Gauge Group of the Standard Model in $Cl_{1,5}$

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Abstract. Describing a wave with spin $1/2$, the Dirac equation is form invariant under $SL(2, \mathbb{C})$, subgroup of $Cl_3^* = GL(2, \mathbb{C})$ which is the true group of form invariance of the Dirac equation. Firstly we use the Cl_3 algebra to read all features of the Dirac equation for a wave with spin $1/2$. We extend this to electromagnetic laws. Next we get the gauge group of electro-weak interactions, first in the leptonic case, electron+neutrino, next in the quark case. The complete wave for all objects of the first generation uses the Clifford algebra $Cl_{1,5}$. The gauge group is then enlarged into a $U(1) \times SU(2) \times SU(3)$ Lie group. We consolidate both the standard model and the use of Clifford algebras, true mathematical frame of quantum physics.

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1. Form Invariance of the Dirac Equation

The standard model uses fermions and bosons. All fermions are described with a Dirac equation. The Dirac wave ψ_e of the electron is made of two Pauli waves

$$\psi_e = \begin{pmatrix} \xi_e \\ \eta_e \end{pmatrix}; \quad \xi_e = \begin{pmatrix} \xi_{1e} \\ \xi_{2e} \end{pmatrix}; \quad \eta_e = \begin{pmatrix} \eta_{1e} \\ \eta_{2e} \end{pmatrix} \quad (1.1)$$

where $\xi_{1e}, \xi_{2e}, \eta_{1e}, \eta_{2e}$ are four functions of space and time with value in the complex field. In its first complex frame the Dirac equation reads

$$0 = [\gamma^\mu (\partial_\mu + iqA_\mu) + im]\psi_e; \quad q = \frac{e}{\hbar c}; \quad m = \frac{m_0 c}{\hbar} \quad (1.2)$$

where γ^μ , $\mu = 0, 1, 2, 3$ are four complex matrices. Relativistic theory uses:

$$\begin{aligned} \gamma_0 = \gamma^0 &= \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}; \quad \gamma_j = -\gamma^j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}; \\ I = \sigma_0 = \sigma^0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \quad (1.3)$$

$$\begin{aligned}\sigma_1 &= -\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_2 = -\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \\ \sigma_3 &= -\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.\end{aligned}\quad (1.4)$$

The electro-weak theory uses also

$$\begin{aligned}\gamma_5 &= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}; \quad \psi_R = \psi_e \frac{1}{2}(1 + \gamma_5) = \begin{pmatrix} \xi_e \\ 0 \end{pmatrix}; \\ \psi_L &= \psi_e \frac{1}{2}(1 - \gamma_5) = \begin{pmatrix} 0 \\ \eta_e \end{pmatrix}\end{aligned}\quad (1.5)$$

To get the relativistic invariance the Dirac theory uses $x = x^\mu \sigma_\mu$ and x' satisfying

$$x = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}; \quad x' = x'^\mu \sigma_\mu = \begin{pmatrix} x'^0 + x'^3 & x'^1 - ix'^2 \\ x'^1 + ix'^2 & x'^0 - x'^3 \end{pmatrix}\quad (1.6)$$

This is equivalent to identify the space algebra Cl_3 to the matrix algebra $M_2(\mathbb{C})$ ¹. Noting a^* the conjugate of a and with

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \quad M^\dagger = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}; \quad \overline{M} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}; \quad \widehat{M} = \overline{M}^\dagger \quad (1.7)$$

$$\det(M) = M\overline{M} = \overline{M}M = ad - bc = re^{i\theta} \quad (1.8)$$

the Dirac theory associates to M the R transformation satisfying

$$x' = R(x) = MxM^\dagger \quad (1.9)$$

because we get, for any M :

$$\begin{aligned}(x'^0)^2 - (x'^1)^2 - (x'^2)^2 - (x'^3)^2 &= \det(x') = \det(M) \det(x) \det(M^\dagger) \\ &= re^{i\theta} \det(x) re^{-i\theta} = r^2 [(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2].\end{aligned}\quad (1.10)$$

Then R is a Lorentz dilation, product of a Lorentz rotation and an homothety with ratio r . Moreover with

$$x'^\mu = R_\nu^\mu x^\nu; \quad N = \begin{pmatrix} M & 0 \\ 0 & \widehat{M} \end{pmatrix}; \quad \widetilde{N} = \begin{pmatrix} \overline{M} & 0 \\ 0 & M^\dagger \end{pmatrix} \quad (1.11)$$

we get [5, 10]:

$$R_0^0 = \frac{1}{2}(aa^* + bb^* + cc^* + dd^*) > 0 \quad (1.12)$$

$$\det(R_\nu^\mu) = r^4 \quad (1.13)$$

$$R_\nu^\mu \gamma^\nu = \widetilde{N} \gamma^\mu N, \quad \mu = 0, 1, 2, 3. \quad (1.14)$$

The form invariance of the Dirac equation comes from

$$\begin{aligned}0 &= [\gamma^\nu (\partial_\nu + iqA_\nu) + im]\psi_e = [\gamma^\nu R_\nu^\mu (\partial'_\mu + iqA'_\mu) + im]\psi_e \\ &= [\widetilde{N} \gamma^\mu (\partial'_\mu + iqA'_\mu) N + im]\psi_e.\end{aligned}\quad (1.15)$$

¹ This identification is not mine, it has been made 88 years ago. It is necessary to get the form invariance of the Dirac equation. This form invariance is incorrectly known as “relativistic invariance”. I enlarged the invariance group into Cl_3^* .

If $\det(M) = 1$, N satisfies

$$M\overline{M} = \overline{M}M = 1; \quad \overline{M} = M^{-1}; \quad \widetilde{N} = N^{-1} \quad (1.16)$$

where $\widetilde{N}$ is the reverse of N . This gives

$$0 = [\gamma^\mu(\partial_\mu + iqA_\mu) + im]\psi_e = N^{-1}[\gamma^\mu(\partial'_\mu + iqA'_\mu) + im]N\psi_e. \quad (1.17)$$

The Dirac theory lets then

$$\psi'_e(x') = N\psi_e(x) \quad (1.18)$$

and since

$$0 = [\gamma^\mu(\partial_\mu + iqA_\mu) + im]\psi_e \iff 0 = \gamma^\mu(\partial'_\mu + iqA'_\mu) + im]\psi'_e \quad (1.19)$$

the Dirac equation is said “form invariant”. Ten years ago, I noticed that relations (1.12), (1.13) and (1.14) are true even if $\det(M) \neq 1$. Then the fundamental group of form invariance of the Dirac wave is the group of the M , usually named $GL(2, \mathbb{C})$. This group is also the multiplicative group Cl_3^* of the invertible elements in Cl_3 .

2. Dirac Equation and Electromagnetism in Cl_3

We have previously [3, 5, 10] let

$$\phi_e = \sqrt{2}(\xi_e - i\sigma_2\eta_e^*) = \sqrt{2} \begin{pmatrix} \xi_{1e} & -\eta_{2e}^* \\ \xi_{2e} & \eta_{1e}^* \end{pmatrix} \quad (2.1)$$

which implies

$$\det(\phi_e) = 2(\xi_{1e}\eta_{1e}^* + \xi_{2e}\eta_{2e}^*) = \Omega_1 + i\Omega_2 = \rho e^{i\beta} \quad (2.2)$$

The ϕ_e wave is then a function of space-time into $Cl_3 = M_2(\mathbb{C})$. We get

$$\widehat{\phi}_e = \sqrt{2}(\eta_e - i\sigma_2\xi_e^*) = \sqrt{2} \begin{pmatrix} \eta_{1e} & -\xi_{2e}^* \\ \eta_{2e} & \xi_{1e}^* \end{pmatrix}; \quad \overline{\phi}_e = \widehat{\phi}_e^\dagger \quad (2.3)$$

$$\phi_e\overline{\phi}_e = \overline{\phi}_e\phi_e = \det(\phi_e) = \rho e^{i\beta} \quad (2.4)$$

where β is the Yvon-Takabayasi angle. $\phi_e^\dagger$ is the reverse of ϕ_e . (1.18) reads

$$\begin{pmatrix} \xi'_e \\ \eta'_e \end{pmatrix} = \begin{pmatrix} M & 0 \\ 0 & \widehat{M} \end{pmatrix} \begin{pmatrix} \xi_e \\ \eta_e \end{pmatrix}; \quad \xi'_e = M\xi_e; \quad \eta'_e = \widehat{M}\eta_e. \quad (2.5)$$

It happens that we get, with any M and any η :

$$(-i\sigma_2)\eta'^* = (-i\sigma_2)\widehat{M}^*\eta^* = M(-i\sigma_2)\eta^* \quad (2.6)$$

The link that I made in (2.1) between ϕ_e and the Weyl spinors ξ_e , η_e is therefore invariant under Cl_3^* : if

$$\phi'_e(x') = \sqrt{2}(\xi'_e - i\sigma_2)\eta'^* = \sqrt{2} \begin{pmatrix} \xi'_{1e} & -\eta'_{2e}^* \\ \xi'_{2e} & \eta'_{1e}^* \end{pmatrix} \quad (2.7)$$

we simply get

$$\phi'_e = M\phi_e; \quad \widehat{\phi}'_e = \widehat{M}\widehat{\phi}_e. \quad (2.8)$$

We have explained [5, 10] how the Dirac equation reads in Cl_3 :

$$\nabla\widehat{\phi}_e\sigma_{21} + qA\widehat{\phi}_e + m\phi_e = 0; \quad \nabla = \sigma^\mu\partial_\mu; \quad A = \sigma^\mu A_\mu; \quad \sigma_{21} = \sigma_2\sigma_1. \quad (2.9)$$

Multiplying on the left by $\bar{\phi}_e$, I got [5, 6]:

$$\bar{\phi}_e(\nabla\hat{\phi}_e)\sigma_{21} + \bar{\phi}_eqA\hat{\phi}_e + m\rho e^{i\beta} = 0. \quad (2.10)$$

The first term is form invariant because, for any M in Cl_3 , with

$$\nabla' = \sigma^\mu \partial'_\mu \quad (2.11)$$

I got the following general relation [3]:

$$\nabla = \overline{M}\nabla'\widehat{M} \quad (2.12)$$

which gives with (2.8):

$$\bar{\phi}_e(\nabla\hat{\phi}_e)\sigma_{21} = \bar{\phi}_e \overline{M}\nabla'\widehat{M}\hat{\phi}_e\sigma_{21} = \bar{\phi}'_e(\nabla'\hat{\phi}'_e)\sigma_{21}. \quad (2.13)$$

Next the form invariance of $\bar{\phi}_eqA\hat{\phi}_e$ is necessary to satisfy both the form invariance and the electric gauge invariance of the Dirac equation. This means

$$\begin{aligned} \bar{\phi}_eqA\hat{\phi}_e &= \bar{\phi}_e \overline{M}q'A'\widehat{M}\hat{\phi}_e \\ qA &= \overline{M}q'A'\widehat{M}. \end{aligned} \quad (2.14)$$

Then qA which transforms like ∇ is named a “covariant vector” (in space-time), while vectors transforming like x , for instance $J = \phi_e\phi_e^\dagger$, are named “contravariant”. This is now a physical distinction because, if $\det(M) \neq 1$ then $\overline{M} \neq M^{-1}$. The non-linear homogeneous equation studied in my thesis [2] reads in Cl_3 :

$$\nabla\hat{\phi}_e\sigma_{21} + qA\hat{\phi}_e + me^{-i\beta}\phi_e = 0 \quad (2.15)$$

and is then equivalent to the form invariant equation

$$\bar{\phi}_e(\nabla\hat{\phi}_e)\sigma_{21} + \bar{\phi}_eqA\hat{\phi}_e + m\rho = 0. \quad (2.16)$$

This equation is the starting point to get the gauge group of the standard model in the frame of Clifford algebra. Two of the eight numeric equations equivalent to the form invariant equation are remarkable and well known: the law of conservation of the current of probability $\partial_\mu J^\mu = 0$ and, still more important, the scalar part of (2.16) or (2.10) reads simply: $\mathcal{L} = 0$ where $\mathcal{L}$ is the Lagrangian density of the Dirac equation. This Lagrangian density and the whole equation (2.16) are form invariant under Cl_3^* because (2.8) implies

$$\rho'e^{i\beta'} = \det(\phi'_e) = \det(M\phi_e) = \det(M)\det(\phi_e) = r\rho e^{i(\beta+\theta)} \quad (2.17)$$

$$\rho' = r\rho; \quad \beta' = \beta + \theta \quad \text{mod } 2\pi. \quad (2.18)$$

The form invariance of the wave equation is satisfied if and only if the mass term satisfies

$$m\rho = m'\rho' = m'r\rho; \quad m = rm'. \quad (2.19)$$

I replace the $\mathcal{L}_+^\dagger$ group (but actually the $SL(2, \mathbb{C})$ group) by $Cl_3^* = GL(2, \mathbb{C})$. Therefore m and ρ are no more invariant. Only the $m\rho$ product is invariant, this is linked to the existence of the Planck constant [10].

The electromagnetism uses

$$F = \vec{E} + i\vec{H}, \quad A = A^0 + \vec{A}; \quad B = B^0 + \vec{B} \quad (2.20)$$

$$j = j^0 + \vec{j}; \quad k = k^0 + \vec{k} \quad (2.21)$$

where $\vec{E}$ is the electric field, $\vec{H}$ is the magnetic field, A the space-time vector potential, B the magnetic potential, j the electric density of charge and current, k the magnetic density of charge and current. Laws of electromagnetism in the void with magnetic monopoles read (See [10], chap. 4)

$$F = \nabla \widehat{A + iB}; \quad \widehat{\nabla} F = \frac{4\pi}{c} \widehat{j + ik}. \quad (2.22)$$

The electromagnetic field F satisfies [6]:

$$F'(x') = MF(x)M^{-1} \quad (2.23)$$

$$A'(x') = MA(x)M^\dagger; \quad B'(x') = MB(x)M^\dagger \quad (2.24)$$

$$j(x) = \overline{M}j'(x')\widehat{M}; \quad k(x) = \overline{M}k'(x')\widehat{M} \quad (2.25)$$

The contravariance (2.24) means that potentials move with sources. The covariance of qA , j and k is compatible with all laws of electromagnetism and relativistic mechanics [5, 6, 10]. The transformation (2.23) explains by itself why only the $SL(2, \mathbb{C})$ part of Cl_3^* was previously seen. The P rotor which plays a central role in the Hestenes' work [11] and in the Boudet's work [1], actually an element of $SL(2, \mathbb{C})$, is defined such as

$$M = \sqrt{r}e^{i\theta/2}P. \quad (2.26)$$

This gives

$$\overline{M} = \sqrt{r}e^{i\theta/2}\overline{P} = \sqrt{r}e^{i\theta/2}P^{-1} \quad (2.27)$$

$$M^{-1} = \frac{1}{\sqrt{r}}e^{-i\theta/2}\overline{P} \quad (2.28)$$

$$F' = \sqrt{r}e^{i\theta/2}PF\frac{1}{\sqrt{r}}e^{-i\theta/2}\overline{P} = PF\overline{P} \quad (2.29)$$

and F transforms as if $r = 1$ and $\theta = 0$. The electromagnetic field (and more generally all gauge fields) transforms in such a way that we see only the relativistic Lorentz rotation induced by the P term. Velocities are also independent from r . Then under the dilation R , with ratio r induced by any M in Cl_3^* , I get [5]:

$$e' = r^2e; \quad \hbar' = r^4\hbar; \quad m'_0 = r^3m_0; \quad m' = r^{-1}m. \quad (2.30)$$

Electric charge, proper mass and Planck "constant" are changed in the dilation R induced by a M in Cl_3^* if $r \neq 1$. The dilation is the composition of the Lorentz rotation induced by P and an homothety with ratio r , in any order. We evidently get the results of restricted relativity if $r = 1$.

3. Electro-weak Interactions

The electro-weak theory needs three spinorial waves in the electron-neutrino case: the right ξ_e and the left η_e of the electron and the left spinor η_n of the electronic neutrino. The form invariance of the Dirac theory imposes to use a wave Ψ_l satisfying

$$\Psi_l = \begin{pmatrix} \phi_e & \phi_n \\ \widehat{\phi}_n & \widehat{\phi}_e \end{pmatrix}; \quad \phi_e = \sqrt{2} \begin{pmatrix} \xi_{1e} & -\eta_{2e}^* \\ \xi_{2e} & \eta_{1e}^* \end{pmatrix}; \quad \phi_n = \sqrt{2} \begin{pmatrix} 0 & -\eta_{2n}^* \\ 0 & \eta_{1n}^* \end{pmatrix} \quad (3.1)$$

$$\widehat{\phi}_e = \sqrt{2} \begin{pmatrix} \eta_{1e} & -\xi_{2e}^* \\ \eta_{2e} & \xi_{1e}^* \end{pmatrix}; \quad \widehat{\phi}_n = \sqrt{2} \begin{pmatrix} \eta_{1n} & 0 \\ \eta_{2n} & 0 \end{pmatrix}; \quad \eta_n = \begin{pmatrix} \eta_{1n} \\ \eta_{2n} \end{pmatrix}. \quad (3.2)$$

The wave is a function of space and time with value into the space-time algebra $Cl_{1,3}$. The standard model uses only a left wave for the neutrino, this may be seen in (3.1)². I use the old matrix representation (1.3). Under the dilation R with ratio r induced by M we have

$$\xi' = M\xi; \quad \eta' = \widehat{M}\eta; \quad \eta'_n = \widehat{M}\eta_n; \quad \phi'_e = M\phi_e; \quad \phi'_n = M\phi_n \quad (3.3)$$

$$\Psi'_l = \begin{pmatrix} \phi'_e & \phi'_n \\ \widehat{\phi}'_n & \widehat{\phi}'_e \end{pmatrix} = \begin{pmatrix} M & 0 \\ 0 & \widehat{M} \end{pmatrix} \begin{pmatrix} \phi_e & \phi_n \\ \widehat{\phi}_n & \widehat{\phi}_e \end{pmatrix} = N\Psi_l \quad (3.4)$$

The form (3.1) of the wave is compatible with the charge conjugation used in the standard model: the positron wave $\psi_{\bar{e}}$ satisfies

$$\psi_{\bar{e}} = i\gamma_2\psi_e^* \Leftrightarrow \widehat{\phi}_{\bar{e}} = \widehat{\phi}_e\sigma_1 \quad (3.5)$$

Then Ψ_l contains the electron wave ϕ_e , the neutrino wave ϕ_n and also the positron wave $\phi_{\bar{e}}$ and the antineutrino wave $\phi_{\bar{n}}$:

$$\Psi_l = \begin{pmatrix} \phi_e & \phi_n \\ \widehat{\phi}_{\bar{e}}\sigma_1 & \widehat{\phi}_{\bar{n}}\sigma_1 \end{pmatrix}; \quad \phi_{\bar{e}} = \sqrt{2} \begin{pmatrix} \xi_{1\bar{e}} & -\eta_{2\bar{e}}^* \\ \xi_{2\bar{e}} & \eta_{1\bar{e}}^* \end{pmatrix}; \quad \phi_{\bar{n}} = \sqrt{2} \begin{pmatrix} \xi_{1\bar{n}} & 0 \\ \xi_{2\bar{n}} & 0 \end{pmatrix} \quad (3.6)$$

And the antineutrino has consequently only a right wave. With:

$$a_1 = \det(\phi_e) = \phi_e\bar{\phi}_e = 2(\xi_{1e}\eta_{1e}^* + \xi_{2e}\eta_{2e}^*) \quad (3.7)$$

$$a_2 = 2(\xi_{1\bar{e}}\eta_{1n}^* + \xi_{2\bar{e}}\eta_{2n}^*) = 2(\eta_{2e}^*\eta_{1n}^* - \eta_{1e}^*\eta_{2n}^*) \quad (3.8)$$

$$a_3 = 2(\xi_{1e}\eta_{1n}^* + \xi_{2e}\eta_{2n}^*) \quad (3.9)$$

Ψ_l satisfies:

$$\det(\Psi_l) = a_1a_1^* + a_2a_2^*. \quad (3.10)$$

and the $\Psi_l(x)$ matrix is usually invertible. To get the gauge group of electroweak interactions I used [6] two projectors $P_{\pm}$ and four operators P_{μ} , where $\mu = 0, 1, 2, 3$, satisfying:

$$P_{\pm}(\Psi) = \frac{1}{2}(\Psi \pm \mathbf{i}\Psi\gamma_{21}); \quad \mathbf{i} = \gamma_0\gamma_1\gamma_2\gamma_3; \quad \gamma_{21} = \gamma_2\gamma_1 \quad (3.11)$$

$$P_0(\Psi) = \Psi\gamma_{21} + P_{-}(\Psi)\mathbf{i} \quad (3.12)$$

$$P_1(\Psi) = P_{+}(\Psi)\gamma_3\mathbf{i} \quad (3.13)$$

$$P_2(\Psi) = P_{+}(\Psi)\gamma_3 \quad (3.14)$$

$$P_3(\Psi) = P_{+}(\Psi)(-\mathbf{i}) \quad (3.15)$$

These operators generate the Lie algebra of $U(1) \times SU(2)$. The covariant derivative of the Weinberg-Salam model:

$$D_{\mu} = \partial_{\mu} - ig_1 \frac{Y}{2} B_{\mu} + ig_2 T_j W_{\mu}^j \quad (3.16)$$

² Then only a 12-dimensional subspace of $Cl_{1,3}$ is used. The reward is the remarkable identity $\det(\Psi_l) = |a_1|^2 + |a_2|^2$. The isomorphism between Cl_3 and $M_2(\mathbb{C})$ is an isomorphism of real algebras. All dimensions of linear spaces are here dimensions on $\mathbb{R}$, not dimensions on $\mathbb{C}$. Quantum physics only needs real Clifford algebras.

where Y is the weak hypercharge ($Y_L = -1$, $Y_R = -2$ for the electron), has a very simple translation in the $Cl_{1,3}$ frame:

$$\mathbf{D} = \boldsymbol{\partial} + \frac{g_1}{2} \mathbf{B} P_0 + \frac{g_2}{2} (\mathbf{W}^1 P_1 + \mathbf{W}^2 P_2 + \mathbf{W}^3 P_3) ; \quad \mathbf{D} = \tilde{N} \mathbf{D}' N \quad (3.17)$$

$$\mathbf{D} = \gamma^\mu D_\mu ; \quad \boldsymbol{\partial} = \gamma^\mu \partial_\mu ; \quad \mathbf{B} = \gamma^\mu B_\mu ; \quad \mathbf{W}^j = \gamma^\mu W_\mu^j \quad (3.18)$$

Because we get from these definitions:

$$D_\mu \xi_e = \partial_\mu \xi_e + i g_1 B_\mu \xi_e \quad (3.19)$$

$$D_\mu \eta_e = \partial_\mu \eta_e + i \frac{g_1}{2} B_\mu \eta_e - i \frac{g_2}{2} [(W_\mu^1 + i W_\mu^2) \eta_n - W_\mu^3 \eta_e] \quad (3.20)$$

$$D_\mu \eta_n = \partial_\mu \eta_n + i \frac{g_1}{2} B_\mu \eta_n - i \frac{g_2}{2} [(W_\mu^1 - i W_\mu^2) \eta_e + W_\mu^3 \eta_n] \quad (3.21)$$

which is equivalent to (3.16). The Weinberg–Salam θ_W angle satisfies

$$B + i W^3 = e^{i\theta_W} (A + i Z^0). \quad (3.22)$$

The $U(1) \times SU(2)$ gauge group is obtained by exponentiation. If a^μ are four real parameters we use the gauge transformation

$$\Psi'_l = [\exp(a^\mu P_\mu)](\Psi_l). \quad (3.23)$$

The wave Eq. [8] reads

$$\mathbf{D}\Psi_l \gamma_{012} + m \rho_1 \chi_l = 0; \quad \gamma_{012} = \gamma_0 \gamma_1 \gamma_2 \quad (3.24)$$

where

$$\rho_1 = \sqrt{a_1 a_1^* + a_2 a_2^* + a_3 a_3^*} \quad (3.25)$$

$$\chi_l = \frac{1}{\rho_1^2} \begin{pmatrix} a_1^* \phi_e + a_2^* \phi_n \sigma_1 + a_3^* \phi_n & -a_2^* \phi_{eL} \sigma_1 + a_3^* \phi_{eR} \\ a_1 \hat{\phi}_{eL} \sigma_1 + a_3 \hat{\phi}_{eR} & a_1 \hat{\phi}_e - a_2 \hat{\phi}_n \sigma_1 + a_3 \hat{\phi}_n \end{pmatrix} \quad (3.26)$$

$$\phi_{eR} = \phi_e \frac{1 + \sigma_3}{2}; \quad \phi_{eL} = \phi_e \frac{1 - \sigma_3}{2}. \quad (3.27)$$

This wave equation is equivalent to the invariant equation:

$$\tilde{\Psi}_l (\mathbf{D}\Psi_l) \gamma_{012} + m \rho_1 \tilde{\Psi}_l \chi_l = 0; \quad \tilde{\Psi}_l = \begin{pmatrix} \bar{\phi}_e & \phi_n^\dagger \\ \bar{\phi}_n & \phi_e^\dagger \end{pmatrix}. \quad (3.28)$$

The form invariance under Cl_3^* of this equation results from (1.11), (2.8), (2.11), (2.12), (3.4) and (3.17). We get:

$$\tilde{\Psi}'_l (\mathbf{D}' \Psi'_l) \gamma_{012} = \tilde{\Psi}_l (\mathbf{D}\Psi_l) \gamma_{012} \quad (3.29)$$

and also [8]

$$m \rho_1 = m' r \rho_1 = m' \rho'_1 ; \quad \tilde{\Psi}'_l \chi'_l = \tilde{\Psi}_l \chi_l. \quad (3.30)$$

The wave equation is also gauge invariant under the gauge transformation (3.17) [8], becoming:

$$0 = \tilde{\Psi}'_l(\mathbf{D}'\Psi'_l)\gamma_{012} + m\rho\tilde{\Psi}'_l\chi' \quad (3.31)$$

$$\mathbf{D}' = \boldsymbol{\partial} + \frac{g_1}{2}\mathbf{B}'P_0 + \frac{g_2}{2}(\mathbf{W}'^1P_1 + \mathbf{W}'^2P_2 + \mathbf{W}'^3P_3) \quad (3.32)$$

$$B'_\mu = B_\mu - \frac{2}{g_1}\partial_\mu a^0 ; \mathbf{B}' = \gamma^\mu B'_\mu \quad (3.33)$$

$$W'^j_\mu P_j = \left[\exp(a^k P_k) W^j_\mu P_j - \frac{2}{g_2} \partial_\mu [\exp(a^k P_k)] \right] \exp(-a^k P_k) \quad (3.34)$$

$$\mathbf{W}'^j = \gamma^\mu W'^j_\mu. \quad (3.35)$$

Two amongst the fourteen numeric equations equivalent to (3.29) are remarkable: the real part of (3.29) reads:

$$\mathcal{L} = 0 ; \mathcal{L} = \langle \tilde{\Psi}_l(\mathbf{D}\Psi_l)\gamma_{012} \rangle + m\rho_1 \quad (3.36)$$

We have then, for the pair electron-neutrino as for the alone electron, a double link between wave equation and Lagrangian formalism: the wave equation may be obtained by the Lagrange equations from a Lagrangian density. Reciprocal relation: the Lagrangian density comes as real part of the invariant wave equation. This explains why there is a principle of minimum.

The other remarkable numeric equation reads

$$\partial_\mu(D_0^\mu + D_n^\mu) = 0 ; D_0 = \phi_e\phi_e^\dagger ; D_n = \phi_n\phi_n^\dagger. \quad (3.37)$$

A conservative current exists, the total current $D_0 + D_n$ ³.

4. Electro-weak and Strong Interactions

The standard model adds to the leptons (electron and its neutrino) in the first “generation” two quarks u and d with three states each. Weak interactions acting only on left waves of quarks (and right waves of antiquarks) we have 8 left spinors instead of 2. It is enough to add 2 dimensions to the space⁴. With our matrix representation it is enough to work with 8×8 matrices. So I read the wave of all fermions of the first generation as follows:

$$\Psi = \begin{pmatrix} \Psi_l & \Psi_r \\ \Psi_g & \Psi_b \end{pmatrix} ; \Psi_r = \begin{pmatrix} \phi_{dr} & \phi_{ur} \\ \hat{\phi}_{ur} & \hat{\phi}_{dr} \end{pmatrix} = \begin{pmatrix} \phi_{dr} & \phi_{ur} \\ \hat{\phi}_{\bar{u}r}\sigma_1 & \hat{\phi}_{\bar{d}r}\sigma_1 \end{pmatrix} \quad (4.1)$$

$$\Psi_g = \begin{pmatrix} \phi_{dg} & \phi_{ug} \\ \hat{\phi}_{ug} & \hat{\phi}_{dg} \end{pmatrix} ; \Psi_b = \begin{pmatrix} \phi_{db} & \phi_{ub} \\ \hat{\phi}_{ub} & \hat{\phi}_{db} \end{pmatrix}. \quad (4.2)$$

The Ψ wave is now a function of space and time with value into $Cl_{1,5} = Cl_{5,1}$ which is a sub-algebra (on the real field) of $Cl_{5,2} = M_8(\mathbb{C})$. The covariant derivative (3.16) becomes

³ This current generalizes the probability current.

⁴Needing a link between the reversion in space algebra, in space-time algebra and in the extended algebra, we cannot use other algebras, only Cl_3 , $Cl_{1,3}$ and $Cl_{1,5} = Cl_{5,1}$. Calculations are simpler in $Cl_{1,5}$ than in $Cl_{5,1}$. The signature of space-time is physically determined in General Relativity.

$$\underline{\mathbf{D}} = \underline{\mathbf{\partial}} + \frac{g_1}{2} \underline{\mathbf{B}} P_0 + \frac{g_2}{2} (\underline{\mathbf{W}}^1 P_1 + \underline{\mathbf{W}}^2 P_2 + \underline{\mathbf{W}}^3 P_3) \quad (4.3)$$

$$\underline{\mathbf{D}} = \sum_{\mu=0}^3 L^\mu D_\mu; \quad \underline{\mathbf{\partial}} = \sum_{\mu=0}^3 L^\mu \partial_\mu; \quad \underline{\mathbf{B}} = \sum_{\mu=0}^3 L^\mu B_\mu; \quad \underline{\mathbf{W}}^j = \sum_{\mu=0}^3 L^\mu W_\mu^j \quad (4.4)$$

$$L_\mu = \begin{pmatrix} 0 & \gamma_\mu \\ \gamma_\mu & 0 \end{pmatrix}, \quad \mu = 0, 1, 2, 3; \quad L_4 = \begin{pmatrix} 0 & -I_4 \\ I_4 & 0 \end{pmatrix}; \quad L_5 = \begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix} \quad (4.5)$$

We use two projectors satisfying

$$P_\pm(\Psi) = \frac{1}{2}(\Psi \pm \mathbf{i}\Psi L_{21}); \quad \mathbf{i} = L_{0123} \quad (4.6)$$

Three operators act on quarks like on leptons:

$$\underline{P}_1(\Psi) = \underline{P}_+(\Psi) L_{35}; \quad \underline{P}_2(\Psi) = \underline{P}_+(\Psi) L_{5012}; \quad \underline{P}_3(\Psi) = \underline{P}_+(\Psi)(-\mathbf{i}). \quad (4.7)$$

The fourth operator acts differently on the lepton and on the quark sector:

$$\underline{P}_0(\Psi) = \begin{pmatrix} P_0(\Psi_l) & P'_0(\Psi_r) \\ P'_0(\Psi_g) & P'_0(\Psi_b) \end{pmatrix}; \quad P_0(\Psi_l) = \Psi_l \gamma_{21} + P_-(\Psi_l) \mathbf{i} \quad (4.8)$$

$$P'_0(\Psi_c) = -\frac{1}{3} \Psi_c \gamma_{21} + P_-(\Psi_c) \mathbf{i}, \quad c = r, g, b. \quad (4.9)$$

These definitions are absolutely all that you have to change to go from the lepton case into the quark case, to get the gauge group of electro-weak interactions.

To get the generators of the $SU(3)$ gauge group of chromodynamics I consider two new projectors:

$$P^+ = \frac{1}{2}(I_8 + L_{012345}); \quad P^- = \frac{1}{2}(I_8 - L_{012345}) \quad (4.10)$$

and eight operators Γ_k , $k = 1, 2, \dots, 8$ so defined (shortening Ψ_c into c):

$$\Gamma_1(\Psi) = \frac{1}{2}(L_4 \Psi L_4 + L_{01235} \Psi L_{01235}) = \begin{pmatrix} 0 & g \\ r & 0 \end{pmatrix} \quad (4.11)$$

$$\Gamma_2(\Psi) = \frac{1}{2}(L_5 \Psi L_4 - L_{01234} \Psi L_{01235}) = \begin{pmatrix} 0 & -\mathbf{i}g \\ \mathbf{i}r & 0 \end{pmatrix} \quad (4.12)$$

$$\Gamma_3(\Psi) = P^+ \Psi P^- - P^- \Psi P^+ = \begin{pmatrix} 0 & r \\ -g & 0 \end{pmatrix} \quad (4.13)$$

$$\Gamma_4(\Psi) = L_{01253} \Psi P^- = \begin{pmatrix} 0 & b \\ 0 & r \end{pmatrix}; \quad \Gamma_5(\Psi) = L_{01234} \Psi P^- = \begin{pmatrix} 0 & -\mathbf{i}b \\ 0 & \mathbf{i}r \end{pmatrix} \quad (4.14)$$

$$\Gamma_6(\Psi) = P^- \Psi L_{01253} = \begin{pmatrix} 0 & 0 \\ b & g \end{pmatrix}; \quad \Gamma_7(\Psi) = -\mathbf{i}P^- \Psi L_4 = \begin{pmatrix} 0 & 0 \\ -\mathbf{i}b & \mathbf{i}g \end{pmatrix} \quad (4.15)$$

$$\Gamma_8(\Psi) = \frac{1}{\sqrt{3}}(P^- \Psi L_{012345} + L_{012345} \Psi P^-) = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & r \\ g & -2b \end{pmatrix}. \quad (4.16)$$

We explained in [6] how this is equivalent to the eight generators λ_k of $SU(3)$. Everywhere in (4.11) to (4.16) the eight matrices $\Gamma_k(\Psi)$ have a zero left up term, therefore all Γ_k project the wave on its quark sector. The physical

translation is: leptons do not interact by strong interactions, this comes from the structure itself of the quantum wave. Now with

$$\begin{aligned}\underline{\mathbf{D}} &= \sum_{\mu=0}^3 L^\mu D_\mu; \quad \underline{\boldsymbol{\partial}} = \sum_{\mu=0}^3 L^\mu \partial_\mu; \quad \underline{\mathbf{B}} = \sum_{\mu=0}^3 L^\mu B_\mu \\ \underline{\mathbf{W}}^j &= \sum_{\mu=0}^3 L^\mu W_\mu^j; \quad \underline{\mathbf{G}}^k = \sum_{\mu=0}^3 L^\mu G_\mu^k\end{aligned}\quad (4.17)$$

where the eight $\underline{\mathbf{G}}^k$ are named “gluons”, the covariant derivative reads

$$\underline{\mathbf{D}} = \underline{\boldsymbol{\partial}} + \frac{g_1}{2} \underline{\mathbf{B}} P_0 + \frac{g_2}{2} \sum_{j=1}^3 \underline{\mathbf{W}}^j P_j + \frac{g_3}{2} \sum_{k=1}^8 \underline{\mathbf{G}}^k \mathbf{i}\Gamma_k. \quad (4.18)$$

The gauge group is obtained by exponentiation. We use four numbers a^μ and eight numbers b^k . We let

$$S = S_0 + S_1 + S_2; \quad S_0 = a^0 P_0; \quad S_1 = \sum_{j=1}^3 a^j P_j; \quad S_2 = \sum_{k=1}^8 b^k \mathbf{i}\Gamma_k. \quad (4.19)$$

We get

$$\begin{aligned}\exp(S) &= \exp(S_0) \exp(S_1) \exp(S_2) = \exp(S_0) \exp(S_2) \exp(S_1) \\ &= \exp(S_2) \exp(S_1) \exp(S_0) = \dots\end{aligned}\quad (4.20)$$

in any order, because:

$$\underline{P}_0 \underline{P}_j = \underline{P}_j \underline{P}_0, \quad j = 1, 2, 3 \quad (4.21)$$

$$\underline{P}_\mu \mathbf{i}\Gamma_k = \mathbf{i}\Gamma_k \underline{P}_\mu, \quad \mu = 0, 1, 2, 3, \quad k = 1, 2, \dots, 8. \quad (4.22)$$

Therefore the set $G = \{\exp(S)\}$ is a $U(1) \times SU(2) \times SU(3)$ Lie group. The gauge transformation reads

$$\Psi' = [\exp(S)](\Psi); \quad \underline{\mathbf{D}} = L^\mu D_\mu; \quad \underline{\mathbf{D}}' = L^\mu D'_\mu; \quad B'_\mu = B_\mu - \frac{2}{g_1} \partial_\mu a^0 \quad (4.23)$$

$$\underline{\mathbf{D}}' = \underline{\boldsymbol{\partial}} + \frac{g_1}{2} \underline{\mathbf{B}}' P_0 + \frac{g_2}{2} \sum_{j=1}^3 \underline{\mathbf{W}}'^j P_j + \frac{g_3}{2} \sum_{k=1}^8 \underline{\mathbf{G}}'^k \mathbf{i}\Gamma_k \quad (4.24)$$

$$W'^j_\mu P_j = [\exp(S_1) W^j_\mu P_j - \frac{2}{g_2} \partial_\mu [\exp(S_1)]] \exp(-S_1) \quad (4.25)$$

$$G'^k_\mu \mathbf{i}\Gamma_k = [\exp(S_2) G^k_\mu \mathbf{i}\Gamma_k - \frac{2}{g_3} \partial_\mu [\exp(S_2)]] \exp(-S_2). \quad (4.26)$$

We then get the gauge group of the standard model, automatically, and not another group. It is possible to get operators exchanging Ψ_l and Ψ_c , $c = r, g, b$ like Γ_1 exchanging Ψ_r and Ψ_g but the difference between P_0 and P'_0 forbids the commutativity. Then we cannot get a greater group than the preceding $U(1) \times SU(2) \times SU(3)$ gauge group.

I got also a remarkable identity [10] allowing $\det(\Psi) \neq 0$ and $\Psi(x)$ is usually invertible. The existence of the inverse allows the construction of the wave of systems of fermions (See [4] and [6] 4.4.1). We got the wave equation for electron+neutrino+quarks u and d [9].

We know three generations of leptons and quarks and the standard model study separately these three generations. The reason is simply that our physical space is 3-dimensional, and we get the wave equation of leptons three times. One of the three is (3.28) that reads:

$$0 = \tilde{\Psi}_3(\mathbf{D}_3\Psi_3)\gamma_{012} + m_3\rho\tilde{\Psi}_3\chi_3$$

$$\mathbf{D}_3 = \mathbf{D}; \Psi_3 = \Psi_l; \chi_3 = \chi_l; m_3 = m; \rho = \rho_1 \quad (4.27)$$

$$0 = \tilde{\Psi}_1(\mathbf{D}_1\Psi_1)\gamma_{023} + m_1\rho\tilde{\Psi}_1\chi_1 \quad (4.28)$$

$$0 = \tilde{\Psi}_2(\mathbf{D}_2\Psi_3)\gamma_{031} + m_2\rho\tilde{\Psi}_2\chi_2. \quad (4.29)$$

To go from one generation to another one is simple: I permute indices 1,2,3 of σ_j everywhere in all preceding formulas with the circular permutation p or p^2 : $p: 1 \mapsto 2 \mapsto 3 \mapsto 1$; $p^2: 1 \mapsto 3 \mapsto 2 \mapsto 1$. If p gives the muon, the wave of the pair muon-muonic neutrino follows (4.28) and this explains why a muon is like an electron, generally. But the covariant derivative is different, because in the place of (3.11) to (3.15) we must use

$$P_{\pm}^1(\Psi) = \frac{1}{2}(\Psi \pm \mathbf{i}\Psi\gamma_{32}) \quad (4.30)$$

$$P_0^1(\Psi) = \Psi\gamma_{32} + P_-(\Psi)\mathbf{i}; P_1^1(\Psi) = P_+(\Psi)\gamma_1\mathbf{i} \quad (4.31)$$

$$P_2^1(\Psi) = P_+(\Psi)\gamma_1; P_3^1(\Psi) = P_+(\Psi)(-\mathbf{i}). \quad (4.32)$$

To add two quarks with three colors each we need

$$P_0'^1(\Psi_c) = -\frac{1}{3}\Psi_c\gamma_{32} + P_-^1(\Psi_c)\mathbf{i}; c = r, g, b. \quad (4.33)$$

We must also change the link (3.5) between the wave of the particle and the wave of the antiparticle. The wave of the anti-muon must satisfy $\hat{\phi}_{\bar{\mu}} = \hat{\phi}_{\mu}\sigma_2$ and we shall have a 3 index in the case of the third generation. We must also change the definition of left and right wave. For the second generation this becomes $\phi_{\mu L} = \phi_{\mu}\frac{1}{2}(1 - \sigma_1)$; $\phi_{\mu R} = \phi_{\mu}\frac{1}{2}(1 + \sigma_1)$ and so on. The Lagrangian density, which is the scalar part of the invariant equation, must be calculated separately. Now since the Γ_k operators, generators of the $SU(3)$ group of chromodynamics, are unchanged by the circular permutation p used to pass from one generation to another one, strong interactions are unperturbed by the change of generation. This allows physical quarks composing particles to mix the generations. The mixing of waves of different generations, and the difference between what we call “left” and “right” in each generation, induce the wave of physical quarks to have both a left and a right wave.

If there are only three objects like σ_{21} , there is one other term with square -1 in Cl_3 , $i = \sigma_1\sigma_2\sigma_3$. This fourth term allows a fourth neutrino [7]. More explanations shall be available soon in [12], where we explain also how inertia and gravitation take place aside electro-weak and strong interactions. The form invariance under Cl_3^* rules all physical interactions.

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