1}{M_1} + b_2 \|K_0\|_{L_\infty(I)} [\|a_3\|_1 + \frac{b_3}{M_2} R]] \cdot c(X). \tag{3.2}$$

Recall that $L = \frac{b_1}{M_1} + b_2 \|K_0\|_{L_\infty(I)} (\|a_3\|_1 + \frac{b_3}{M_2} R) < 1$ and then the inequality obtained above together with the properties of the operator H and the fact that the set Q_R is compact in measure allows us to apply Theorem 2.13 which completes the proof. □

Remark 3.2. Let us recall that in the proof we utilize the following fact: U maps $L_1(I)$ into $L_\infty(I)$ and F_2 maps $L_1(I)$ into itself. This allows us to use the Hölder inequality. In this situation, we prove the existence of a.e. monotonic

solutions which are integrable. Sometimes we need more information about the solution, namely if a solution is in some subspace of $L_1(I)$ (the space L_p , for instance). In such a case we are able to use also the same type of inequality. Namely we need only to modify the growth conditions and consequently the spaces in which our operators act. As claimed in the introductory part of our paper we can repeat our proof with appropriate changes for considered operators: F_2 maps $L_p(I)$ into $L_q(I)$ and U maps $L_p(I)$ into $L_r(I)$, where $\frac{1}{r} + \frac{1}{q} = \frac{1}{p}$. Whence we obtain an existence result for L_p -solutions.

It should be noted that in some papers, their authors consider the existence of solutions in L_p spaces simultaneously for $p \geq 1$. As claimed above it cannot be done for quadratic equations. Here is a version for $p > 1$. An interesting (and motivating) remark about the solutions in L_p spaces for integral equations (by using similar method of the proof) can be found in [31, page 93]. However, by considering the measure of noncompactness $c(X) = \limsup_{meas D \rightarrow 0} \{\sup_{x \in X} \|x\chi_D\|_{L_p(I)}\}$ introduced by Erzakova ([32]) (restricted to the family of sets compact in measure) instead of usually considered ones based on Kolomogorov or Riesz criteria of compactness (cf. [13]) we are able to examine by the same manner the case of $L_p(I)$ spaces.

Assume that $p > 1$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. Denote by q the value $\min(p_1, p_2)$ and by r the value $\max(p_1, p_2)$. This implies, in particular, that $q \leq 2p$. We shall treat the equation (1.1) under the following set of assumptions presented below.

- (i)' Assume that functions $f_i : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ satisfy Carathéodory conditions and there are positive constants b_i ($i = 1, 2$) and positive functions $a_1 \in L_p(I)$, $a_2 \in L_q(I)$ such that

$$|f_1(t, x)| \leq a_1(t) + b_1|x|,$$

$$|f_2(t, x)| \leq a_2(t) + b_2|x|^{\frac{p}{q}},$$

for all $t \in I$ and $x \in \mathbb{R}$. Moreover, f_i ($i = 1, 2$) are assumed to be nonincreasing with respect to both variable t and x separately.

- (ii)' $u : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies Carathéodory conditions. The function u is nonincreasing with respect to each variable, separately. Suppose that for arbitrary non-negative $z(t) \in L_q(I)$

$$\lim_{\delta \rightarrow 0} \left\| \int_D \sup_{|x| \leq z} \int_D u(t, s, x(s)) ds \right\|_{L_r(I)} = 0.$$

and that

$$|u(t, s, x)| \leq k(t, s)(a_3(s) + b_3|x|^{\frac{p}{q}}), \text{ for all } t, s \geq 0 \text{ and } x \in \mathbb{R},$$

where the function k is measurable in (t, s) , $a_3 \in L_q(I)$ and a constant $b_3 > 0$. Assume that the linear integral operator K_0 with the kernel $k(t, s)$ maps $L_q(I)$ into $L_r(I)$.

- (iii)' $\phi_i : I \rightarrow I$ are increasing, absolutely continuous functions (for $i = 1, 2$). Moreover, there are constants $M_i > 0$ such that $\phi'_i \geq M_i$ a.e on $(0, 1)$ (for $i = 1, 2$).

(iv)' $\int_0^b k(t_1, s) ds \geq \int_0^b k(t_2, s) ds$ for $t_1, t_2 \in I$ with $t_1 < t_2$ and for any $b \in [0, 1]$.

(v)' Assume, that the following equation

$$\|a_1\|_{L_p(I)} + \|K_0\| \|a_2\|_{L_q(I)} \|a_3\|_{L_q(I)} + \left(\frac{b_1}{M_1^{\frac{1}{p}}} - 1 \right) t + \|K_0\| [b_2 \|a_3\|_{L_q(I)} + \frac{b_3 \|a_2\|_{L_q(I)}}{M_2^{\frac{1}{q}}}] t^{\frac{p}{q}} + \frac{b_2 b_3 \|K_0\|}{M_2^{\frac{1}{q}}} t^{\frac{2p}{q}} = 0.$$

has a positive solution s in $(0, 1]$.

By L' we will denote a number

$$\frac{b_1}{M_1^{\frac{1}{p}}} + b_2 s^{\frac{p}{q}-1} \|K_0\| \left(\|a_3\|_{L_q(I)} + \frac{b_3}{M_2^{\frac{1}{q}}} s^{\frac{p}{q}} \right).$$

Theorem 3.3. *Let the assumptions (i)'-(v)' be satisfied. If $L' < 1$, then the equation (1.1) has at least one $L_p(I)$ -solution a.e. nonincreasing on I .*

Let us note, that in the assumption (v)' we consider the equation of the type $A + Bt + Ct^{\frac{p}{q}} + Dt^{\frac{2p}{q}} = t$. The case $p = q$ leads to the quadratic equation (considered in our first theorem). Although the case $p < q$ seems to be more complicated, it should be noted that since $\frac{p}{q} < 1$ and $\frac{2p}{q} < 2$ this equation has a solution in $(0, 1]$. In some papers the assumption of this type is described by using auxiliary functions. In such a formulation the problem of existence of functions is unclear. Let us note, that for arbitrary pair of spaces $L_p(I)$ and $L_q(I)$ we are able to solve our problem.

Indeed, if $\frac{2p}{q} \geq 1$, then for $t \in I$ we have $A + Bt + Ct^{\frac{p}{q}} + Dt^{\frac{2p}{q}} \leq A + Bt + C + Dt$ and our inequality has a solution in $(0, 1]$ whenever $\frac{A+C}{1-B-D} < 1$. In the case $\frac{2p}{q} < 1$, we have the following estimation: $A + Bt + Ct^{\frac{p}{q}} + Dt^{\frac{2p}{q}} \leq A + Bt + C + D$ and then $\frac{A+C+D}{1-B} < 1$ form a sufficient condition for the existence of solutions of our inequality in $(0, 1]$. Thus the set of functions satisfying our assumptions is nonempty (cf. also some interesting Examples in [11]). Let us recall that the first case is considered in the paper.

We would like to pay attention, that the condition (ii)' implies that the kernels $k(t, s)$ should be of Hille-Tamarkin classes i.e. $\| \|k(t, \cdot) \|_{q'} \|_r$ and $\| \|k(\cdot, s) \|_q \|_{r'}$ it is sufficient to assume that they are finite being at the same time the upper bounds for $\|K_0\|$, where q' and r' are conjugated with q and r , respectively.

Moreover, it is worthwhile to note that by the same manner we can extend our main result for other subspaces of $L_1(I)$ for which we are able to check the required properties of considered operators (some Orlicz spaces, for instance) cf. [25].

Remark 3.4. Till now, we are interested in finding monotonic solutions of our problem. Assume that we have the decomposition of the interval I into the disjoint subsets T_1 and T_2 with $T_1 \cup T_2 = I$, such that $f_i(\cdot, x)$ are a.e. non-decreasing on T_1 and a.e. nonincreasing on T_2 . By an appropriate change of the monotonicity assumptions we are able to prove the existence of solutions belonging to the class of functions described above (similarly like in [9]). In such a case we need to consider the operators preserving this property, too.

4. Examples

We need to show an example for which our main result is useful and allows us to extend the existing theorems. Let us recall that we are looking for monotonic solutions for the considered problems in the interval I .

But first, let us recall that the quadratic equations have numerous applications in the theories of radiative transfer, neutron transport and in the kinetic theory of gases [5, 15, 22, 23]. In order to apply earlier results of the considered type, we have to impose an additional condition that the so-called “characteristic” function ψ is continuous (cf. [23]) or even Hölder continuous ([5]). In the theory of radiative transfer this function is immediately related to the angular pattern for single scattering and then our results allow to consider some peculiar states of the atmosphere. In astrophysical applications of the Chandrasekhar equation $x(t) = 1 + x(t) \int_0^1 \frac{t}{t+s} \psi(s)x(s) ds$ the only restriction that $\int_0^1 \psi(s) ds \leq 1/2$ is treated as necessary (cf. [22, Chapter VIII; Corollary 2 p. 187]. An interesting discussion about this condition and the applicability of such equations can be found in [22]. Recall that to ensure the existence of solutions normally one assumes that $\psi(t)$ is an even polynomial (as in the book of Chandrasekhar [23, Chapter 5]) or continuous ([22]). The using of different solution spaces in the current paper allow us to remove this restriction and then we give a partial answer to the problem from [22]. The continuity assumption for ψ implies the continuity of solutions for the considered equation (cf. [22]) and then seems to be too restrictive even from the theoretical point of view.

Let us consider now the following integral equation

$$\begin{aligned}
 x(t) = a(t) + \frac{-\ln(1 + x^2(\frac{t}{3} + \frac{t^2}{2}))}{3 + t} & \tag{4.1} \\
 + \arctan\left(\frac{1 + h(x)}{\sqrt{t + 2}}\right) \int_0^1 \frac{\lambda}{t^2 + s^2} \left[\frac{1}{\sqrt{s + 1}} + \frac{x(s)}{1 + x^2(s)}\right] ds, &
 \end{aligned}$$

where

$$a(t) = \begin{cases} 0 & t \text{ is rational,} \\ 1 - t & t \text{ is irrational} \end{cases}, \quad h(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ \frac{\sin x}{1 + e^x} & \text{for } x > 0. \end{cases}$$

It can be easily seen that equation (4.1) is a particular case of the equation (1.1), where

$$f_1(t, x) = a(t) + \frac{-\ln(1 + x^2(\frac{t}{3} + \frac{t^2}{2}))}{3 + t}, \quad f_2(t, x) = \arctan\left(\frac{1 + h(x)}{\sqrt{t+2}}\right)$$

and

$$u(t, s, x) = \frac{\lambda}{t^2 + s^2} \left[\frac{1}{\sqrt{s+1}} + \frac{x(s)}{1 + x^2(s)} \right].$$

In view of the inequalities $\ln(1 + x^2) \leq x$ ($x > 0$) and $\arctan\left(\frac{1+h(x)}{\sqrt{t+2}}\right) < \frac{1+h(x)}{\sqrt{t+2}}$, the functions f_1 , f_2 and u are nonincreasing in each variable separately. Moreover, $|f_1(t, x)| \leq a(t) + \frac{1}{4}|x|$, $|f_2(t, x)| \leq \frac{1}{\sqrt{t+2}} + \frac{1}{3}h(x)$ and

$$|u(t, s, x)| \leq \frac{\lambda}{t^2 + s^2} \left[\frac{1}{\sqrt{s+1}} + \frac{1}{2}|x| \right],$$

with $a_1(t) = a(t)$, $a_2(t) = \frac{1}{\sqrt{t+2}}$, $a_3(s) = \frac{1}{\sqrt{s+1}}$ and $k(t, s) = \frac{\lambda}{t^2+s^2}$. Here we have the constants $b_1 = \frac{1}{4}$, $b_2 = \frac{1}{3}$ and $b_3 = \frac{1}{2}$.

Since $\int_0^1 \frac{\lambda}{t^2+s^2} ds = \lambda \arctan \frac{1}{t}$, $|k(t, s)| \leq \lambda$, thus the expected property for the operator K_0 holds true. Moreover, for given arbitrary $h > 0$ and $|x_2 - x_1| \leq \delta$ we have

$$\begin{aligned} |u(t, s, x_1) - u(t, s, x_2)| &= \frac{1}{t^2 + s^2} \left| \frac{x_1(1 + x_2^2) - x_2(1 + x_1^2)}{(1 + x_1^2)(1 + x_2^2)} \right| \\ &= \frac{1}{t^2 + s^2} \left| \frac{(x_1 - x_2) + x_1x_2(x_2 - x_1)}{(1 + x_1^2)(1 + x_2^2)} \right| \\ &\leq \frac{1}{t^2 + s^2} \frac{\delta(1 + h^2)}{(1 + x_1^2)(1 + x_2^2)}. \end{aligned}$$

Put $\phi_1(t) = \frac{t}{3} + \frac{t^2}{2}$ and $\phi_2(t) = t$, then $\phi_1'(t) = \frac{1}{3} + t > \frac{1}{3} = M_1$ and $\phi_2'(t) = 1 > \frac{1}{2} = M_2$. Thus our assumptions (i)-(iv) are satisfied. Since $\frac{3}{4} + \frac{1}{3}\lambda\frac{\pi}{2}(1+R) < 1$ for small $\lambda > 0$, assumption (v) holds true for sufficiently small λ .

Taking into account all the above observations we are able to deduce from Theorem 1.1 that for sufficiently small λ equation (4.1) has at least one integrable solution x which is a.e. nonincreasing on I .

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