Jürgen Bokowski*

*Corresponding author

Technical University Darmstadt
Department of Mathematics
64289 Darmstadt GERMANY
juergen.bokowski@googlemail.com

Heike Matcha

Technical University Darmstadt
Department of Architecture
Institute of Design
64289 Darmstadt GERMANY
hm@hmgmb.net

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Didactics

Möbius Strip Segmented into Flat Trapezoids: Design-Build Project by the Departments of Architecture and Mathematics of the Technische Universität Darmstadt

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Abstract. The Möbius strip trade fair stand was designed and built by students of the Faculty of Architecture of Darmstadt Technical University in collaboration with students of the Faculty of Mathematics. The project focused on two main issues: to introduce computer-driven parametric design and production strategies, techniques and technologies into architectural education, and to build and operate the final result. This required knowledge of different software packages and readiness to become familiar quickly with unknown data formats, flexibility in the time schedule and disposition to transport the material to the machinery. It introduced students to the possibilities of designing with parametric modeling software and familiarized them with the initial difficulties and ample payback in actual production.

1. Introduction: the Möbius strip as a trade fair stand

The Möbius strip trade fair stand is part of a series of three trade fair stands that were designed and built by students of the Faculty of Architecture of Darmstadt Technical University (fig. 1). This series of 1:1 design and build projects focused on two main issues: firstly, to introduce computer-driven parametric design and production strategies, techniques and technologies into architectural education; secondly, to not just design a project, but to actually build and operate it.

Parametric design and production enables architects to construct their designs from elements that are neither completely identical nor totally different – as was dictated by the principles of serialized mass production. Instead, elements can be similar and varied, like members of a family. This permits greater complexity and diversification within the built structures and promises more appropriate designs that can respond better to the specific individual characteristics of users, program, site, etc. Most educational projects in architecture schools never leave the drawing board, so to speak. While this is understandable with large-scale projects like large cultural buildings or residential complexes, it excludes from architectural education the very important development a design project undergoes as it is built: ideas have to adapt and change according to the restraints and possibilities of budget, availability of material, manufacturing and building machines and issues of transportation. In this development, quite different skills are needed than the conceptualising and presentation skills necessary for designing. Instead, organization, management, craftsmanship and aptitude with materials and tools become essential. The annual Hobit fair of the Technical University Darmstadt, where prospective students become acquainted with the university and its faculties and

departments, provided a perfect occasion to conduct a course that merged our two educational ambitions: the architecture faculty needed a fair stand that would provide information about the courses and, through its sculptural qualities, would also communicate what the study of architecture is about. The Möbius strip was chosen as basis for the stand's shape because it inherently displays fascinating aspects of geometry, and because constructing it on the scale of a fair stand requires geometrical and constructive effort that would illustrate important issues of the study of architecture.

Fig. 1. Series of three trade fair stands clockwise: Möbius strip, Parametric Origami and Cardboard Wave. Photos top and bottom right by Stefan Daub; photo bottom left by Thomas Ott

Moreover, the Faculty of Mathematics had expressed its interest in collaborating on the stand, and the Möbius strip is one of the most iconic and well-known mathematical objects. The collaboration between the two faculties would also become an important part of the exhibition, as collaboration with different disciplines is an important part of architectural work.

Fig. 2. Transformation of Möbius strip to segmentation into flat trapezoids. Images courtesy of Technische Universität Darmstadt

We chose to segment the Möbius strip into several flat trapezoids because it would provide us with different possibilities of use: it could become a counter, a shelf, a table and other things. As it turned out, the segmentation was much more difficult than expected (figs. 2, 3).

Fig. 3. Flat trapezoids provide different possibilities of use.
Images courtesy of Technische Universität Darmstadt

2. Mathematical problem: constructing a Möbius strip out of regular flat trapezoids

We introduced the topic of Möbius strips through a talk from a mathematical point of view. Its focus, though, lay not in a discussion of topological issues – an area that would usually be opened up through such an introduction. Instead, we focussed on geometrical issues, partly because of the task at hand, and partly because past experiences with students of architecture have shown that projective geometry, let alone higher-dimensional aspects are not a very suitable topic for them. This time, though, a higher-dimensional model came into being naturally from the very beginning. The final mathematical way to see the problem can be written as follows. We prepare a rectangular paper strip with endpoints A,B,C, and D, say in counter-clockwise order. We identify or we glue line segment AB with line segment CD, such that A equals C and B equals D. The resulting surface with boundary becomes non-orientable, and we obtain a Möbius strip. Now we draw a regular zig-zag with end points on the boundary BC and AD of the Möbius strip such that the strip will be partitioned into isosceles triangles. We next use a remaining smaller strip by cutting off another strip parallel along the Möbius strip's boundary. Thus from each triangle we are left with a trapezoid that we then require to be of flat shape. The edges of adjacent trapezoids are considered to function as hinges. Of course, we require to have these trapezoids without self-intersections (fig. 4).

Fig. 4. Möbius strip when antipodal line segments identified (left), knotted Möbius strip (middle), Möbius strip partitioned into isomorphic trapezoids (right).
Images courtesy of Technische Universität Darmstadt

Mathematical questions immediately arose that were close to what the architects were interested in. Under which conditions does such a model exist? There is, for example, a triangulation of the Möbius strip by Ulrich Brehm that cannot be realized with flat

triangles and without self-intersections. Of course, the students in our “trapezoidulation problem” already had a lot of models showing their existence, didn’t they? However, when it comes to calculating the angles between adjacent trapezoids, we cannot trust the paper versions, can we? When the trapezoids become in a certain limit rectangles, we are sure that a solution cannot exist. And what happens near such a rectangular limit case? Let us assume that we have n trapezoids and therefore n angles between adjacent ones. Mathematicians introduce for each angle between adjacent trapezoids a dimension and they describe the problem as an n -dimensional one. The conditions $A=C$ and $B=D$ reduce the dimension n of the realization space by five: three conditions for the three coordinates of the requirement $A=C$, and only two for the condition $B=D$ because the distance from A to B equals the distance from C to D . The resulting manifold of dimension $n-5$ that describes our solutions in this n -dimensional space is not connected. We have to assume in general that there are solutions with knots. A knot solution cannot be transformed continuously into an unknotted version. Has the reader noticed that n has to be odd and that we cannot choose n to be very small? Any suggested solution of a polyhedral Möbius strip of the architects with certain angles measured from their models is very unlikely to be a correct point on our $(n-5)$ -dimensional solution manifold for polyhedral Möbius strips. Can we use nevertheless such an “angle point” of the architects in n -space to find in the vicinity of it a true solution on the solution manifold? Can we somehow project this point onto this manifold? It was good for the architects to have some mathematical advice for realizing their 1:1 model. The following formulation might tell the non-mathematician what the problem looked like for a mathematician. For simplicity, we assume for that the exact solution manifold is a curve in space, i.e., a one-dimensional manifold. In other words, only points on this curve can be accepted for a model. The students then suggested only “points” near this curve and the mathematical problem was to correct their data in order to get an acceptable solution point near the given one on the solution curve.

3. Designing and constructing the trade fair stand

The students had three months to design, manufacture and build the stand. We started with a design competition that produced a large number of alternative designs, all based on Möbius strips, and all accommodating different situations of use, like information desk and various display possibilities (fig. 5). One of these alternatives was elected by a jury for its use of parametrics and its promise in terms of buildability.

Fig. 5. Various Möbius strip designs for the trade fair stand.
Photo courtesy of Technische Universität Darmstadt

The students then worked as a group to develop the winning project further. Physical models were produced and tested as to their qualities as fair stands. The parametrics in this design stage consisted of varying the number and precise positioning of trapezoids to

produce different shapes and possibilities for use. An important issue in producing and discussing these studies and their qualities consisted of making implicit factors explicit. Architects very often unconsciously set themselves a set of implicit rules that govern their designs but which are also often broken, leading to much confusion. We urged the students to make their implicit rules explicit so that their being broken would surface and their productive qualities be discussed. To be able to actually build the Möbius strip, the trapezoids were further broken up into straight ribs. This was to resolve the complex 3D joint of the trapezoids, and the ribs provided additional means of display. The ribs were to be manufactured of glass and wood. The joints between the different strips or parts of the original trapezoids were designed to be adjustable so that the geometry of the Möbius strip closing in on itself could be realized on site. Without a mathematical solution or procedure, segmenting the strip geometry into flat trapezoids became very difficult and cumbersome as the last neighbouring trapezoids never matched up. The collaboration with mathematicians proved extremely fruitful as they provided the architects with an interactive software application that could generate many variations of perfectly segmented strips and export them in a file format that the architects' CAD packages were able to use, which will be discussed in the next section.

4. An interactive application that offers solutions to the mathematical / geometrical problem

Without an attempt to explain the mathematical background of the problem that we tried to describe in Section 2, the mathematical help for the architects was provided by a mathematics student, Sebastian Stammmler. He used the mathematical software package MATLAB's built-in optimization software and wrote corresponding software parts in close collaboration with the architects. Wikipedia provides the following description:

MATLAB (matrix laboratory) is a numerical computing environment and fourth-generation programming language. Developed by MathWorks, MATLAB allows matrix manipulations, plotting of functions and data, implementation of algorithms, creation of user interfaces, and interfacing with programs written in other languages.

The set of angles between adjacent trapezoids served as the input of the software. An animation visualized how the starting strip could be moved to its final position as a polyhedral Möbius strip. This helped a lot to imagine what the precise solution would finally look like (fig. 6).

Fig. 6. Examples of configurations of Möbius strips with varying number of trapezoids.
Images courtesy of Technische Universität Darmstadt

Furthermore, all spatial information for manufacturing the trapezoids were produced and the interfaces for other production software were provided. It was possible to play with parameters like the number of trapezoids and the shape of the trapezoids defined via its angles or the width of the strip (fig. 7).

Fig. 7. The interactive MATLAB programme provides different segmentations.
Images courtesy of Technische Universität Darmstadt

5. Building and operating the trade fair stand

Building the trade fair stand with students made it necessary to limit the crafts involved to no more than three or four. Furthermore, since the university does not provide CNC tools or CAAM machinery that can produce objects larger than simple models, the students had to find regional companies that would provide machinery and material.

Fig. 8. Cut-out ribs of plexiglass and timber (left) and hinge detail made from PVC spacers and steel (right). Image left courtesy of Technische Universität Darmstadt; photo right by Stefan Daub

The two-dimensional segments made from plexiglass and timber were cut by a local joiner's CNC laser cutting machine. The three-dimensional spacing elements between the ribs were cut from a simple PVC pipe (fig. 8). The specific geometry for the spacers was produced by five-dimensional CNC milling. The data for both elements was generated in Rhino. The transfer to the laser cutter was handled through AutoCAD; the transfer to the milling machine through the mechanical engineering file-format Parasolid.

Whenever possible, the students operated the machines themselves and handled transportation and storage of the construction components. The assembly of the parts was tested twice: first before manufacturing the components, in a small-scale model (fig. 9), and a second time a few days before the opening of the fair in a kind of dress rehearsal at the architecture faculty over the weekend (fig. 10). Both tests proved invaluable and provided information that influenced both the construction details and the final assembly procedure on site.

After the final assembly in the spaces of the fair, the students took shifts to operate the stand, to explain its construction and principles to the visiting high-school students.

Fig. 9. Physical model in scale 1:5 (left) and pre-assembly of segmented parts (right).
Photos courtesy of Technische Universität Darmstadt

Fig. 10. Pre-assembly of the Möbius strip. Photos courtesy of Technische Universität Darmstadt

Fig. 11. Möbius strip made from ribs of plexiglass and timber. Photo by Stefan Daub

Fig. 12. Display detail (left) and group of representatives at the Hobit fair (right).
Photo courtesy of Technische Universität Darmstadt

6. Conclusion

The idea of the design of the Möbius strip consisted not only of one specific object: parameterizing the idea meant that a whole family of related objects was designed. Together with a fully integrated digital production chain, this would in principle allow for the production of a large number of fair stands, all stemming from the same design, but adjusted to situational specifics like place, size, arrangement of different functions. Realizing design-build projects in a non-CAAM environment constitutes a big challenge for all participants: knowledge of different software packages and readiness to become familiar quickly with unknown data formats, flexibility in the time schedule and disposition to transport the material to the machinery. The project not only introduced the students to the possibilities of designing with parametric modeling software, but also familiarized them with the initial difficulties and the following large payback in employing them for actual production. The finished stands went far beyond what they initially had believed possible. Employing the Möbius strip geometry proved indeed to communicate issues of geometry and the fascination of solving geometrical and technical problems and challenges. As we had hoped, the fair stand in itself became a showcase for architecture and mathematics. Although the stand contained a number of different exhibition objects, the Möbius strip geometry became an exhibit in itself.

Fig. 13. Lighting installation of Möbius strip. Photo by Stefan Daub

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Further information

Movie showing segmented Möbius strip:

<http://hmgb.net/files/hmgb-expo15-2-film-matlab-1.mov>

Movie showing interactive programme:

<http://hmgb.net/files/hmgb-expo15-2-film-matlab-2.mov>

Movie showing set-up of the built Möbius strip:

<http://hmgb.net/files/hmgb-expo15-2-film-aufbau.mov>

About the authors

Jürgen Bokowski, Electrical Engineer, Berlin 1964, studied mathematics, Ph.D. at Technical University Berlin, 1972. Habilitation at University Bochum 1979, Professor of Mathematics at Technical University of Darmstadt since 1981, Guest Professor at Université 6, Paris, 2004, and at Universidad Nacional Autónoma de México, 2008/2009. Teaching activities included descriptive geometry for architects. For research interests, mainly convexity and combinatorial geometry, see his book entitled *Computational Oriented Matroids, Equivalence classes of matrices within a natural framework* (Cambridge University Press, 2006), which also shows many self-made mathematical pottery models.

Heike Matcha is an architect based in Berlin. After studying in Stuttgart and Aachen, she worked for practices in London and Berlin before setting up her own practice HMGB Architects in Berlin (www.hmgb.net) together with Günter Barczik. As Assistant Professor she taught at the Technical University of Braunschweig and Darmstadt. She researches new possibilities for architectural design and construction through parametric modelling and customized mass production. Results of her research have been presented in several conferences and publications.