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Research

Octagonal Geometry of the Cimbório in Burgos Cathedral

Abstract. This paper is a geometric analysis of elements of Burgos Cathedral, featuring the Cordovan proportion. It invites to mathematical and decorative creativity with rosettes and tesseracts, based on other remarkable proportions as well.

Keywords: Burgos Cathedral, geometric analysis, Cordovan proportion, regular polygons, geometric shapes, stars, tesseracts, silver mean

Fig. 1. Burgos Cathedral

Introduction

Burgos is a city in Spain in the region of Castilla and León (fig. 2). The stylized spires and stately towers of its cathedral are among the most beautiful achievements of Gothic art in Spain (fig. 1). Burgos Cathedral was declared a World Heritage Site by UNESCO [Rico Santamaria 1985].

Fig. 2. Castilla-León (grey) in Spain

This magnificent work of art was pieced together and erected above an ancient Romanesque cathedral of the eleventh century (begun in 1075 and completed in 1096), built by Alfonso VI, named Santa María, where the famous Spanish nobleman and military leader Rodrigo Díaz de Vivar, called Cid Campeador, departed for exile. This historic event and the heroic deeds of that Castilian knight are narrated in the Spanish

epic poem "Cantar del Mío Cid", which mixes the history of the medieval wars between Spanish Christian Kingdoms and Arabs with legends.

The current cathedral is the result of the efforts and interventions of a collective: kings, bishops, architects, sculptors, artisans, marble workers, carvers, blacksmiths, etc. Many circumstances promoted the construction of the new cathedral at the beginning of the thirteenth century: a historic dynamism, the support of the city, the strategic situation of the town of Burgos in the Camino de Santiago, and, mainly, the appointment of bishop Mauricio, a cultured man trusted by King Ferdinand III the Saint. After travelling to France and Germany, the prelate urged Ferdinand King to construct a new cathedral, vast and stately. On 20 July 1221, the foundation stone of the new cathedral was laid.

The architectural solutions in Burgos Cathedral were influenced by French Gothic design and, after the fifteenth century, by Germanic artistic inspirations as well. Its construction went through many phases and styles: from classic Gothic, (from 1221 through the first half of the fourteenth century), to late Gothic, (from the second half of the fourteenth century through the first half of the sixteenth century). Some elements of Renaissance and Baroque styles are present as well, but they are less prominent.

Geometric patterns

Some significant elements of the cathedral were built in the second phase: the octagonal spires, the octagonal Condestable Chapel (a *condestable* held the highest rank in the military profession during the Middle Ages), and the most significant construction of that time, an octagonal tower in Gothic-Plateresque style, known as the cimborio. This tower, the highest in the Cathedral, corresponds to the lantern of the transept and is configured as an octagonal prism within a squared prism with four towers on the vertices of the latter. On the vertices of the octagonal prism also rise eight smaller towers. The octagonal tower is placed over the crossing of the nave where the tomb of Cid Campeador is located.

The geometry of the cimborio becomes more discernable when entering through the door of the Sarmental façade that opens to the transept of the cathedral. Then the tower can be observed from the inside, in its entire splendor. It is an octagonal prism supported by means of pendentives in another quadrangular prism, held up by four great columns (fig. 3). In the middle there is a well-designed rosette (fig. 4).

Fig. 3. Vault of the cimborio

Fig. 4. Pattern against the light (left); geometric pattern of the rosette (right)

The initiative for the building of the first tower came from Bishop Luis de Acuña, who had the idea for the construction of a lantern for the transept in 1460. The master builder was Juan de Colonia (Köln) and he incorporated the German building trends and methods of that time.

In 1481 Juan died, and his son, Simón de Colonia, continued the project, finishing the tower in 1489. However, the current cimborio doesn't look like the original tower. The plains on the plateau of Castile are prone to terrible winter weather, especially snowfall and wind. The snow increased the weight of the tower, thus unbalancing its center of gravity. These conditions caused it to collapse in 1539.

Architect Juan de Vallejo overtook the restoration in the year 1544, but he may have been helped by Francisco de Colonia, the son of Simón who had died in 1542. De Vallejo repaired the tower, coated its columns, and increased the section of the base. This way he decreased its slenderness, that is, the ratio between the height and the supporting surface on the floor. The tower was finished in 1573, but in 1642 a furious hurricane destroyed the eight exterior towers of the cimborio, and seriously damaged the domes around. The repair was also disrupted by a fire in 1644.

Finally, in 1981, another raging storm destroyed some pinnacles and ornaments of the domes. Marcos Rico Santamaría, the architect who restored the cathedral in the 1980s, also reduced the weight of the covering, thus slightly lowering the center of gravity and guaranteeing the stability of the dome. According to this architect, *“the Cimborio is a Cathedral on top of another Cathedral”* [Rico Santamaria 1985].

The vault of the tower is formed by smaller and smaller convex octagons ending in an eight-pointed star polygon. It will be denoted by “ $8/3$ ”, to point out it is a figure formed by straight lines connecting every third point out of the eight regularly spaced vertices of an octagon. The points of another star $8/3$ appear on the concave vertices of the polygon. The eight concave vertices of this last star can be joined by segments forming an eight-armed cross (see fig. 4).

Beyond the two main stars on the vault are various triangles, trapezoids and connected squares, forming a grill that covers the roof of the vault. Small circles show trefoils and Greek crosses carved onto them (fig. 5).

Fig. 5. Details of the roof of the vault. Photos by J. Arroyo, reproduced by permission

Mathematical interpretations

A notable proportion in a regular octagon is obtained by dividing its second diagonal and its side (fig. 6), resulting in the $1+\sqrt{2}$ or “silver” proportion or number, here denoted by θ (see [Kappraff 1996] and [Spinadel 1998]). A second proportion is obtained when the length of an octagon's radius is divided by the length of its side (fig. 7), resulting in

$(\sqrt{2}-\sqrt{2})^{-1}$, denoted by c (see [Redondo Buitrago and Reyes 2008a, 2008b, 2009]). The proportion is called the “*Cordovan proportion*” because it was introduced by Spanish architect Rafael de la Hoz Arderius (see [Hoz 1976]), who investigated the architecture of the Spanish city of Cordoba. θ and c are related by $c = \sqrt{((1+\theta)/2)}$ or $\theta=\sqrt{2}c^2$.

Fig. 6. Silver rectangle

Fig. 7. Cordovan triangles

Note the second (big) rosette is similar to the first (the smaller one), as it results from a central dilation with similarity ratio $c\sqrt{2} = \theta/c$ (fig. 8).

Fig. 8. Construction of the vault using reflections and dilatations

If a “silver” rectangle whose sides are in a $\theta/1$ proportion, is rotated 45° around its centre, an $8/3$ star polygon inscribed in the octagon is obtained, as well as an inner octagon (fig. 9).

Fig. 9. Rotating a silver rectangle

The ratio AB/CD is the *silver number*, θ . The same star $8/3$ follows from a rotation of the two equal sides of a Cordovan triangle (fig. 10).

Fig. 10. Rotating a Cordovan Triangle

An $8/2$ star is a figure formed by connecting with straight lines every second point of the eight regularly spaced vertices of an octagon. In fig. 11, this $8/2$ star is drawn in red, inscribed in a green octagon. The green octagon appears between two blue octagons, whose sides form a geometric sequence with a θ/c ratio.

Fig. 11. Star $8/2$ in the Cimborio and nested octagons.
Photo at left by J. Arroyo, reproduced by permission

The roof also shows two nested squares inscribed in the quadrilateral formed by the union of a Cordovan triangle and an isosceles right triangle. It illustrates a geometric property:

Given a quadrilateral PQSR that is the union of a Cordovan triangle PQR, and an isosceles right triangle QRS, the common side QR divides the square ABCD in a $\sqrt{2}$ rectangle and a $1+\theta=2c^2$ rectangle. The ratio of the areas of the two pieces is equal to θ (fig. 12).

When a square is placed such that three and only three of its vertices intersect a Cordovan triangle ABC , another interesting shape is obtained: seven rotations of 45° about its circumcenter O generate a pattern like a grid of stars (fig. 13). It splits in a rosette and a sequence of four nested 8/2 stars (fig. 14).

Fig. 12. Proportions in the quadrilateral details.
Photo at left by J. Arroyo, reproduced by permission

Fig. 13. Generator of the starred grid

Fig. 14. The grid of the stars and its subdivision

Placing the orange and blue stars on top of the rosette, the smallest squares of the decorative elements of the stained-glass are locked between them (fig. 15). The two outer stars, in green and orange, define a geometric contraction of ratio $\sqrt{2}/2$, though the two inner stars, in blue and purple, do not belong to this sequence (fig. 16).

Fig. 15. Decorative elements of the vault

Fig. 16. Contracting the stars

The vertex H of the blue star divides the segment BJ in three segments BH, HI and IJ such that $BH=IJ$ and $BJ/HI=\theta$, since BJ and HI are respectively the second diagonal and the side of the orange octagon with lengths in ratio θ (fig. 17). The purple star plays a similar role when we consider the radius AD and the points B and C. This is a direct consequence of Thales' "intercept" theorem since $AF/FG=\theta$ (fig. 18).

Fig. 17. The blue star and θ

Fig. 18. The purple star and θ

The dissection of the Cordovan triangle can be extended by adding more squares (see figs. 19 and 20), allowing to inscribe a rosette inside a tesseract, a tesseract in a rosette, and so on. Fig. 21 shows the fractal character of the pattern.

Fig. 19. Trapezoid and triangle on the rosette

Fig. 20. Fractal dissection of the Cordovan triangle

Fig. 21. Nested rosettes and tesseracts

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Dirk Huylebrouck's career in Congo ended after an incident between Belgium and the President of Congo, and so he went teaching in Portugal and later to European Divisions of American Universities, until the first Iraq war drew away his students. He returned to Africa, until a coup d'état during the 1994 genocide epoch ended his contract in Burundi and so he began teaching at the Department for Architecture Sint-Lucas in Brussels and Ghent. Author of the books *Africa + mathematics* about the now famous Ishango rod (in Dutch, translated in French, VUBPress, Brussels, 2004-2008) and *De Codes Van Da Vinci, Bach, Pi & Co* (Academia, 2009), he edits the column "The Mathematical Tourist" in *The Mathematical Intelligencer*.

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