Touring Persia with a Guide Named ... Hermann Weyl

Abstract. A journey across the lands that were part of Persia long ago offers a friendly introduction to symmetry and symmetry groups, as presented in Hermann Weyl's seminal and popular book, *Symmetry* (1952). Weyl's intent was to show how geometrical transformations first, then mathematical structures, could be better understood from a cultural point of view through art and architecture. Our intent is to provide a complementary set of selected pictures of Persian monuments to illustrate Weyl's ideas. Following the master, we have focused on different kinds of symmetries, starting from the simplest and oldest to those that are more complex, disregarding chronology or geography within the lands of Persia.

Introduction

Just before he retired from the Institute for Advanced Study in Princeton, New Jersey, Hermann Weyl (1885-1955) gave four "semi-popular" lectures on symmetry. These were gathered in a book simply entitled *Symmetry*, first published in 1952, "a masterful and fascinating survey of symmetry in sculpture, painting, architecture, ornament and design," according to *Scientific American*. At the time it was and still remains the kind of book that professional mathematicians seldom write: Weyl planned to demonstrate to people who were not acquainted with mathematics how familiar the mathematical concepts of symmetries and groups had been, from the dawn of civilization to the then brand new advances in physics, such as relativity and quantum mechanics. He explained in the foreword :

I aim at two things: on the one hand to display the great variety of applications of the principle of symmetry in the arts, in inorganic and organic nature, on the other hand to clarify step by step the philosophico-mathematical significance of the idea of symmetry. As readers of this book I had a wider circle in mind than that of learned specialists [Weyl 1952: preface (not numbered)].

Consequently, his book surprises some readers: one will not find mathematical proofs nor even a bibliography. Definitions are precise, but the text runs along without pausing to introduce them. Although organized very carefully, dealing with symmetry from its common meaning to sophisticated achievements in contemporary mathematics and physics, this small book offers a stroll through places and time with unexpected jumps and backtracks. The backbone of the book demonstrates the growing complexity of geometrical and algebraic concepts; it must be kept in mind that the idea of *symmetry* is one of the oldest in mankind's development, while the concept of *group* is one of the newest, having emerged in the first half of the nineteenth century, although it was not applied to geometry until the second half of the century.

It seems evident that Weyl mostly journeyed virtually through libraries to locate relevant pictures and, as a result of then contemporary standards, the book contained only black and white snapshots. Half a century later, however, there is place for a work to

combine new techniques and to extend the area of investigation geographically. Even given greater facilities for traveling, this was nonetheless challenging in two directions at least:

- Attempting to obtain a representative collection of illustrations not restricted to European or ancient Mediterranean civilizations (I have undertaken this on my website, and it is a work in progress; see [Juhel]);
- Choosing a geographic region that seemed underrated in Weyl's book – not due to the author's taste or insufficient knowledge, but perhaps to a lack of images – and devising an itinerary through it that is parallel to that of Weyl. From this point of view, Persia was an ideal and quite evident choice.

Persia has had such a long and varied history and has covered such a large territory over time that it can be considered living testimony to Weyl's ideas about symmetry. It offers all kinds of examples to illustrate his points. Roughly speaking, from the time of Darius and the Achemenids (550-330 BCE) to Shah Abbas and the Safavids (1501-1736 CE), Middle Eastern and Central Asian countries comprised different parts of an empire, the extent of which varied through time, but always included a broad zone from Anatolia to the Indus. We shall choose our examples as widely as possible in this area, approximately corresponding to the extent of the Sasanian Empire (224-651 BCE) (fig. 1), keeping in mind that the borders of past civilizations are often quite different from political and contemporary boundaries. In our selection we also include a few monuments that attest to Persian influence in eastern China and northern India.

Fig. 1. The Sasanian Empire (about 500 CE). Map by F. Dany

We follow the plan of Weyl's book and classify examples according to the symmetries involved rather than by geographical or chronological considerations.

From harmony to symmetry

At the beginning of his book, Weyl points out an older common meaning of the word *symmetry* as a general feeling of harmony and balance. Even someone who is not a

mathematician or architect will experience the symmetry of Naqsh-e-Jahan Square in Isfahan or Registan Square in Samarkand as harmony and balance. But Weyl turns at once to the precise meaning of a geometric transformation and starts studying the simplest kind, a reflection across a line or a plane. It is the very first geometric transformation that artists used after having observed that human or animal bodies are invariant through it, so it is found in the oldest representation of idols: no fidelity to reality forced sculptors to carve two figures facing one another, each the reflected image of the other. One of two examples of Persian art in Weyl's book is taken from Darius's palace in Susa (490 BCE). The selected picture did not show the Mazdean winged sun above, but, as our image shows, its own reflection invariance respects this general symmetry, too (fig. 2).

Fig 2. Susa Relief, glazed bricks (Teheran Museum)

Fig 3. Hittite Relief from Gaziantep (Ankara Museum)

It is well known that Persian art was nourished by, and later incorporated, many previous elements from all the peoples in the Achaemenid Empire. Reliefs with similar symmetry also existed in the Babylonian Empire, and in the Hittite world (fig. 3). Symmetry remained a key feature in the plans and elevations of palaces and places of worship after the Arab conquests. The elaborately decorated and enlarged portals of

mosques (fig. 4), called *pishtaqs*, celebrate reflection at the highest degree, where symmetry exists with respect to an axial plane, no longer a straight line.

Fig. 4. Pishtaq of the Friday Mosque, Yazd (Iran)

Perfect symmetry or not?

Sometimes symmetry seems to be perfect at first glance, then some infringement may appear upon a more careful inspection. There must be a higher reason, and in Persia more than elsewhere it is always a revealing sign of a social or religious order. First consider an old door in Yazd (fig. 5): in order not to violate symmetry, it has two knockers! But take a closer look: although the placement of the knockers preserves symmetry, their shapes don't. In addition, they do not produce the same sound, a feature designed to make it possible to know whether a man or a woman was knocking, so that the door would be opened by a person of same gender!

Fig. 5. Traditional door, Yazd (Iran)

The Shah Mosque (1611-1629), on the south side of Naghsh-e Jahan Square, Isfahan, may be considered in connection with symmetry and adjustments to symmetry in at least two ways. First, the entrance *pishtaq* is carefully aligned with the axis of the rectangular square, but the domed sanctuary is not similarly aligned (fig. 6a). The reason for this is the requirement that the building be oriented toward Mecca, which clearly was of the highest priority. The architect solved the problem in a very elegant way by building two successive *pishtaqs*, one along the side facing Naghsh-e Jahan Square, and a second one facing the courtyard, perpendicular to the Mecca direction. Between the two *pishtaqs* lies the magnificent blue-tiled courtyard. In this manner, symmetries are preserved both in the square and within the mosque. Visitors reorient their steps towards Mecca in the covered passage between the first *pishtaq* and the courtyard, leaving one symmetry behind and entering another. The same device has been used in the Sheikh Lotfollah Mosque on the eastern side of Naghsh-e Jahan Square.

Fig. 6a. Shah Mosque, Isfahan (Iran)

Fig. 6b. Entrance detail

Secondly, it is often said that barely perceptible changes were made in the ornamentation as a mark of respect for God: pure symmetry would mean that man is capable of the same perfection (fig. 6b). Weyl reports a similar legend regarding ancient temples. In figure 6b, arrows point to a pattern that has not been symmetrized; it is the name “Ali” (the Prophet Muhammad’s son-in-law, who is highly regarded in Shi’ah Islam).

Our third example goes back to the foundation of the Sasanian Empire: in 226 CE, when Ardashir I became king. A stone relief, sculpted a few years later, tells the story [Bier 1993] as if it were a legend: King Ardashir takes a ring from the Persian sky god Ahura-Mazda (fig. 7).

Fig. 7. Ahura Mazda (right) and Ardashir I (left), Naqsh-e Rostam (Iran)

Symmetry predominates, depicting the king as an equal to the God; but this geometrical rule was bent twice:

- first, in contrast with the forelegs of the horses, seen as if in a mirror (the raised left leg on Ardashir's horse facing the raised right leg on Ahura-Mazda's horse), the ring is given from right hand to right hand, the strength of the symbol having a higher priority. Weyl discusses this "mathematical philosophy of left and right," as he calls it, for a few pages [1952:20-25];
- second, a more subtle sign appears on closer scrutiny: the emperor is actually a little smaller than Ahura-Mazda, as a sign of respect. This is politically very clever!

Let us go back to a theoretical kind of perfect symmetry, the one generated by water as a mirror. Nobody has mastered this concept better than Persian architects! Their influence cannot be denied in the most perfect example in the world, the Taj Mahal, built in 1632 in Agra, India.

Fig. 8. Nadir Divan Beg mosque, Bukhara (Uzbekistan)

Fig. 9. Friday mosque, Kerman (Iran)

Fig. 10. Ateshkadeh Zoroastrian Temple, near Yazd (Iran)

In semi-desert zones, the scarcity of water gives even higher value to this daring plan. We can also notice that it is a friendly introduction to polygonal symmetries considered below, for in all these cases the resulting two orthogonal line-reflections generate a four element group (a so-called Klein Four-Group $K = R_2 \times R_2$; see the Appendix for precise definition of a group, and a Cayley's table of the Klein Four-Group) which preserves the frontal view through all its transformations (figs. 8, 9, 10). In Isfahan, the Chehel Sutun Palace (completed in 1647 under Shāh Abbas II, see fig. 19a below) got its nickname “40 columns Palace” from this symmetry property, as it has only 20 columns with a mirror reflection in the reflecting pool at the front!. What other trick could give us an opportunity to see this group act on the front elevation of a building?

Translations and rotations

Translations are the simplest geometric transformations after reflections, so Weyl examines this next. As an example of a figure that is invariant under a translation and its iterations, he refers to Darius's Palace in Susa [Weyl 1952: 49, fig. 25], but he could have chosen Persepolis as well: the Apadana stairways offer various examples, from the most basic one (fig. 11a, horizontal repetition of a single form) to the most unusual one, in which the direction of the translating vector is neither horizontal nor vertical (fig. 11b)!

Fig. 11a, b. Stairways at Persepolis (Iran)

In fig. 11b, the pattern is not quite identical, of course, because the tributaries are carrying various gifts for the Great King.

In the relief at Susa, Weyl presents the case of a “fundamental” pattern (i.e., the smallest unit that generates the whole drawing through the iterations of one translation) made of two soldiers, differentiated by their garments. We can find a new implementation of this pattern of symmetry in the Apadana, where Persian soldiers

holding shields alternate with Median warriors (fig. 12a). This pattern of symmetry may also be seen in fig. 11b if we take the length of garments and the shape of caps into consideration. An older but more complex example was found in Susa at the time of Elamite civilization (thirteenth century BCE), where the pattern unit consists of three standing figures (fig. 12b).

ig. 12a. Persepolis (Iran), Apadana, Achaemenid period

Fig. 12b. Susa (Iran), Apadana, Elamite period (Louvre Museum, Paris)

Fig. 13. Sasanian Relief, Bishapur (Iran)

Reliefs with similar patterns of symmetry were created centuries earlier by peoples who lived in areas that would later become parts of the Hittite Empire. The reliefs of the Hittites in central Anatolia, those of their successors in the eastern part of what is now Turkey, and the reliefs of Assyrians in Mesopotamia were among the sources for Persian art of the Achaemenid Empire, which assimilated the artistic traditions of its forerunners. Further along the time axis, we find beautiful translated horses among the Sasanian reliefs of Bishapur (fig.13).

However, the art of perfectly translated designs goes much farther back in time to Mesopotomian civilizations: translations are the natural outcome of rolling a cylinder-seal, whose function was to impress an authentic signature on a clay document. Median and Persian cylinder seals used symmetry similarly. From a geometric point of view, a single complete turn prints the pattern and each successive rotation prints translations. Fig. 14 shows an example with monstrous lions.

Fig.14. Cylinder Seal (4000 to 3000 BCE) from Uruk, Iraq (Louvre Museum, Paris)

This example shows a blend of line-reflection and translation, for the fundamental pattern has its own vertical line-symmetry: as this symmetry cannot be handled by rotation of the seal, the carving has been designed to include two animals facing one another with their necks interlocking.

Polygonal symmetries

In fact, Weyl proceeds in the opposite direction. Starting with a picture of translated patterns, each of length a , he imagines the result of applying it to a cylinder whose circumference is a multiple of a , say na , to introduce the finite groups R_n and D_n . The first one is a cyclic group of n elements, $Id, r, r^2, \dots, r^{n-1}$ (r being a rotation such that $r^n = Id$); the second one is the dihedral group, that is, the group of all symmetries leaving an n -sided polygonal invariant: it has $2n$ elements, the n rotations in R_n and n line-reflections whose axes pass through the fixed point of the rotations. Weyl credits Leonardo da Vinci for discovering the fact that these are the only finite symmetry groups in the plane [1952: 66] while studying the symmetries in churches. Weyl adds, "In architecture the symmetry of 4 prevails" [1952: 65].

On second thought, this may be surprising: there are many more monuments based on a rectangular than there are based on a square. But of course, it must be understood as "among the regular polygons". Square monuments are found in nearly all civilizations, even those that are totally unrelated, but Central Asia offers two gems of this type: the

ziggurat at Chogha Zambil, near Susa in Iran (fig. 15) and the Samanid Mausoleum, which is the tomb of Ismaïl I and his successors in Bukhara, Uzbekistan (fig. 16). Chogha Zambil was built by the Elamites about 1250 BCE, and the tomb in Bukhara was built in 905 CE; both feature the D_4 symmetry group.

Fig. 15. Chogha Zambil, near Susa, (Iran)

Fig. 16. The mausoleum of Ismaïl Samani, Bukhara (Uzbekistan)

Other occurrences of D_n may be found, provided we restrict ourselves to the study of the ornamentation rather than the general shape of the buildings. The D_6 group, for instance, is encountered in the ceiling of the famous mausoleum of Oljeitu in Soltaniyeh, Iran, which was built in 1312 CE (fig. 17).

Fig. 17. Oljeitu's Mausoleum, Soltaniyeh (Iran)

We can also observe a D_{12} -invariant pattern at the center of the six-pointed star: the artist implicitly uses the fact that D_{12} is a subgroup of the D_6 group! The mausoleum itself is based on a D_8 -invariant octagon, perhaps the regular polygon most frequently used by architects after the square; a beautiful example of D_8 is the base of the Burana tower (fig. 18) in the Tien-Shan mountains of the Kyrgyz Republic. Erected at the end of the ninth century, part of a minaret half-destroyed by an earthquake, it shows how far to the east the influence of Persian architecture spread.

Fig. 18. Burana tower (Kyrgyz Republic)

As Weyl says, occurrences of an odd n are far more scarce, which seems a bit strange, as the means of constructing a pentagon were well known from both the Greek tradition and in medieval Persia. Pentagons and decagons appear relatively frequently as ornamental patterns on the ceilings and walls of civilian and religious monuments. Fig. 19a shows an example from the ceiling of the Chehel Sutun Palace in Isfahan (completed in 1647 under Shāh Abbas II).

Fig. 19. a, (left) Chehel Sutun Palace, Isfahan (Iran); b, (right) City Palace, Jaipur (India)

This pattern with pentagons and decagons is indeed classical; it is documented from Turkey in the west to Xinkiang, China in the east. It is known in India, as shown in this *jali* (a type of window with carved wood or stone lattice openwork called *mashrabiya* in North Africa and the Middle East) (fig. 19b). This is a blend of D_{10} -invariant patterns (the large ten-pointed stars) and D_5 -stars, cleverly completed with quadrilaterals (tiling the plane with pentagons being impossible).

An *iwan*, a kind of vaulted hall in an Islamic monument, of the Friday Mosque in Yazd, Iran, shows several seven-pointed stars (Fig 20). The walls themselves are ornamented with a D_{10} pattern, but the trick was to insert the D_7 pattern inside the roof, just where the space is reduced so a ten-sided pattern could no longer be inserted. Hardly anyone – maybe only a mathematician? – would observe this “minor” change; rather, they perceive a perfect tiling. We might wonder why heptagons or seven-pointed stars are so rare, for approximate constructions were certainly known. The two early examples below (figs. 21 and 22) demonstrate perfect mastery of these figures: the piece from the Louvre (fig. 21) shows the construction of a heptagon on one side, and the construction of a hexagon, with cuneiform comments on the other side.

Fig. 20. Friday Mosque, Yazd (Iran) with several seven-pointed stars above and ten-pointed stars below as indicated

Fig. 21 (left). Constructing the heptagon, Susa clay tablet (Louvre, Paris).
Fig. 22 (right). Bronze Bowl, Nimrud, Neo-Assyrian period, ninth-eighth century BCE (Metropolitan Museum of Art, New York)

Rotation without reflection

Chasing R_n means going one step further in scarcity. However, patterns R_n but not D_n -invariant have fascinated people since the dawn of humanity: “It seems that the origin of the magic power ascribed to these patterns lies in their startling incomplete symmetry—rotations without reflections” [Weyl 1952: 67].

When you enter Yazd, you often hear a boast from natives: there are no mosques anywhere like the ones in Yazd. From our geometrical point of view, this is certainly true: we found an R_6 -invariant pattern on the mihrab of the Friday Mosque (fig. 23a), and, located similarly in the spandrel above the arch, on the impressive *pishtaq* of the Amir Chakhmagh Mosque (fig. 23b).

Fig. 23. a, above) Friday Mosque, detail of the mihrab, Yazd (Iran); b, right) Amir Chakhmagh Mosque, Yazd (Iran)

In the same monument, we can find the most frequent R_n -invariant patterns, with $n = 4$ (fig. 24). Another nice one appears in the corridor between the two *pishtaqs* of the Shah Mosque in Isfahan (fig 25).

Fig. 24 (left). Amir Chakhmagh Mosque, Yazd (Iran) Fig. 25 (right). Shah Mosque, Isfahan (Iran)

Fig. 26a. Friday Mosque, Isfahan (Iran)

Fig. 26b. Friday Mosque, Isfahan (Iran), detail of pattern in the western *iwan*

Of course, for a mathematician, the most fascinating pattern lies in the western *iwan* of the Friday Mosque in Isfahan (fig 26a): in addition to its inherent beauty, it offers a famous demonstration of the Pythagorean theorem (fig 26b) by Abu Al-Wafa (940-998). His celebrated book, *Geometrical Constructions useful to Craftmen* (*Kitab Fi Ma Yahtaju Al-Sani Min Al-a Mal Al-Handasiyya*), became so popular that it could have been used by an architect or craftsman without help from a professional mathematician, and it had been precisely intended for that purpose.

Ornamental symmetries

The combination of two-directional translations and the finite groups enumerated above gives rise to *wallpaper groups*; these are associated with the different ways of tiling a plane by repetition of a single design. If the tiles are regular polygons, they can only be equilateral triangles, squares or hexagons; the most beautiful example of the latter may be found in Ulugh Beg's Madrasa (a *madrasa* is an Islamic academy, where not only theology, but also literature, poetry and sciences were taught) in Samarkand, built in 1420 CE. The open-work brick balustrade (fig. 27) was indeed a revolutionary prince's idea, because it took the place of the usual high, solid wall that made it impossible to see in or out: its architectural function was to keep the scholars somehow isolated from the noisy marketplace, while keeping some symbolic social link between the elite and the people.

Fig. 27. Shir Dor Madrasa (built 1636 CE) as seen through the open brickwork of Ulugh Beg's Madrasa (built 1420 CE), Samarkand (Uzbekistan)

Do not think that this is the only item of geometrical interest in this madrasa: the artists who worked on the walls inside showed a deep understanding of ornamental geometry in creating arrangements of different regular shapes, none of which can be used to tile the plane by itself, but could do so when gathered together in a pattern that can be translated in two directions: this has already been observed in figs. 19a and 19b. We find a fine example on the interior wall of an *iwan* where the fundamental pattern is a combination of pentagons, hexagons and enneagons, with heptagons inside the enneagons (fig. 28b)!

Despite the fact that very sophisticated tilings were used in purely religious monuments, it would be hard to believe that these works had no connection with the Prince's deep appreciation of astronomy and mathematics: Ulugh Beg (1393-1449), a grandson of the warrior Timur-Leng, devoted his patronage to science, and, as a Viceroy of Samarkand from 1409 to 1447, he managed to obtain the funding for an observatory, schedule the building of the madrasa, and gather a research team of seventy astronomers to work in it! Among the distinguished scholars were the astronomer, Qadi-Zada-al-Rumi (1364-1436), and the mathematician Al-Kashi (1380-1450), who may be regarded as the father of the famous successive approximations method. He designed this to compute $\sin(1^\circ)$ with an accuracy of ten sexagesimal places, about 200 years ahead of Kepler (1618). Six centuries later, this remains the standard method of computing roots of more or less intricate equations arising from astronomy!

Fig. 28. a, above) Statues of Ulugh Beg and his astronomical team, Ulugh Beg Madrasa, Samarkand (Uzbekistan);
b, right) Detail of the tiling, inside wall of the *iwān*

Fig. 29. a, left). Blind arch of the Burana Tower (Kyrgyz Republic);
b, above). Detail of brick ornament

A tiling may have an underlying lattice based on a square, but this does not mean that the subgroup of the transformations that fix one point must be the symmetry group of the square, namely, D_4 . For example, in figs. 29 a, b the blind arches of the Burana tower (see fig. 18) show only rotation and translation invariance, without any reflection: if we consider the red dot in fig. 29b as a corner of the fundamental square, then the symmetries leaving it invariant are the R_4 group.

Though not located in Persia, the Burana tower exhibits brickwork typical of Islamic Central Asia around the year 1000 CE. The extent of Persian influence is evident when it is compared, for instance, to the brick patterns of the Samanid Mausoleum in Bukhara. The patterns of symmetry are not so far from the “mauresque” patterns of Weyl’s fig. 66 [1952: 114], having the same invariance group, classically named $p4$ in the theory of wallpaper groups (see [Frieze Groups] and [Groups] for notations and details).

Fig. 30. Akbar Mausoleum in Sikandra, near Agra (India)

In a less severe style, many such combinations of patterns may be observed in Mughal monuments, for instance, in Akbar’s Mausoleum (built 1613 CE) near Agra, where the Persian influence is easily recognized in both the general plan and in ornamentation (fig. 30). Here, the artist mixes a taste for geometry with a taste for nature and flowers. As Weyl points out [1952: 58], flowers frequently offer rotational symmetries of order 5 or 6! A careful inspection reveals, indeed, that local occurrences of D_5 , D_6 and R_6 are present. Check the slight rotation of the petals! And notice the vertical invariance by translation.

Another example is offered by the world-famous Taj Mahal in Agra, which reflects the influence of Timurid mausoleums; its chief architect, Ustad Ahmad Lahauri, was said to be Persian.

A last example is the caravanserai of No-Gonbad (fig. 31), where we can see translations and symmetries and the winding of patterns around the corner towers. Note that the “arrows” at the top of the towers have been directed according to a plane reflection with respect to a (virtual) vertical plane through the main door.

Fig. 31. No-Gonbad, on the road from Yazd to Isfahan (Iran)

Scale changes

When he turns to three-dimensional space and similarity groups, Hermann Weyl no longer refers to architecture, but to the works of nature, flowers and seashells. There are, however, examples from architecture: a wonderful example in brick is offered by the Emin minaret in Turfan (Xinkiang, China), built in 1778 CE (fig. 32). Its conical shape ensures its invariance by a continuous group of similarities (the common center of which is a virtual point, namely the apex of the cone; the tower is only a truncated part of it) and the brick protrusions trace conical helices on it. There seems to be a constant angle with the vertical axis of the tower, which the architect has taken care to maintain. This is the fundamental property of a helix, each family being invariant through a discrete subgroup of the group – one growing while turning right, the other while turning left.

Fig. 32. Two views of the Emin minaret, Turfan, Xinkiang (China)

From geometry to abstract groups

Of course, we shall not find on the roads of old Persia any connection with one of the topics mentioned in the final chapter of Weyl's book, the use of group theory in relativity or quantum mechanics. Nor do we find the extraordinary development of group theory in itself, which occurred in the twentieth century. The struggle with the classification of finite groups ended more than thirty years after Weyl's death in 1955. Nonetheless, there is a spiritual link with Weyl's final topic, Galois theory, because it deals with the resolubility of algebraic equations: don't forget that in order to illustrate these symmetry groups in Persia, we have traveled in the native lands of algebra, the land of its father al-

Khwarizmi (780-850 CE) from Khwarizm, and the land of Omar Khayyam (1048-1131 CE), who came from Nishapur in Khorasan. Omar Khayyam was the first to make valuable attempts to solve cubic equations, that is, equations of the third degree.

But, as extraordinary as it may seem, we encounter in Persia a major and recent advance in the science of symmetries. Weyl's last chapter is entitled "Crystals. The General Idea of Symmetry". Thirty years after the publication of his book, quasicrystals were discovered in 1982, and in 2011, another three decades later, the Nobel Prize in Chemistry was awarded to Dan Shechtman for this discovery. The growth of quasicrystals can be modeled using Penrose tilings (1976), that is, structured tilings without any translation invariance. These tilings use a finite number of tile shapes, including pentagons and decagons. It is elementary to show that no regular tiling can use only this kind of tiles: just think of the sum of the angles at one vertex. Now comes the most astonishing fact: several researchers (first Makovicky [1992] and Bonner [2003], later Peter Lu and Peter Steinhardt [2007]) have found images of such tilings (with a few minor adjustments) in monuments of medieval Islamic architecture dating back to the Gonbad-e Kabud in Maragha (twelfth century); the most frequently cited is the later Darb-i-Imam shrine (fig. 33), Isfahan, Iran (1453)!

Fig. 33. Darb-i-Imam shrine, Isfahan (Iran).
Photo by K. Dudley and M. Elliff, reproduced by permission

Conclusion

Weyl's project was to show to his reader, who may not have been acquainted with mathematics, that there was a somehow secret but deep link between the history of civilizations and the more recent advances in science and the study of symmetry. In our times of highly specialized research and compartmentalized knowledge, interactions between scholars in science, on one hand, and history, art, and architecture, on the other, still remain a statistical exception; there is no doubt that not only does the general public not know them, but even selected audiences can hardly imagine them. Weyl was one of the very few who tried to bridge the gap between mathematicians and laymen and this mission must be carried on. A trip to Persia is a relaxed, pleasant way of staying in touch

with the spirit of Weyl's book, so let us hope that those who study group theory will want to discover Persian architecture and that tourists in Persia will yearn for a basic understanding of group theory!

Appendix: About groups

A (very) brief history of (the birth of) groups

Groups were introduced first by Lagrange (1770), Cauchy (1815) and Galois (1830). All were concerned with the solubility of algebraic equations of the fifth degree and higher. They were led to the study of *permutations* of the roots of a polynomial, and Galois was the first to link the existence of explicit formulas for solving this kind of equation to the study of a *group of permutations*. It is interesting to notice that, despite his brilliant, although tedious solution, Galois did not give a precise definition of the word *group*, which was not provided until Cayley (1854).

An other source for understanding group theory was the study of geometrical transformations, mainly undertaken by Klein (1870).

What is a (symmetry) group?

It turns out that all isometric transformations preserving a given figure – for instance, a polygon – have the following properties:

- i. Applying two isometric transformations in succession, say r first, and then s , leads to a transformation sharing the preservation property; we denote it $s \circ r$, such that, M being an arbitrary point in the plane (or space),

$$s \circ r (M) = s(r(M));$$

- ii. The identical transformation, Id , preserves the given figure;
- iii. any transformation of this kind has a reciprocal one, that is, for any r there exists an s such that $r(M) = P$ is equivalent to $s(P) = M$; as $s \circ r = r \circ s = Id$, we write it $s = r^{-1}$;
- iv. The order of “packing” when applying a third transformation t is irrelevant; one can apply first $s \circ r$ then t , or first r , then $t \circ s$.

$$t \circ (s \circ r) = (t \circ s) \circ r.$$

These four properties are Cayley's definition. A group *does not have*, generally speaking, the property:

- v. $s \circ r = r \circ s$ for any r, s .

If it does, it is said to be *commutative* or *Abelian*, but as we said, most transformation groups do not have this property.

Building Cayley's tables: the square group

It follows easily from the preservation of distances that a symmetry preserving a regular polygon may be regarded as a permutation of its vertices. This links geometric groups to permutations groups as studied by the pioneers, and it is of great help in computing $s \circ r$, as it is sufficient to look at what each vertex is transformed in.

For instance, here is Cayley's table for the square-preserving group: it has eight elements that act on the vertices A, B, C, D of a square as eight of the sixteen permutations on these four points. (Thus, it may also be considered as a subgroup of the group of all permutations on four elements as well). This group, denoted as D_4 , is made

of four rotations Id, r (a 90° turn), $r^2 = r \circ r = -Id, r^3 = r \circ r^2$, (the R_4 subgroup, as can easily be seen on the table by checking property (i)) and four reflections s_H, s_V, s_I, s_2 with respective axes $(H), (V), (\mathcal{A}_1), (\mathcal{A}_2)$.

It must be read in the following way: $a \circ b$ is computed from a in column and b in row. Notice that $s_H \circ r$ and $r \circ s_H$ are not equal, so one has to be careful in building and reading this table.

A square with four axes of symmetry

	Id	r	r^2	r^3	s_H	s_V	s_I	s_2
Id	Id	r	r^2	r^3	s_H	s_V	s_I	s_2
r	r	r^2	r^3	Id	s_I	s_2	s_V	s_H
r^2	r^2	r^3	Id	Ir	s_V	s_H	s_2	s_I
r^3	r^3	Id	r	r^2	s_2	s_I	s_H	s_V
s_H	s_H	s_2	s_V	s_I	Id	r^2	r	r^3
s_V	s_V	s_I	s_H	s_2	r^2	Id	r^3	r
s_I	s_I	s_H	s_2	s_V	r^3	r	Id	r^2
s_2	s_2	s_V	s_I	s_H	r	r^3	r^2	Id

Cayley’s table for D_4 , featuring R_4 in the top left corner

Building Cayley’s tables: the rectangle group

A rectangle has fewer symmetries than a square: obviously $(\mathcal{A}_1), (\mathcal{A}_2)$ are no longer axes of symmetry, $(H), (V)$ are the only remaining ones. As for the rotations preserving it, there are just two, Id and $-Id$; $R_2 = \{Id, -Id\}$ is a group with two elements only.

	Id	$-Id$	s_H	s_V
Id	Id	$-Id$	s_H	s_V
$-Id$	$-Id$	Id	s_V	s_H
s_H	s_H	$-s_H$	Id	$-Id$
s_V	s_V	$-s_V$	$-Id$	Id

Cayley's table for Klein Four-Group, featuring R_2 in the top left corner.

This model is of some interest, because it naturally occurs when a symmetric facade is mirrored in a reflecting pond in front of it. It is a four element Abelian group, as R_4 is; however, they are different, because in Klein group, each element has the property $x \circ x = Id$, which is obviously false in R_4 . Another way of looking at it is to think of $K = R_2 \times R_2$ with

$$(a, b) \circ (c, d) = (a \circ c, b \circ d).$$

When looking up how to build a four elements group from scratch (by writing down the different kind of tables), one can discover that these are the only existing four elements groups.

Polygonal symmetries and associated groups

Generalizing the case of the square, a regular n -sided polygon is preserved through the transformations of the *dihedral* group D_n , which has $2n$ elements, n axial symmetries and a subgroup of n rotations, the *cyclic* group R_n , (and, for this reason, called C_n by Weyl):

... we have the two following possibilities for finite groups of [symmetries] ... : (1) the group consisting of the repetitions of a single proper rotation by an aliquot part $\alpha = 360^\circ/n$ of 360° ; (2) the group of these rotations combined with the reflections in n axes forming angles $1/(2\alpha)$. The first group is called the cyclic group C_n and the second the dihedral group D_n . Thus these are the only possible central symmetries in two-dimensions:

$$(1) \quad C_1, C_2, C_3, \dots; D_1, D_2, D_3, \dots$$

... Leonardo da Vinci engaged in systematically determining the possible symmetries of a central building and how to attach chapels and niches without destroying the symmetry of the nucleus ... [1952: 65].

As one may expect, the famous artist stated more than proved this property, and of course, he did not mention the group properties.

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The reader may be surprised by the lack of a bibliography in Weyl's book, but we must insist on the fact that it was not written for scholars in the field. In keeping with this idea, we offer only a limited selection of introductory works, together with a few references by the master himself. The interested reader will easily find further references to landmark papers by great mathematicians who contributed to the birth and evolution of group theory, such as Lagrange, Cauchy, Galois, Jordan, Klein, Lie, Cartan, and others.

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