Preface

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This volume contains a selection of papers written on the basis of presentations given at the 10th International Conference on Clifford Algebras and their Applications in Mathematical Physics (ICCA10), August 4–9, 2014, in Tartu, Estonia. The conference presentations can be grouped into four sessions in Clifford analysis, Clifford algebras, geometry and physics, and, according to this classification, the volume consists of four parts which have the same titles.

This conference was a continuation of tradition to bring together the leading scientists in the field of Clifford analysis, Clifford algebras, geometry and their applications in various fields of theoretical physics by organizing scientific meetings. Previous conferences on Clifford algebras and their applications were held at the University of Kent, Canterbury, UK, 1985; University of Montpellier, Montpellier, France, 1989; University of Gent, Gent, Belgium, 1993; University of Aachen, Germany, 1996; Ixtapa, Mexico, 1999; Tennessee Technological University, Cookeville, USA, 2002; Université Paul Sabatier, Toulouse, France, 2005; IMECC-UNICAMP, Campinas, Brazil, 2008; and Bauhaus-University Weimar, Germany, 2011.

In order to emphasize the important directions of development in the field of Clifford algebras and their applications, the following workshops were organized within the framework of the conference:

- Discrete and Continuous Quaternionic and Clifford Analysis: Swanhild Bernstein (TU Bergakademie Freiberg, Germany) and Uwe Kähler (Universidade de Aveiro, Portugal);
- Conformal Structures and Conformal Spin Structures: Pierre Anglès (Institut de Mathématiques de Toulouse, Université Paul Sabatier, Toulouse, France) and Jacques Helmstetter (Institut Fourier, Saint Martin d'Hères, France);
- From Signals to Consciousness using Clifford and Geometric Algebras: Eduardo Bayro-Corrochano (CINVESTAV, Campus Guadalajara, Mexico; Elio Conte, University of Bari, Neurological and Psychiatric Sciences, Italy) and David Hestenes (Arizona State University, Tempe, Arizona, USA);

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- Geometric Algebra and Calculus in the Standard Model of Particle Physics: Gene McClellan (Arlington, Virginia, USA);
- Geometric Algebra in the High School and Undergraduate Curriculum: Charles Gunn (Technical University Berlin, Germany) and Ramon González Calvet (Institute Pere Calders, Spain);
- Quaternion and Clifford Fourier Transforms and Wavelets 2: Eckhard Hitzer (International Christian University, Tokyo, Japan) and Roxana Bujack (Leipzig University, Germany).

During the conference there was a presentation of the second Clifford Prize to David Eelbode of the University of Antwerp, Belgium, who was selected by the 2014 jury as the recipient of the second Clifford Prize for his outstanding mathematical research achievements in the fields of harmonic and Clifford analysis with applications in theoretical physics. The prize for the best PhD student presentation at ICCA10 was awarded to Cohl Furey (Perimeter Institute for Theoretical Physics, Canada) for her presentation “Charge Quantization from a Number Operator”. The prize for the best postdoctoral research presentation at ICCA10 was awarded to Hilde De Ridder (Ghent University, Belgium) for her presentation “Fueter’s theorem in discrete Clifford analysis” (joint work with F. Sommen).

1. Part I. Clifford Algebras

Abramov The author proposes a $\mathbb{Z}_2$ -graded (super) extension of a concept of n -Lie algebra. It is shown that a method to construct $(n+1)$ -Lie algebra by means of n -Lie algebra endowed with an analog of trace can be extended to n -Lie superalgebras if instead of an analog of trace one uses an analog of super trace. This extension of a method of induced n -Lie algebras is applied to Lie superalgebra of supermodule of spinors of Clifford algebra with even number of generators. As a result the author obtains a series of 3-Lie superalgebras.

Dechant The author of this paper discusses applications of a Clifford algebra framework to root systems and to groups connected with reflections. A versor formalism for these is described and the Coxeter group A_3 is explored. The author proves the induction theorem for 4D root systems and describes their automorphism groups. Finally, he discusses the reflections in conformal space and shows how to extend the conformal geometric algebra treatment to the conformal and modular groups.

Helmstetter The author of this paper describes some facts about the conformal group $\text{Conf}(V, Q)$ of a quadratic space (V, Q) , where V is a finite dimensional vector space over the field of real numbers and Q is a non-degenerate quadratic form $V \rightarrow \mathbb{R}$. The author shows that there exists a surjective group morphism $\text{O}(V^\dagger, Q^\dagger) \rightarrow \text{Conf}(V, Q)$, where $(V^\dagger, Q^\dagger)$ is a direct orthogonal sum of (V, Q) and a hyperbolic plane spanned by two isotropic vectors. He describes how the elements of $\text{O}(V^\dagger, Q^\dagger)$ and $\text{Conf}(V, Q)$ can be represented by Vahlen matrices. The author studies the properties of Vahlen matrices.

Schott and Staples The authors introduce the notion of generalized zeon algebras, which are due to their inherent combinatorial properties very useful for routing problems by implicitly pruning the underlying tree structures. These algebras occur as subalgebras of Clifford algebras, lending them a natural connection to quantum probability and quantum computing. The aim of this paper is to find Pareto paths satisfying multiple constraints. This algebraic approach can be applied to sieve out the collection of feasible paths and recover all single-source Pareto paths remaining.

Trovan and Suzuki The idea of noncommutative Galois extension for a class of associative algebras is presented in this paper. The authors show how binary and ternary Clifford algebras can be recovered by Galois extensions where the binary or ternary character is described by particular properties of the extension. The nonion algebra as the simplest example of a ternary Clifford algebra is characterized by Galois extensions. The authors observe ternary Dirac and Laplace operators. The authors consider a representation of the algebra $\mathfrak{su}(3)$ in terms of 3×3 complex matrices.

2. Part II. Clifford Analysis

Caçã, Falcão and Malonek The authors show that Clifford algebra-valued orthogonal polynomials and orthogonal polynomials of one real variable share some similar properties. They prove that there exist some three-term recurrence relations which can be considered as building blocks for Clifford algebra-valued orthogonal polynomials. As a surprising byproduct, it can be seen that all the results are in a way connected through the application of Standard Appell Sequence.

Cerejeiras, Kähler and Ku The authors study the Hilbert boundary value problem with Fermionic weight for the Dirac operator on the smooth surfaces of $\mathbb{R}^3$. The authors solve the Hilbert boundary value problem both on the half space and on the unit ball of $\mathbb{R}^3$, and as special cases they solve the Hilbert boundary value problem with Fermionic weight for analytic functions on the half plane and the unit circle. Finally, they provide sufficient and necessary conditions for the solvability of the Hilbert boundary value problem in more general domains that have smooth boundary of $\mathbb{R}^3$.

Eriksson and Orelma The authors study function theory in higher dimensions based on the hyperbolic metric of the Poincaré upper half space model. In this paper they investigate function theory of k -hypermonogenic functions connected to k -hyperbolic harmonic functions that are harmonic with respect to a given hyperbolic Riemannian metric in the upper half space $\mathbb{R}_+^{n+1}$. The authors continue with their previous endeavor and provide a fundamental k -hyperbolic farmonic function that is used to find kernels and integral formulae for k -hyperbolic functions.

Eriksson, Orelma and Vieira In this paper the authors develop an analogous of the class of two-sided axial monogenic functions for the case of κ -hypermonogenic functions. The authors present some new properties of the

two-sided κ -monogenic functions. This is done by solving Vekua-type system in terms of Bessel functions.

Gal and Sabadini In this paper the authors continue their studies about slice regular functions of one quaternionic variable, now using approximation theory. Using the notion of quaternionic ellipsoid the authors prove that the Chebyshev and Legendre expansions converge uniformly to a slice regular function in certain compact sets and that convergence has the same order as geometric series.

García, Teodoro and Vanegas The authors of this paper observe initial value problems where the underlying algebra is of *Clifford type*. The authors study the problem of finding first order differential operators associated to the given space of q -monogenic functions, where q is a constant vector, or a vector valued function. From the paper we can find necessary and sufficient conditions on the coefficients of the aforementioned first order differential operator. An example of application of the results is given – it turns out that the initial value problems that involve such operators are solvable if the initial *Clifford type* algebra valued function is q -monogenic.

Shirokov In this paper the author observes a class of operators that act on real Clifford algebras. As a concrete case, Reynolds operator of Salingaros'ee group is considered and examined in more detail. It turns out that the Reynolds operators, in a sense, “average” a group action on a Clifford algebra, and with the use of this method of averaging the author provides solution to the system of commutator equations. The author also presents a relation between Reynolds operators and projection operators onto a fixed subspaces of Clifford algebras.

3. Part III. Geometry

Anglès The study of a real conformal spin structure in a strict sense over a pseudo-Riemannian or Riemannian $2r$ -dimensional manifold V developed early by the author in previous publications is extended here to construct fiber bundles which will be useful in order to construct conformal spin and conformal $U(1)$ -spin schemes of prequantization and quantization. The author define the fundamental diagram of $U(1)$ -spin geometry and conformal fundamental diagrams of conformal $U(1)$ -spin geometry. He studies the obstruction classes for the existence of a conformal spin structure and a conformal $U(1)$ -spin structure over V .

Gunn The author introduces the dual projectivized Clifford algebra $\mathbf{P}(\mathbb{R}_{n,0,1}^*)$ as homogeneous model for Euclidean geometry. The author compares $\mathbf{P}(\mathbb{R}_{n,0,1}^*)$ with conformal geometric algebra and establishes that the dual projectivized Clifford algebra is the smallest structure-preserving Euclidean geometric algebra.

Leão, Rodrigues and Wainer In this paper the authors give a geometrical motivated definition for the Lie derivative of spinor fields in a Lorentzian

structure $(M, \mathbf{g})$, where M is 4-dimensional manifold and $\mathbf{g}$ is Lorentzian of signature $(1, 3)$, using the Clifford and spin-Clifford bundles formalism. The authors call this operator the *spinor Lie derivative* and denote it by $\overset{s}{\mathcal{L}}_{\xi}$. They prove that $\overset{s}{\mathcal{L}}_{\xi} \mathbf{g} = 0$ for arbitrary vector field ξ . The authors compare their results with the many others appearing in literature on the subject.

Snygg In this paper the author shows the possibility of using the structures of Clifford algebra in order to study the symmetries of the Platonic solids. This presentation is devoted to high school and college students. The author studies the proper symmetry group for the cube more precisely and discusses the other four solids.

4. Part IV. Physics

Acus A modern approach to quantum mechanics, which is based on Clifford's geometric algebra, is used instead of abstract Hilbert space. The author shows with the help of fully elaborated examples how geometric algebra can be used to find eigenvalues and eigenmultivectors in 3D Euclidean space with $Cl_{3,0}$ and in relativistic Minkowski space with $Cl_{3,1}$ algebra without explicitly involving Hamiltonian matrix diagonalization procedure.

Castro The author presents some of the novel physical consequences of the Extended Relativity Theory in Clifford spaces (C-spaces), which can be viewed as a natural extension of the ordinary Relativity theory. An extensive analysis of the C-space generalized Lorentz transformations and their physical implications are provided. Based on the notion of "extended events" the author finds a very different physical explanation of the phenomenon of "relativity of locality" than the one described by the Doubly Special Relativity framework.

Cavalcanti, da Rocha and Hoff da Silva The authors revisit the basics on the DKP (meson) algebra and its relation to the Clifford algebras. Then they briefly discuss Elko spinor fields and their symmetry groups, which are the very special relativity (VSR) groups. The linear spaces composed exclusively by Elko or Dirac spinors, which enable the authors to take the tensor product of those spinor spaces, are found. Finally Elko spinor field is shown to carry signature of VSR symmetries when Elko spinor field is considered as constituent element of DKP theory in the Clifford algebras background.

Daviau The Clifford algebra Cl_3 is used to derive all features of the Dirac equation for a wave with spin $1/2$. Then this is extended to electromagnetic laws. Next the author gets the gauge group of electro-weak interactions, first in the leptonic case, electron+neutrino, next in the quark case. The complete wave for all objects of the first generation uses the Clifford algebra $Cl_{1,5}$. The author consolidates both the standard model and the use of Clifford algebras, which can be viewed as a true mathematical frame of quantum physics.

Dvoeglazov The author re-examines the theory of antisymmetric tensor fields and 4-vector potentials. The massless limits are studied. Taking into account mass dimensions of the notoph and the photon the author analyzes the

quantum field theory. The equations for the symmetric tensor of the second rank are derived by means of the Bargmann-Wigner formalism. It is shown that these equations are consistent with the general relativity. The correct definitions of the energy-momentum tensor and other Nöther currents are proposed.

Dzhunushaliev The author proposes a nonassociative generalization of supersymmetry. For this purpose he gives exact definitions of three and four associators and shows that for a special definition of a four associator, this associator is connected with the angular momentum operator. Some interesting problems arising from the obtained decomposition of quantum operators are discussed.

Socroun The author of this paper tries to unify General Relativity and Electromagnetism. He starts with a symmetrization of 2 particle Lagrangian and a General Relativity reminder about the Reissner-Nordstrom solution. He shows that the usual integration constants are not coherent with the “classic” case. These 2 constants are then changed to fulfill this property and that leads to a new metric. The use of a Clifford square root of the obtained metric will lead to the Unification equation, a new theory contains the Dirac equation and the Reissner-Nordstrom solution, both as limit case.

Vilson The author clarifies some mathematical issues concerning the invariant quantities in general scalar-tensor theories of gravity, characterized by four arbitrary local functionals of the scalar field, and the translation rules. The author proves a theorem that allows to link an object invariant under local rescaling of the metric to each quantity in a theory, where two out of the four arbitrary local functionals of the scalar field are specified in a suitable manner.

Zenczykowski This paper is a review of a Clifford algebra approach to elementary particles. The author discusses the philosophical need for a maximally symmetric treatment of position and momentum coordinates. This fundamental argument and the analogy with the Dirac linearization procedure will lead to the 64-dimensional Clifford algebra $Cl_{6,0}$ as a structure that should naturally describe some aspects of nonrelativistic phase space. He describes rotationally noninvariant generalization of the concept of mass. The author shows that the Clifford algebra describes features of a single generation of elementary particles in the Standard Model. He discusses the ways of restoring rotational and translational invariances at the level of individually observable particles composed of quarks.

Acknowledgements

We wish to express our gratitude to the member of Organizing Committee of ICCA10 Jaan Vajakas, who did a great job in the preparatory phase of the conference, during the conference and preparing this volume. We also thank the reviewers for their valuable comments and recommendations, which allowed to significantly improve the level of the present volume. The

contributions of the invited speakers, the session chairs and the workshops organizers are also greatly appreciated. Finally, we gratefully acknowledge the partial financial support of the conference ICCA10 by the Estonian Doctoral School in Mathematics and Statistics.

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