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Research

Gothic Construction and the Traça of a Heptagonal Apse: The Problem of the Heptagon

Abstract. The heptagonal shape and its geometric layout have been the subject of a great deal of speculation. Because some apses in Gothic cathedrals are heptagonal, there must be a methodology implicit in the layout of the geometric shape. Two particularly important sources help us arrive at an understanding: the exceptional of the caputular archive of the Cathedral of Tortosa, which contains the main neo-Platonic sources among its codices dating from thirteenth and fourteenth centuries, and the parchment known as *la traça de Guarc* (c.1345-1380), which shows the layout of the non-constructed cathedral. These sources show a heptagonal apse with an arithmetical and geometric dimension, based on a metrological and tonal musical proportion of 9/8, which is perfectly compatible with the bases of the quadrivium. The lateral and radial chapel, as the basic unit and feature element in fourteenth-century Gothic cathedral design, can be used as a pattern, and its measurement established as the basic unit for the overall proportions of the cathedral.

Gothic apses and the heptagon

In medieval Christian culture, since the days of the Fathers of the Church, the number seven had represented the Creation: the first six days of work (Gen. 1:1), and the seventh, when God rested. In the Pythagorean tradition, the number seven was the perfect and religious number, known as *telesphoros*. The humanist and mathematician Charles Bouvelles (1478-1567),¹ the author of *Geometricum Introductorium* (1503), and *Geometrie pratique* (published in Paris in 1542 with several subsequent editions), acknowledged that such an important figure for Christian symbolism as the heptagon did not appear in the *Elements* (Chap. 2.57) by Euclid (ca. 325-265 BC) [Bouvelles 1542; 1551: 25; 1608: 82-83].

The *Elementa* as translated by Adelardo of Bath (1075-1166) in 1142 considers the layout of regular polygons in Book IV,² but fails to mention either the heptagon, or the mathematical syntax in Ptolemy's (ca. 85-165) *Almagesto*, which was translated by Gerardo of Cremona (1114-1187) in around 1175.³

Only a few works were known that mentioned the heptagon: the Heptagon Book by Archimedes (287-212 B.C.) [Hogendijk 1984: 197-330; Knorr 1998: 257-271], transmitted by Arab sources – in this case, by Abu Ali al-Hasan ibn al-Haytham (ca. 965-1040) [Rashed 1976: 387-409] through Archimedes *lema*; the text written by Abu Sahl Waijan ibn Rustam al-Quhi (ca. 940-1000) [Hogendijk 1984: 231-234] on the proportional division of the segment into three parts; and finally another text on the

trisection of the right angle written by Abu Said Ahmad ibn Muhammad Al-Sijzi (ca. 945-1020) [Hogendijk 1984: 290-316 and Appendix II].

The angle trisection method influenced the construction of the heptagon discussed in the earliest medieval Latin texts, such as the *Geometria vel de triangulis libri IV* by Jordanus Nemonarius (1225-1260) [Curtze 1887: 25-32] (*Liber IV*, 23), or *Varii de Rebus Mathematicis Responsi, Liber VIII* (1593) by François Viète (1540-1603) and his *Protasis IV. Theorema* [Viète 1696: 362-363]. All these methods are completely inapplicable for the construction of Gothic apses.

The heptagonal geometric construction which we have inherited determines the side of the regular heptagon inscribed within a circumference as the height of an equilateral triangle with a side equal to the circumference's radius. It was described by Albrecht Dürer (1471-1528) in his 1525 *Underweysung der Messung, mit dem Zirckel und Richtscheit: in Linien Ebren vo gantzen Corporen*, as a corollary to the layout of the pentagon (LII.15), rather than of the heptagon (LII.11) [Dürer 1525: 27v-28v]. The method has the same matrix as the one mentioned by Mohammad Abu'l-Wafa Al-Buzjani, (940-998) in *El libro de construcciones geométricas que son necesarios para los artesanos* (ca. 990) (CII.6) and (CIII.13), which was also completed in the instructions in his *Aritmética* (P2, C.IV-V).⁴

Fig. 1. Layout of heptagon: 1) Abu'l-Wafa (ca. 990); 2) Dürer (1525).

Drawing: authors

The heptagonal layout by means of geometrical instrumentation was first refuted by Kepler (1571-1630) in his *Harmonices Mundi, Libri V*, in his *Primus Geometricus, De figurarum regularium, quae proportionibus harmonicas pariunt, ortu, classibus, ordine et differentiis, causa scientiae et deomonstraciones* (1619), in the *Propositio. Heptagonus et figurae ab eo omnes* (LI.45) [Kepler 1864: vol. V, 101-108]; and later by Gauss (1777-1855), at the end of his *Disquisitiones Arithmeticae* [1801: Section VII, Propositions 361-366].

This means that it is one of the apparently most enigmatic geometric figures, particularly if we take into account that it is the basic layout for the apses of some Gothic cathedrals. The medieval architect therefore had to consider the problem of how to lay out seven chapels around the arc of a semicircle. The problem was aggravated because the centre of the circumference necessary for laying out the seven radial chapels was inaccessible, since it was physically located inside the old cathedral's presbytery, which were replaced by the Gothic building and which were still functional. The *magister operis* therefore had to resolve three geometrical problems when building the heptagonal apse: first, finding a method to lay out the heptagon, as this method did not appear in the texts available to him at the time; second, he had to build the geometric figure without knowing where its centre was; and finally, he had to commensurably resolve the proportional ratio between the radial chapels, situated along the curved section of the apse, and the side chapels, situated along the straight section of the choir and the side aisles.

The masonry project, the magister operis and its promoters

Tortosa's Gothic cathedral, which is equidistant between those in Barcelona and Valencia, transformed and replaced the Romanic cathedral from 1345 onwards. The Cathedral as a whole has an important Capitular Archive [ACTo] in which some neoplatonic codices [Bayerri 1962: 122-503] are preserved, as well as most of the new cathedral's construction books – *llibres d'obra* [Bayerri 1962: 122-503].

Fig. 2. Apse and presbytery of Tortosa Cathedral. Photo: authors

Tortosa's cathedral chevet, with a heptagonal plan, was roofed between 1383 and 1441. It has a double ambulatory apse, which was constructed around the old Romanic cathedral. The first construction phase of the belt of radial chapels took place between 1383 and 1424, with a very low sectional proportion of 9/5. During the second phase of construction, the ambulatory, the sectional proportion increased to a proportion of 9/6. Unlike the previous vaults, it was built symmetrically (1424-1435), closing inwards from

ambulatory's mouth to the presbytery. Finally, the presbytery was covered (1435-1441), with the placing of a very large keystone with a diameter of 2.323 m and a weight of over 8.746 Kg.

The double ambulatory (1383) resulted in the structural liberation of the wall between the radial chapels of the apse, creating a concentration of the load in the abutment. This problem subsequently emerged in the controversy over Milan's cathedral in 1392. The wall suppression led to a structural change in Gothic masonry mechanics, affecting the construction's intermediate phases, and changing the buttressing conditions. In Tortosa, the chapel's geometric solution consists of a square ribbed vault organized with nine chapels around a heptagonal chevet.

The basic pattern of measurements appearing in the *Llibres d'Obra* [ACTo] is the *cana* of eight palms and the palm of twelve fingers. The *cana* of Tortosa is defined in Book IX, no. 15.5 of the *Consuetudines Dertosae* (1272) [AHCTE: cod. 53, fol. 256r], and in the copy of 1346 when the Gothic cathedral was started, *Llibre de les Costums Generals feutes de la insigne ciutat de Tortosa* [FBMPM: fol. 100r]. By comparing the unifying documents of the *cana* of Tortosa with that of Barcelona (24-VII-1593), we concluded that the *cana* of Tortosa used for the cathedral measures 1.858 m, and the palm 0.2323 m.⁵

The general drawings of the Cathedral are obtained by a combined operative method, using both direct and instrumental measurements. In the first method, most of the points chosen for the measurement are accessible, while in the second one, mainly used to determine the section, a great amount of points aren't accessible.

The direct measurements are obtained manually using a measuring tape, by means of a system of triangulation of the optimal triangle from a few setpoints. Those were previously situated using a tachometer to eliminate deviations on distances greater than 30 meters. The setpoints are located on the vertices of two polygonal lines, one drawn outside the Cathedral with nine vertices, and another one inside with four vertices.

It was used for the work a SOKKIA total station, model SET 3B11, with 10cc of angular precision in 2cc resolution mode, and a maximum range of 2500 meters using reflector (WILD GPR-1), with a precision of +/- (3mm +3 ppm).

Data are recorded with a PSION ORGANISER II electronic field book, model LZ64, with a Datapack of 32K and a Flashpack of 128K with capacity for 2000 tachymeter points. The automatic upload of data is made with SDRCOMMS V4, and the tachymetric calculation and process of codes with PROTOPO V5.00. Finally, AUTOCAD MAP is used as edition platform.

The drawings obtained with this methodology could have a deviation of at most 1%. Also, it must have taken into account that measures exposed below corresponds to the geometrical assessment performed by overlapping stakeout lines on the cathedral's plant, as some of the points were inaccessible.

Thus, cathedral chevet has the following metrological proportions: it is 150 palms wide, 100 palms deep and the radial chapels have a square plan of 21 x 21 palms. Using computer technology, it was determined that the centre points of the pillars, around which the work was raised, are equidistant by 24 palms, that is, 3 Tortosa *canas*. The metric in the apse section is 45 palms in the radial chapels, 72 palms at the ambulatory and 100 palms at the presbytery, where the keystone has a diameter of 10 palms.

Fig. 3. Metrical structure of the apse of Tortosa Cathedral. 1module = 3 palms
Drawing: authors

The importance of radial chapels, as a modular element in the cathedral's construction originates in the Gothic liturgy *Prochiron, vulgo Rationale divinorum officiorum* (1291) by the bishop of Mende, Gulielmus Durandus (1230-1296). The correspondence between the physical *ecclesia materialis* and the heavenly *ecclesia spiritalis* is defined [Sebastián 1994, 352-355]. The new pattern was first used at Clermont-Ferrand, then Narbonne, and came to Catalunya through Girona, where the Chapter established a programme of nine chapels at the apse [Street 1926: 318, 339].⁶ Direct knowledge of Durando's text in Tortosa Cathedral is by the *código* ACTo no. 58, dating from the end of the thirteenth century, the Roman incunabula dating from 1477 [ACTo: no. 258], and the Venetian incunabula from 1482 [ACTo: no. 290].

The radial chapels were placed 54 palms from the centre of the presbytery and the pillars are also equidistant at 24 palms (3 *canas*). There is therefore a relationship between the circumference radius (54 palms = 18 modulus), and the side of the fourteen-sided polygon (24 palms = 8 modulus), with a proportion of 9/8.

The *traça de Guarc* (ca. 1345-1380), ink on parchment (917x682 mm) [ACTo: Fábrika no. 49], is a plan for a Gothic cathedral chevet different to the one that was built. The existence of a modular structure with a reiterative transfer of measures – starting from the lateral chapels – was deduced from an analysis of the parchment’s auxiliary lines, traced with styllet and compass. The methodology allowed the method used to trace the octagon of the ciborium, and the double heptagon at the apse to be determined.

Fig. 4. The parchment of Antoni Guarc (ca. 1345-1380): left) original [ACTo: Fábrika no. 49]; right) authors’ tracing

Guarc’s structure is based on a modular system in base nine: this provides the proportions of 18 (9+9) for the central nave, 9 for the side aisles, and 8 for the chapels. The 3x3 structure was often used at that time. *Raymundus Lullus* (1232-1315) based his constructions on this structure, while Juan de Herrera (1530-1597), in *El trazado de la figura cúbica. Útil y necesario para entender los Principios de las cosas naturales y sus excelentes y admirables operaciones maxime según el <<Arte>> de Raumundo Lulio* (ca. 1580), still recognized the 3x3-based structures [Herrera 1988: 19-106].

The assessment of Guarc’s layout is based on the analysis of points, compass layouts and auxiliary lines. It may lead to establish an interpretive methodology of the possible graphic construction of the main geometrical figures. Over the parchment we can find some holes through the surface. Some of them appear on the perimeter where the parchment were fixed to the support (Pa1) and others are compass marks. Some of them were used to transport measures (Pa2) and others were used to draw the circumference (Pa3). The assessment of the points will give the layout sequence and the number of operations. Also, some points are used only once, others two times, and others until five times (PO.1). This last one is the main point of the auxiliary layout.

The lines are drawn with styllet of straight line (La4), or with a double-headed compass (La5). About the typology of the graphic technique, some of the lines are drawn with sharps, which mark the surface, and others are stroke-like graphite. The first ones fit

the pattern and layout the proportions, while the others are auxiliary lines, fixed later with the final ink strokes (La6).

The layout sequence was:

- T0. Fixing points: Guarc fixed the support by means of an element over the parchment edges, piercing the leather.
- T1. Fitting lines of the layout: The measurement of lines T1.1 and T1.3 are the modulus used in the rest of the drawing.
- T2. Fitting lines of the plant: The line T2.1 is the starting point to fix the proportion of the naves width. An external line is drawn, T2.2, which is divided into six equal parts.
- T3. Vertical lines, composition of the plant.
- T4. Horizontal lines, composition of the plant: Vertical lines T2 cross with the horizontal T4.1 and T4.2, which will form the structure of the nave. At this point, it is possible to choose to draw the ciborium or the apse.
- T5. Composition lines of the chevet: Draws the arch T5.1 with center on PO.1, from T4.1 to T2.1. After the compass line T5.1, the line T5.2 is drawn, which ending determines the situation of the semicircular chevet. The layout of lines T5 may be interpreted as the methodology for the heptagonal layout.
- T6. Layout of the apse lines: the line T5.2 is as long as the distance between the points (T2.1-T3.3) and (T2.1-T5.1). It determines the depth of the straight section of the presbytery, T6.1. The keystone of the presbytery, point PO.3, establishes the center of the ambulatory arcs T6.2 and its correction T6.3 from the points (T3.5-T6.1), and finally T6.4.
- The lines T7 and T8 are the main lines of the auxiliary operations to determine the construction of the radial chapels of the apse.

Fig. 5. Layout sequence: points and auxiliary lines. Images: authors

From the parchment, it has been possible to determine that module 9 on the axis of the lateral chapels was divided between 8 units for the width and one unit for the separation wall. This module of 8 in the lateral chapel was transferred to the ambulatory arc, and determined the radial chapels. In order to trace the heptagon, an arch was lowered over the straight section of the presbytery, dividing it into 8 units and 10 modules. Guarc transfers the measure of 8 to the straight section of the presbytery, creating the first chapels in the apse, and then places the measure on the arc seven times, with the opening of 8 units. From here, he establishes the interior construction of the radial chapels. He then establishes a relationship between the two axes; the width of the side nave = 9; the width of the radial chapel = 8, and its proportional ratio is therefore $9/8$.

Fig. 6. Antoni Guarc's parchment metrical structure (ca. 1345-1380). Drawings: authors

The ratio ($9/8$) between the geometric layout of the *Traça de Guarc* (ca. 1345-1380) and the general laying out of the masonry in the radial chapels (1383-1424) – 24 palms wide and traced at 54 palms – is the same. The solution between the radius of 18 modules of the circumference and 8 on the fourteen-sided polygon in turn establishes a geometric, arithmetic and metrological solution.

The method used may mean that considering a circumference with radius of 18 units, half the side of the heptagon is 8 units. Using the theory of proportions, if the presbytery width is 18, the radial chapels must be 8 modules. In order to build a chapel of 3 *canas* (24 palms) it is necessary to have a radius of 6 *canas* and 6 palms (54 palms). This proportion accentuates the neoplatonic metrological character, and establishes the ratio of the musical tone ($9/8$) between the width of the chapel and the width of the aisles, making it possible to relate the four *ars* of the *quadrivium*.

The cathedral's main measures use the Patristics of Saint Augustine (354-420), in his *Civitas Dei* [ACTo: no. 20] as a direct reference. The importance of the archive as a direct source of the promoter's knowledge means that a hypothetical syntax between Tortosa's Gothic cathedral and the philosophy of Chartre can be established. Among the neo-Platonic codices are Calcidius (fol. 350) *Timaeus translatus commentarioque instructus* [ACTo: no. 80, fols. 146r-155v]; Macrobius (fol. 400) with *Comentarii In*

Somnium Scipionis [ACTo: no. 236], with commentary from the Chartres School as *Glosae super Macrobius*; a portion of the Geometry of Book VII of *De Nuptiis Philologiae et Mercvrii*, Martianus of Capella (f. 430) [ACTo: no. 80]. From the Chartres School there is the *Geometria incerti auctoris* (ca. 1000) from Gerbertus Remensis (ca. 940-1003) [ACTo: no. 80, 159r-160v]. From Gulielmus of Conchis (1080-1145) there is the *Dragmaton Philosophiae* [ACTo: no. 144, fols. 38r-90v and part of no. 80, fols.163r-188v].

Fig. 7. left) Calcidius *Timaues*; right) Gerbertus *Geometria* [ACTo: no. 80]

The promoter's scientific knowledge was completed with the Aristotelian sources of *Physics* in codices nos. 24 and 142, and its scholastic commentators, Aegidius Romanus (1247-1316), with the *Quaestiones super librum phisicorum de Aristoteles*, no. 200, and Gualterius Burlaeus (1277-1345), with *Expositio in libros octo Physicorum Aristotelis* [ACTo: no. 226].

Also present was the physics of *impetus* from the Paris school with: Joannes Buridanus (ca. 1300-1358), *Quaestiones magistri Ioanis Buridani*, ACTo no. 108; Albertus Saxoniae (ca. 1316-1390), with *Quaestiones Alberti de Saxonia et Quaest*, no.108; Marsilius de Inghen (ca. 1330-1396), no.114 with the *Incipiunt quaestiones suppositionum magistri Mercelli de ynghen*; Nicolaus Oresmius (1323-1382), the *Magistri Nicholai Oresmi, De communicatione idiomatum* and Henrico de Hasia (ca. 1350-1427) with the *Tractatus qui intitulator «Dici di omni» a Mgro. Henrico de Hasia* (both from [ACTo: no.143]).

The Augustinian canons of Tortosa catedral undoubtedly knew of or had access to the main works of the theory of medieval science related to the *De scientia doctinali* when work began on the Gothic cathedral.

The methods for tracing the heptagon: design vs. masonry

The method used by Guarco to trace the cathedral apse has nothing in common with the major treatises in the geometrical tradition by the Gothic constructors that have survived.

In general terms, the methods for tracing the heptagon in Gothic geometry work using either angular division – the method used by the manufacturers of geometrical instruments such as the astrolabe and the quadrant – or the division of the circumference by means of the compass, as already established by the gromatic tradition. The angular division method is illustrated by the case of the *Practica Geometriae* (ca. 1125-1130) by Hugh of St. Victor (1096-1141) in the definition of *Altimetria* (pp. 10-14) [Baron 1956: 194-198; Frederick 1991: 31-70], and the *Practica de Geometriae* (1346) by Dominicus of Clavasius (f. 1346), *Instrumentum gnomonicum construer* (Monac. 410, fol. 76r) [Busard 1965: 530-531]. These methods for the angular division of the circumference arch survived until the Renaissance, and were used by Bouvelles (1478-1567) in the *Geometrie pratique* [Bouvelles 1551: 25] and in the *Compendio de Arquitectvra y Simetria de los Templos* by Simón García [1681: fol. 64v], which is the compilation of the knowledge of Rodrigo Gil de Hontañón (1500-1577). These systems led to trigonometric tracings such as the *Problemata geometrica* (1583) by Stevin (1548-1620), in his *Tomus Secundus Mathematicarum hypomnematum de Geometriae Praxi* (Lib 2. pro. 7), using a $360^\circ/7$ angular division, with the ratio of 10000000/8677676 [Stevin 1605: 66].

La siensa de atermenar (1401), by Berand Boyssset (1355-1415) contains a chapter about the layout of the heptagon: *Quapitol d'atermenar posesion partida que fos tota redona*. The graphic solution is achieved using a square shaped landmark with some clefts indicating the heptagon diagonals. This method suggests the angular division of the circumference applied to a geometric template [Portet 1995].⁷

The first geometry disseminated widely beyond the loggias of architecture was in German – the *Geometrie Deutsch* (1488) by Matthäus Roriczer (ca. 1495) – in which the heptagonal construction takes place by means of a combined operation with the compass, starting from the equilateral triangle with a side equal to the circumference radius [Roriczer 1999: 56-60; Hoffstadt 1847: 17-18; Shelby 1977: 118-119; Recht 1980: 24-25]. The major treatise writers during the Renaissance also used: Dürer's *Underweysung der Messung* (L II. 12) and the pentagon collorary (L II. 15); Juan de Arfe (1535-1603) *De varia commensvracion para la escvltvra y architectura* (1585), LI. Cap.II, Heptágono [Arfe 1585: 7v; facsimile ed. 1974: 9]. The same is true of the manuscript from the Arab tradition (L. ICap 15) by Diego López Arenas (ca. 1640); in the *Primera y sigunda parte de las reglas de la carpintería* (1616), published as *Breve compendio de la carpinteria de lo blanco, y tratado de alarifes, con la conclusion de la regla de Nicolas Tartaglia, y otras cosas tocantes a la leometria y pvntas del compas* [Lopez de Arenas 1633: 16-17; Lopez de Arenas 1912: 31-34; Nuere 2001: 155]. This tradition was continued by Fray Lorenzo of San Nicolás (1593-1679) in *Arte y Vso de Architectvra* LI, Chap. XLII [Lorenzo de San Nicolás 1633: 80v]. All of these methods operate using the same adapted matrix as the one appearing in *Libro de construcciones geométricas* (ca. 990) by Abu'l-Wafa (940-998).

Fig. 8. Heptagonal layout: 1) Roriczer (1488); 2) Dürer (1525). Drawing: authors

An approach similar to Guarco's layout (ca. 1345-1380) is found in the *Manifiesto Geométrico del Muy Reverendo Padre Fray Ignacio Muñoz, Maestro en Teología, de la Orden de Predicadores, Catedrático propietario de Matemáticas de la Real Universidad del Imperio Mexicano, y Reformador de la Hidrografía Universal y particular de todo cuanto se navega en el Mar Océano y en el Mediterráneo* (1683), by the Dominican mathematician Fray Ignacio Muñoz (1612-1685), which was published almost two centuries after the parchment [Muñoz 1683]. The *Manifiesto Geométrico* was intended as an addition to Euclid's *Elements Book IV*. It determines that the heptagon's side has a ratio of $4/9$ with the figure's main diagonal, and traces the other five vertices using a method that is different to Guarco's.⁸

The search for a geometrical solution to the heptagon layout problem continues today, through the inscription in other polygons. Miranda Lundy's *Sacred Geometry* [1998] uses a method starting with the regular triangle and hexagon, inscribed in the circumference, seven from three [Lundy 1998: 36-37]. In Mark Reynolds's construction "From Pentagon to Heptagon: A Discovery on the Generation of the Regular Heptagon from the Equilateral Triangle and Pentagon", the construction of the regular heptagon is related to the pentagon, through the inscribed and circumscribed circumference of the pentagon [Reynolds 2001].

Nevertheless, there are other methods in practical geometry which are not passed down in the learned geometry texts which are capable of tracing the heptagon. In the six-knot method, using an extended rope split into five segments, two circumferences are traced with a radius of 4 parts from both edges; their intersection determines the heptagon's side. In the fourteen-knot method, with a triangle (4-4-5) over the side of 5, a line is traced from the first knot to the opposite vertex, obtaining the heptagon side [Lundy 1998: 36-37]. In the seven equal segments method, *nesius* construction, a segment determines the base for the construction of a triangular figure, while the other six intersect two by two over the bisector of the base segment until three of its points are

aligned, forming a triangle. As a template, the method can be constructed easily with a pocket rule, which articulates the instrument in its segments.

Some of the later classical treatises dealt with a proportional solution between the side of the inscribed polygon side and the heptagon's surface, such as the *Metrica* of Heron of Alexandria (ca. 20-62) (I, 29 teorema 54), also known as *Dimetiendi rationes* (I, XX). Starting from the regular hexagon, Heron deduces that the equilateral triangle constructed by the radius – to which he assigns a length of 8 units – is 7 units high (I, IXX). Hence in proposition (I.XX), a heptagon of side 7 has a radius of 8, providing the proportion with the circumference diameter of 16:7. This gives a commensurable solution similar to Guarc's proportion [Schoene 1903: 55-59; Bruins 1964]. A similar method in *Geometria* II of Boethius, *De multiangulis figuris*, *De eptagoni* relates a six foot side with a heptagon surface of 81 feet [Boethius 1867: 420-421].

Fig. 9. Heptagon methods with $R=9$. Drawing: authors

Using a computer simulation of Guarc's method for the division of the circumference of the 18-unit radius into seven equal parts, a side of 8.011 is obtained. Analyzing the different methods, we obtain the results shown in fig. 9:

- a) Ancestral 1, rope with six knots (6/5): six of 8.010 and one of 8.014;
- b) Ancestral 2, ribbon with thirteen knots (4-4-5): six of 7.994 and one of 8.113;
- c) Ancestral 3, Nesius 7 L, equal segments: seven sides of 8.011;
- d) Heron of Alexandria (ca. 20-62) (16/7): six of 7.875 and one of 8.823;
- e) Abu'l-Wafa (ca. 990) and its derivatives: six sides of 7.994 and one of 8.113;
- f) Guarc (ca. 1345-1380) (18/8); six sides of 8.000 and one of 8.075;
- g) Fray Ignacio Muñoz [1683], (9/4): one side of 8.000, four of 8.009 and two of 8.019;
- h) Simon Stevin [1583], angular 360/14 division: seven sides of 8.011;
- i) Miranda Lundy [1998], the triangle: four of 8.046, two of 7.985 and one of 7.920;
- j) Mark Reynolds [2001], the pentagon: six sides of 8.016 and 7.981.

An analysis of the accuracy of the solutions, each of which was acceptable and valid in its own time, shows that Guarc's method has a smaller error than Abu'l-Wafa's and its derivatives, and an accuracy similar to the thirteen-knot layout. Only the ancestral six-knot method and the Fray Ignacio Muñoz's method are more accurate than Guarc's.

The sequence in which Guarc traced his Project (ca. 1345-1380) is similar to the way the cathedral chevet was built. The Gothic master therefore laid out the stonework in the same way as he drew it. A comparison of the beginning of the Gothic cathedral and Guarc's project – despite their considerable formal differences – shows that are very similar both geometrically and arithmetically.

Tracing the Gothic chevet: the heptagon without a measure

The tracing method used in the parchment design (ca. 1345-1380) must have been different to the method used in the workplace: the tracing of the project takes the arc of the ambulatory's circumference into account. When the site of the new Gothic cathedral was being staked out, the geometric centre of the circumference was not known. In Tortosa, as in Barcelona and Girona, the Romanic construction remained standing among the new structures.

At the beginning of the work, the Master asks for a *cana* and a *triangle* for the tracing of the new cathedral. The *cana* is the measuring unit: *Item fiu fer lo maestre de/la obra a-n Antoni ferrer una cana de ferre per pendre mesures de l'obra costa... .V. s (Llibres d'Obra 2 1345-1347, fol. 36v)*. However, a wooden feature, called a *triangle*, was also built: *...costa .I. post de noger per a triangle obrada d-en Francescn Vicent fuster... .VII. s. (Llibres d'Obra 2, 1345-47, fol. 51v)*. These tools were complemented by other items: *manechs de exada, perxa per a mesures, claus per a les gualgues a ferrar* [Almuni 1991: 214].

Fig. 10. The cathedral and the geometric structure of A. Guarc (ca. 1345-1375). Drawings: authors

Fig. 11. Development of Guarc's proportion; the triangle, trapeze and squares. Drawing: authors

There are various possible speculative hypotheses about the layout of the apse using two instruments: the *norma* (*Etymologiarum* XIX.18.1) or the *linea* (*Etymologiarum* XIX.18.3) [Lindsay 1911: vol. 2, 321]. Both may be operative without the centre of the inscribing circumference. The following *item* appears in the capitular library catalogue of 1420: *Ysodorum, Ethimologiarum in pergamino*, the *Etymologiarum* (ca. 634) by Isidorus Hispalensis (560-636). The canons certainly knew of the works of Rabanus Maurus (ca. 780-856) *De Rerum Naturis* (ca. 810-822), and the transmission of the Liber XXI.11 *De instrumentis aedificiorum* [Migne 1851: MPL111 0009 - 0614B].

From Guarc's proportion, it is possible to deduce the knowledge of an isosceles triangle of 18-8-18, with which a template can be built, or a trapezium measuring 9-8-9-4, with which the ambulatory was built. The metric of (18/8) is similar to the proportion in the work *Hibbur ha-Meshihah ve-ha-Tishboret* (1116) by Abraam Bar Hiia (1070-1136) (L II. 77) [Millàs 1931: 62] and in *Incipit de figuris que habent capita abscisa de quipus IIII er. Sunt genera in differencia prima* of the *Practica geometriae* (1223) by Leonardo Pisano (ca. 1180-1250) [Boncompagni 1862: II, 78-81], who built a trapezium with sides of 13-18-13-8 [Levey 1952: 257-264].

The construction of Guarc's method, 18/8, through the *norma* leads to two possible instrumental solutions: the construction by means of a square with a right angle, or a template generating the double heptagon angle. Neither of these solutions needs a geometric center for its trace (fig. 11).

The solution of the square is also twofold: either the direction of the chapel side is found, or the pillar's center point. In the former, from the ambulatory trapeze, it is possible to build a right-angled triangle with sides of 9, 2 and 8.775, thereby creating a square.

The operations by means of a square may be the symmetry of the angle over the chapter radius, as in chapter XIII "*Des Circfels und Richtscheids*", by Gualtherius Hermenius Rivius (ca. 1500-1548) in his *Der furnembsten notwendigen der gantzen Architectur angehörigen mathematischen und mechanischen Künst eygentlicher Bericht und verständliche Unterrichtung* (1548) [Ryff 1548: 13r]. Another way to operate from the diameter is from the hypotenuse, forming a right triangle of 18-7.80-16.22.

With knowledge of Guarc's proportion, the geometric construction of these squares takes place immediately, and it is also possible to operate based on the knowledge of the arithmetic proportion 8/18. However, making any instrument will require a geometric adjustment of the possible building error it creates, operating repeatedly and according to the scale of the object [Nuere 2001: 12-13].

The construction of building elements such as spiral staircases means that a strict arithmetic and geometric control of their architectural form is necessary. In the Gothic spiral staircase in the towers of San Pedro and Santa Caterina, in Tortosa Cathedral, which have 11 to 13 steps in a single turn, the master must split the circumference into thirteen equal parts. It is a more complex problem than in the heptagon, because it does not appear in the medieval treatises. The trace is therefore produced by means of a template adjusted to the circumference's length. The construction of the first steps in the tower of Santa Caterina is solved using a square template. The hypotenuse is the width of the staircase and the minor angle of the square is found adjusting the lower leg by an iterative test.

Fig. 12. Spiral staircase squares, Santa Caterina. Photo: authors

The squares-rotation method has a predecessor in the notebook of Villard of Honnecourt (ca. 1220-1250), with the construction of regular pentagon: *Par chu portait on one toor a chinc arestes* [Villard 1220: fol. 21r; Bechmann 1993: 146-148]. Another possible precedent is the abacus in the tomb of Saint-Ouen, which could be related to the construction of regular polygons [Sarrade 1986: 48, 52-54].

The second speculative hypothesis is on-site staking using a string, *linea*, solvable by the tracing of two arches with radii of 9 and 2, and building an inscribed angle (*arc capable*).

The *capable arch* method appeared in Aristotle's (384-322 B.C.) *Metaphysicorum* (Met.L.IX, 10, 1051^a, 25-30) [ACTo: no. 241]. In a similar timeframe to the beginning of Tortosa's Gothic cathedral, Bertrand Boysset (1335-1415), in *La siensa de destriar et la siensa de atermenar*, uses a methodology that is very similar to the calibration of a square [Boysset 1390: fols. 215-216v].

The Master orders the building of a *triangle*, on which a *post* is situated – a kind of planar surface where templates are traced. It is a different term from the *escayre* [ACTo: NC 1364, 12 July] used in the construction of the right angle of the Chapel of San Juan. The arch could therefore be traced on a *post*, or using a template with the edges of the radius, like the one by Bertrand Boysset in *Quapitol d'atermenar posesion partida que fós tota redona*.

The *magíster operis* could trace the new heptagonal apse above the Romanic cathedral using any of these methods, without knowing the circumference's geometric centre. The Master therefore knew that by staking a radial chapel of 24 palms, and once the chevet was liberated, the opposite chapel would be exactly 54 palms from the presbytery. It was therefore possible to build a Gothic cathedral over the preexisting one, without demolishing it and keeping it operational for worship.

Fig. 13. Staking of the apse without a circumference. Drawings: authors

The commensurability of the chapel and the arithmetic and geometric proportionality

The Prochiron Gothic liturgy confers particular importance on lateral chapels. Moreover, chapels are a financing instrument of the masonry, as well as a support providing balance during the staggered construction of Catalan cathedrals [Font y Carreras 1891: 9-14; Puig i Cadafalch 1923: 65-87; Domenge 1999: figs. 13, 14, 15]. From a metric point of view, the radial and lateral chapels must be equal. Guarc presents a solution which is metrologically commensurable for the division of the semi-circumference. The chapels along the curved section must be equal to the chapels on the straight side (fig. 14).

The commensurability of the $9/8$ ratio applied to the chapel unit, allows an immediate formal display: the apse's width and depth and if necessary, the cathedral's length as a function of the number of lateral chapels.

The system used by Guarc is similar to the one used by some medieval masters when tracing the octagon, following the tradition of Heron of Alexandria's *Metrica* [Cantor 1907: vol. 1, pt. 2, 377-379; Özdurol 2002].

Fig. 14. Construction structure of the chapels of Tortosa Cathedral. Images: authors

The utility not only determines the cathedral's large dimensions, but can define other dimensions for the masonry according to this predetermined order. As mentioned, the keystone of Tortosa's cathedral presbytery has a diameter of 2.32 m, and weighs around 8.746 Kg. The shape of the keystone must solve the liturgical symbolism of the coronation of the Virgin Mary with its diameter of 10 palms. However, in addition the neck of the keystone must accommodate the concurrence of the 9 presbytery arches.

Guarc's criterion determines that there must be a relationship between the length of the circumference and the size of the arch. The stone cutting and placement of the keystone takes place before the construction of the presbytery arches. The measurement of the ribbed vaults is determined by the keystone and the ribbed arches, and the section of the ribs of the presbytery pillars. If the proportional ratio ($9/4$; $18/8$) is known, it is possible to either determine the circumference radius as a function of the crossing nerve, or vice versa. The measurement of the ribs at the base of the pillar yields this metrological ratio; the measurement of the presbytery's ribbed arches is 18 cm and its neck is 80 cm [Lluís i Ginovart and Almuni 2011: 733-742].

The metrological basis (8, 10, 12) is also a useful approximation in the deployment of the Gothic section. The calculation of the height of an equilateral triangle is undertaken by approximations: *Gerbertus* (ca. 950-1003), whose proportion is $(7/6)$ ⁹ used in Pseudo Boethius I, that of Stornalocha (1391) $(8/7)$,¹⁰ and the same approximation by Heron of Alexandria and Guarc ($9/8$). Those approximations are very effective when linking the heptagon side with the height of the equilateral triangle provided by Abu'l-Wafa (ca. 990).

Fig. 15. Commensurability in sections *ad triangulum*. Drawings: authors

Conclusions

The radial chapel is the basic unit in Tortosa Cathedral, and determined on liturgical, economic and constructive grounds. It is the essential modulus which imposes the final form and metrology, defining a neoplatonic proportional relation in terms of the *quadrivium*. The lateral chapels acted as an auxiliary structure during the construction process. They provided the essential buttressing system while the cathedral was being built, when it needed a staggered system of the chapels and lateral naves.

The Tortosa masters solved the heptagon construction in an elementary way. Using the 18/8 ratio, they traced the cathedral chevet without using the centre of the circumference. The method consisted of the 9/8 ratio between the modulus of the lateral nave and the chapel, which is also a tonal expression in medieval music. The proportion of 9/8 also has the same metrological structure as the Tortosa *cana*, in base 8, making it a good approximation, tracing *ad triangulum*, to the height of the equilateral triangle.

The solution in Tortosa cathedral is in keeping with the tradition of Heron's *Metrica*, in which the inscription of some regular polygons is solved by means of the proportional solution of integers. The arithmetical and geometrical expressions used by some Gothic southern Masters are integers within Neoplatonic culture and far from approximations of irrational numbers.

The way in which Guarc traced the heptagon, or staked out the Tortosa apse, does not appear in medieval geometries, but it is been proven to be far more effective than the Renaissance methods because it solves the layout of the enigmatic figure geometrically and arithmetically. The method of proportionality established by Guarc can be implemented in various ways, with the *circinus*, as well as with the *norma* or the *linea*. These methods do not need to know the centre of the circumference to trace the heptagon, and can be used to build new apses around old presbyteries while the latter were still in use. The *magister* knew that if the apse was started from the gospel radial chapel it would end at a particular point – the epistle chapel – according to the measure of the radial chapel.

The canons of the Cathedral of Tortosa commissioned a cathedral with a chapel of 3 *canas* and 24 palms. Everything else opens out automatically from this unit in to a predetermined order: the width of the cathedral is 150 palms; the depth of the apse is 100 palms; if the *sesquitercia* section is opened out, it is 100 palms high, to which the big 10-palm keystone was added

Notes

1. Bouvelles, *Tractatus varii de rebus philosophicis*, conserved in Tortosa [ACTo. 300]. The compilation has two parts; the first part, of a scientific philosophical nature, is composed of: *Liber de intellectu*, *Liber de sensu*, *Liber de nichilio*, *Ars oppositorum*, *Liber de generatione*, *Liber de Sapient*, *Liber de duodecim numeris*, and *Epistole complures*. The second part is dedicated to mathematics, numbers and polygonal figures: *Insup mathematicuopus quadripartitu*, *De Numenis perfectis*, *De Mathematicis Rosis*, *De Geometricis Corporibus*, *De Geometricis Suplementis*. Published in Paris by Henri Estienne, 1510.
2. Euclid, *Elements*, Book IV; the triangle (IV.5), the square (IV.6 to 9), the pentagon (IV.11 to 14), the hexagon (IV.15), and the fifteen-sided polygon (IV.16) in the first edition in Spanish [Euclid 1567: 71-79]. See Book IV [Heath 1908: vol. 2, 88-111].
3. The first edition in Spanish [Ptolemy 1578: 61-81]. On the measures of the straight lines drawn in the circle, (LI, IX); figure of square, pentagon, hexagon, decagon and dodecagon, see book I [Toomer 1984: 35-74].
4. For the geometry, see [Woepcke 1855: 218-256, 309-359] and [Suter 1922: 94-109]; for the arithmetic [Saidan 1974 367-375].
5. Felipe II (1527-1598) Cortes de Monzón (1585). Chapter 89, unification of *metric criteria*. The prosecutors of Tortosa, under oath on 24 July 1593, refer the documentation about the reduction of the cana of Tortosa to Barcelona.
6. *Capitulum Gerundense more solito congregatum statuit, voluit et ordinavit, quod caput ipsius ecclesiae de novo construeretur et hadificaretur, et circum circa ipsum caput, novem capellae fierunt et in dormitorio veteri, fierit sacristia...*
7. *Item, destrador et atermenador, si venies en luoc que aguesas destrat la posesion et aquela partida en 7 l fol.116 r l parts a requesta de gens, la qual posesion fosa tota redona et apres qualie que l'atermenesas. Atermenador, estay avisat que l'atermenes en 7 parts a quascun quap debes deforas 1 terme an 3 aguachons, e per tal seran 7 termes. L'uchen terme, tu, planta en mieg aqui von si jonhon los 7 quaps d'aquelas 7 partidas d'aquesta terra o d'aquela posesion que será tota redona et aquí, tu, en planta 1 terme an 7 aguachons que sien tots 7 partits d'una peira e que si acordon sus la rompedura o sus la joncha, quar d'autramens non serien digs aguachons lials ni vertadies, e planta los tots 7 per la forma que l'enprenta ti most[r]a que s'en siec [81] l fol.116v l.*
8. *Tómese cualquiera recta AB y divídase en 9 partes iguales y sobre esta recta hágase el Triángulo Isósceles ABC, de manera que los lados AB - CB cada uno tenga 9 partes iguales, y la base AC tenga 4 partes iguales. Cíñase este Triángulo con el círculo oculto AFIC que toque los tres puntos angulares A - B - C, buscando el centro K. Después desde el punto A, con extensión de 5 partes iguales como en AB, señálese el punto D y con la misma extensión, desde el punto C, señálese el punto E. Júntese el punto A con D, tirando la recta AD hasta la circunferencia, notando en ella el punto H, asimismo tírese la recta CE, notando en la circunferencia el punto G. Además de esto, el ángulo conocido HAC pártase por medio con la recta oculta AI, y asimismo el ángulo conocido GCA divídase por medio con la recta CF. Últimamente los puntos hallados en la circunferencia júntense consecutivamente con rectas, A con F, éste con G, éste con B, éste con H, éste con I y últimamente I con C. Digo que la figura rectilínea hallada AFGBHIC es Heptágono regular* [Muñoz 1684].
9. *Epistola ad Adelboldum causa diversitatis arearum trigoni aequilaterum* [Olleris 1867: 477-478; Bubnov 1899: 43-45; Riché and Callu 1993: vol. 2, 700-705] and *Geometria Gerberti* Cap. XLIX [Olleris 1867: 450], *Geometria incerti auctoris* IV, 8 [Bubnov 1899: 342-343].
10. The classic references are [Frankl 1945: 45-60; Panofsky 1945: 61-64; Ackerman 1949: 84-111]. See the qualification of [Beaujouan 1963: 555-563].

Abbreviations used

ACTo. Arxiu Capitular Tortosa.

AHCTE. Arxiu Històric Comarcal Terres Ebre.

FBMPM. Fundació Bertomeu March Palma Mallorca.

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