Paul A. Calter | *How to Construct a Logarithmic Rosette
(Without Even Knowing it)*

Paul Calter explains what a logarithmic rosette is and gives some examples of their occurrence in pavements. Then he gives a simple construction method which is totally geometric and requires no calculation. He then proves that it gives a logarithmic rosette, with the exception that the spirals are made up of straight-line segments rather than curved ones.

What is a logarithmic rosette?

Rosettes (Figure 1) are common features of pavements from Roman times through the Renaissance, and have been well documented by Kim Williams.¹ They usually feature a number of concentric circles and other curves whose distance from the center of the rosette continuously increases, their intersections forming a number of curvilinear triangles whose size increases with their distance from the center of the rosette. Those other curves are usually circular arcs that are not concentric with the center of the rosette. But occasionally we see a rosette in which such a curve is a logarithmic spiral, and we call it a logarithmic rosette.

What is a logarithmic spiral?

Figure 4 shows a polar plot of a portion NPQ of a logarithmic spiral. A logarithmic spiral is a plane curve whose vectorial angle q is proportional to the logarithm of the length r of the radius vector OP . This spiral was discovered by the French mathematician René Descartes in 1638. It is seen in nature in the chambered Nautilus, animal horns, and certain plants. In the 19th century the mathematician Bernoulli discovered that the angle ϕ between the tangent T to the spiral at any point P and the radius vector OP to that point does not change as the spiral grows (Figure 2). Hence the logarithmic spiral is also called the equiangular spiral. It may have been this property of keeping its shape that prompted Bernoulli to call it the *spira mirabilis*, or miraculous spiral.²

It is helpful to compare the logarithmic spiral to the circle. The circle intersects its own radii everywhere at the same angle of 90° , while the logarithmic spiral intersects its own radii everywhere at the same angle, which may differ from 90° .

Fig. 1.
The rosette motif.

Fig. 2.
A rosette pattern from the Baptistery of S. Giovanni, Florence.
Photograph, Fratelli Alinari, Florence. Reproduced by permission.

Identifying and constructing a logarithmic rosette

Since the logarithmic spiral maintains a constant angle with the radius vector anywhere on the curve, it follows that the geometric figures formed by the intersection of logarithmic spirals in a rosette will preserve the same shape anywhere in the rosette. They will be similar figures, in the geometric sense, that the angles are preserved under magnification.

Figure 2 shows a rosette from the Baptistry of S. Giovanni in Florence; Figure 3 shows another rosette from Pompeii. In the S. Giovanni rosette, notice that the shapes of the curvilinear triangles change as we move from the center outwards, but that the shapes in the Pompeii rosette do not. That's a sure sign that the spirals in the Pompeii rosette are logarithmic.

Our object is to construct a rosette in which all the white tiles are similar isosceles triangles, and all the black tiles are similar isosceles triangles, but not similar to the white tiles. Later we will show that we indeed get a logarithmic rosette by this procedure, even if we had never heard of the logarithmic spiral.

We have to choose the values of three dimensions: the radius of the inner circle; the number of its subdivisions; the height of the first row of triangles. After that all other dimensions follow automatically.

Fig. 3.
A rosette pattern from Pompeii. Photograph, Museo Nazionale Romano.
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First Band

1. Choose the radius r_1 of the inner circle (Figure 4).
2. Subdivide the circle into n equal intervals. Let w be the width of each interval.
3. Choose the height h for the triangular black tiles in the first band.
4. Cut n black tiles, all isosceles triangles of base w and height h .
5. Install the black tiles, with base vertices on the inner circle and heights in the radial direction.
6. Connect the vertices of the black tiles. This defines the shape of the white tiles in the first band.
7. Cut n white tiles and install these in the spaces between the black tiles.

Fig. 4.
A logarithmic spiral. Drawing by the author.

Second Band

8. Cut and install n black tiles with base equal to the base of the first-band white tiles, and height chosen geometrically similar to the first-band black tiles. Similarity requires that the angles be the same as for those in the first band. A simple template can be cut to transfer the angles.
9. Connect the vertices of the second-band black tiles to give the shape of the second-band white tiles. You should find these to be geometrically similar to the white tiles in the first band. Install between the spaces between the black tiles.
10. Repeat the procedure for the subsequent bands.

Proof that this construction gives a logarithmic spiral

If $r = OP$ (Figure 4), the equation of the spiral in polar form is

$$a\theta = \ln r$$

where a is a constant. Rewriting this equation in exponential³ form gives

$$r = e^{a\theta}$$

For equal increments of $q = 1, 2, 3, \dots$, our values of r are:

$$e^a, e^{2a}, e^{3a} \dots$$

or,

$$ea, (e^a)^2, (e^a)^3 \dots$$

We recognize this as a geometric progression with a common ratio of e^a . Thus, in a logarithmic spiral, the radius vectors r , for equal increments of the polar angle θ , form a geometric progression. Conversely, *if the radius vectors to a curve form a geometric progression for equal increments of the polar angle, the curve is a logarithmic spiral.*

Returning to our construction in Figure 5, let

$$OA = r_1, OB = r_2, OC = r_3, \text{ etc.}$$

The radii r_1 and r_2 are two sides of triangle OAB. Let us denote the ratio of r_2 to r_1 by the letter k .

$$r_2 / r_1 = k$$

or,

$$r_2 = k r_1$$

Fig. 5.
Construction of a logarithmic rosette. Drawing by the author.

Here k is a constant whose value depends upon our initial choices of the radii of the inner circle and the height of the first band. Now triangle OBC is similar to triangle OAB , and since corresponding dimensions of similar figures are in proportion we can write,

$$r_3 / r_2 = k$$

$$r_3 = kr_2 = k(kr_1) = k^2r_1$$

Also, triangle OCD is similar to triangle OAB , so,

$$r_4 = kr_3 = k(k^2r_1) = k^3r_1$$

and so forth. Our radii, for equal increments of angle, thus form the geometric progression

$$r_1, kr_1, k^2r_1, k^3r_1, \dots$$

proving that the points $A, B, C, D, \dots$ lie on a logarithmic spiral.

The straight-line segments connecting these points do not, of course, lie on the spiral, but give its approximate location. If we want a logarithmic rosette with curvilinear triangles, we would lay out the rosette on thin metal or wood, as shown above, and connect the vertices of the triangles with smooth curves. The bases of the curvilinear triangles would be circular arcs concentric with the center of the rosette. Such a layout could then be cut apart and the pieces used as templates for the individual tiles.

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Notes

1. Kim Williams, "Spirals and the Rosette in Architectural Ornament", *Nexus Network Journal* 1 (1999), pp. 129-138.
2. H. E. Huntley, *The Divine Proportion* (New York: Dover, 1970), p. 168.
3. Paul and Michael Calter, *Technical Mathematics with Calculus* (New York: Wiley 2000) p. 571.

The Author

Paul A. Calter is a Visiting Scholar at Dartmouth and Professor Emeritus of Mathematics at Vermont Technical College. He has interests in both the fields of mathematics and art. He received his B.S. from Cooper Union and his M.S. from Columbia University, both in engineering, and his Masters of Fine Arts Degree at Vermont College of Norwich University. Calter has taught mathematics for over twenty-five years and is the author of ten mathematics textbooks and a mystery novel. He has been an active painter and sculptor since 1968, has participated in dozens of art shows, and has permanent outdoor sculptures at a number of locations in Vermont. Calter developed the MATC ("Mathematics Across the curriculum") course "Geometry in Art & Architecture" and has taught it at Dartmouth and Vermont Technical College, as well as giving workshops and lectures on the subject. He is presented papers at the Nexus 96 and Nexus 2000 conferences.