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Research

Mathematical Interpretation of a Thirteen Hundred Year Old Stone Masonry Observatory

Abstract. The present paper discusses various implications suggested by a thirteen hundred year old stone masonry structure, built by Queen Sun-Duk of the Silla Dynasty, regarding astronomy, religion, symbol, mathematics in architectural form and construction. The name of the 9.1 m tall stone tower, Chum-Sung-Dae, implies its use as celestial observatory and it is claimed to be the oldest one existing in Asia. Although there have been some disagreements about its function as an observatory, this claim is generally accepted in the absence of any evidence to the contrary. No matter what the original function was, Chum-Sung-Dae has been unanimously acknowledged as the product of then state-of-the-art technology. What makes Chum-Sung-Dae so unique and particular among coeval buildings in stone masonry is in its elegant, multiple curvatures. Furthermore, the geometry of Chum-Sung-Dae is reported to embody a notable numerical significance. The absence of mortar between stone layers and sides indicates that the load transfer mechanism of the tower is dependent upon the frictional resistance due to gravity. All these features will be discussed, with an emphasis on the mathematical analysis and interpretation.

Introduction

Designated as National Treasure No. 31 of South Korea in 1962, Chum-Sung-Dae's name literally means celestial observatory. It is claimed to be the oldest one existing in Asia. The 9.1 m tall stone masonry tower is located in the southeast outskirts of Kyung-Joo, once the capital city of the Silla Dynasty, from its establishment in 57 B.C. until its collapse in 935 A.D. The tower was built in 647 A.D. during the reign of Queen Sun-Duk, one of three queens in Korean History and the twenty-seventh ruler of Silla [Moon 2006]. The elegant appearance of Chum-Sung-Dae is characterized by circles in plan, double curvatures in elevation and squares at its top, mid-height and bottom (fig. 1). In spite of its modest scale, these particular features make the stone masonry tower unique and distinctive from all the other, similar structures built in that era.

Although various ancient documents attest that Chum-Sung-Dae was built for the purpose of celestial observation, and many people accept this as fact, there has been skepticism regarding its function as an observatory because mention of the tower in the literature is so simple and brief [Moon 2006]. Those who claim other functions suggest several reasons why Chum-Sung-Dae was not suitable to serve as an observatory: the inappropriate location of the tower on the plain close to the royal palace rather than on a high elevated area; a too-high entrance and inconvenient access to the top; a too-low height for efficiently observing the night sky.

Fig. 1. Photograph of Chum-Sung-Dae

Several alternatives are claimed for the function of the tower: to symbolize ancient mathematical principles; to perform religious services to invoke blessings for the royal family and national prosperity; to measure the sun's shadow; and to form Mt. Soomi, the imaginary ideal mountain of Buddhism [Moon 2006; Yoo, Jang and Kim 1995]. Even though these different opinions largely reflect the personal background and point of view

the individuals holding them without concrete evidence, they are not considered to be totally groundless.

In addition to the argument about its function, the stone masonry tower is intriguing, in spite of its modest size, because of the symbolic and mathematical implications of its geometry. The number of stone blocks and layers, proportions between dimensions, shape and orientation seem to be clearly significant in various ways; these are not explicitly mentioned in the ancient literature but can hardly be attributed to accidental coincidence. The construction technique is also a matter of dispute, due to the lack of field records. Before proceeding further, we should point out that we are not the only ones to have investigated Chum-Sung-Dae. Many researchers have been involved in trying to deciphering the stone masonry tower, only a few of whom are cited in the present paper.

The Chum-Sung-Dae structure

The structure of Chum-Sung-Dae consists of a podium, a body and a top (fig. 2). The first field survey of its dimensions was made by the executive staffs of Kyung-Joo National museum in 1962-1963 [Hong 1965]. Subsequent field surveys have been made independently by various researchers and engineers since then [Yoo, Jang and Kim 1995; Son and Lee 2003]. According to the surveyed measurements, the square podium consists of two layers of flat stone segments, one layer of 5.18 m × 5.18 m on the ground and the other layer of 5.36 m × 5.36 m almost buried in the ground, probably due to settlement over the long period of time between its construction and the present day.

Fig. 2. Chum-Sung-Dae structure and dimensions

Fig. 3. Parallel stone rods traversing the circular plane at different layers of the body to compose the #-shaped elements

The body consists of twenty-seven layers of granite stone blocks placed in a circular plan, without the use of mortar between the layers. That is, the stone blocks are in simple contact with each other, side by side and layer on top of layer. The outer diameter of the circular body varies nonlinearly from 4.93 m at layer 1 to 2.85 m at layer 23, and remains constant to the last one, layer 27. Therefore, the overall elevation of the tower displays the three-dimensionally curved surface obtained by rotating a vertically multi-linear curve with double-curvatures about the center of the circular plan. In order to realize the double-curvature surface in the construction, stone blocks were inwardly corbelled at each successively higher layer up to layer 23. Unlike the smoothly finished exterior surface, the interior surfaces of the stone blocks were left untouched, unevenly jagged. As shown in fig. 1, a 910 mm × 910 mm opening with two three-layer high side columns facing southeast by 16° is located from layers 13 through 15. Although that opening has been claimed to be an entrance through which the top of the tower is approached, some opponents point out that its unusual height – 4.16 m from its lower edge to the ground – makes it difficult to access. However, it should be noted that the presence of soil fill up to layer 12, just beneath the opening, implies that men may have passed through it to reach the top of the tower.

A pair of parallel stone rods passing through the circular plan of the body of the tower at layer 19 is intersected by another pair at layer 20 to form a #-shaped element in plan (fig. 3).

The same kind of intersection occurs at layers 25 and 26. The edges of these rods protrude slightly, and are visible on the exterior of the tower, except those at layer 20. It has been claimed that the function of the #-shaped elements is to provide the convenient, safe supports for a ladder to access the top of the tower [Moon 2006]. Another claim is made by a group of researchers and engineers, who insist that they function as structural ties to secure the structural stability for the portion of the tower body above the opening [Lee 1986]. A compromise between the two claims perhaps leads to a more reasonable explanation: that is, the #-shaped elements may have originally been installed to provide the ladder supports and as a result, became structurally beneficial, too. The other probable function of the #-shaped elements is further discussed below.

The top of the tower consists of two layers of carefully finished rectangular-section prismatic stone rods framing into the square whose side dimension is the same as the outer diameter of layer 27 of the body. The prismatic stone rods seem to have been cut and dressed as though of wood, to fit the other stone rod to be connected to form the square. In addition to the two layers of square frames in the top of the tower, a flat stone covering about a half of the interior space of the top of the body is set on the pair of parallel stone rods at layer 26. This floor on the top of the tower is claimed to be a work space for men observing the night sky [Moon 2006].

Summing up the feature of Chum-Sung-Dae structure, the total height of the tower above the ground (including the upper podium layer and the lower one underground), twenty-seven layers of the body and two layers of square frames on the top, is 9.1 m. Therefore, one layer of stone masonry is about 300 mm thick on average.

Symbolic implications

The number of layers and stone blocks composing Chum-Sung-Dae is symbolically significant in various ways. The number of layers of stone blocks in the circular body (27) coincides with the 27 days of the revolutionary period of the moon. Adding the two layers from the prismatic stone rod frames of the top to the 27 layers of the stone blocks in the body gives a total of 29 layers, equivalent to the number of days in a lunar month. The 12 layers of stone blocks respectively above and below the opening in the body may denote 12 months a year and 24 solar seasonal terms [Yoo, Jang and Kim 1995; Park 2002]. The number of stone blocks in the circular body part equals 362, two side columns of the opening, and one flat stone plate covering a half space supported on the parallel stone rods traversing the 26th layer. Summing up all these numbers (excluding the two #-shaped stone elements inside the body at layers 19, 20, 25 and 26) results in $362+2+1=365$, the number of days in a year. 15 or 16 stone blocks placed on each of the layers 1 through 6 agree with the number of days in each of the 24 solar seasonal terms.

There is another possible interpretation regarding the number of layers and stone blocks. Queen Sun-Duk was the twenty-seventh ruler of the Silla Dynasty, and 27 is the number of layers of stone blocks in the circular body. The total number of the stones composing the circular body, excluding the interior #-shaped stone elements and one flat stone plate on

$$(1) \quad 27^2 + 364^2 = 365^2 \text{ [Song 1981].}$$

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The pictorial solution, originally introduced in ancient Chinese mathematics, seems to have been transmitted to Silla. Therefore, the relationship in Eq. (1) may indicate a numeral metaphor in which 365 is the heavenly number, 364 is the number of stones to observe the stars in the heavens, and 27 symbolizes Queen Sun-Duk, the twenty-seventh ruler of Silla. A possible interpretation of this relationship may imply that Queen Sun-Duk on earth looks up at the heavens through Chum-Sung-Dae.

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Mathematical implications

The major constituents of geometry in ancient Asian mathematics were the circle and the square, with the circle denoting the heavens and the square the earth. This mathematical principle was quite thoroughly implemented into the geometrical form of Chum-Sung-Dae with some proportional relationships. Fig. 5 shows that the ratio of outer diameters between layers 1 and 27 of the circular body is approximately 5:3.

Fig. 5. Geometry of Chum-Sung-Dae utilizing the values of sine and cosine values in the 3:4:5 right triangle

The same figure also shows the ratio of height of the tower above the ground to the diagonal dimension of the upper layer of the podium to be 5:4. These ratios are the components of a right triangle having the ratio of adjacent side:opposite side:hypotenuse equal to 3:4:5 [Cho 1999]. In other words, the ratios are the values of sine and cosine in a right triangle. This relationship clearly indicates the fact that the geometry of Chum-Sung-Dae utilized the Pythagorean Theorem.

The angle formed by the diagonal $\overline{AB}$ in fig. 5, connecting the outer surface of the first layer of stone blocks in the body and the exterior surface of the upper layer of prismatic stone rod frame of the top, with the vertical calculated by

$$(2) \quad \tan^{-1} \left[\frac{4.93 - \frac{4.93 - 2.85}{2}}{8.7} \right] \cong 23.5^\circ,$$

where the height 8.7 m to the top of the tower from the upper surface of the podium is obtained by assuming the average thickness of layers to be 300 mm. The resultant angle is approximately coincident with the tilt between the axis of the earth's rotation and the plane perpendicular to its orbit about the sun.

The elegant elevation of Chum-Sung-Dae with double-curvatures can be obtained by manipulating the cosine curve, whose values have already been implemented into the proportion of dimensions of the tower, as per the working steps shown in fig. 6. The executive working steps are as follows.

- Step 1 Construct a cosine curve with about a half period;
- Step 2 Reduce the amplitude appropriately and trim the curve out;
- Step 3 Tilt up the curve on the x -axis and move it to the left of the y -axis;
- Step 4 Rotate the curve about the y -axis

Fig. 6. Construction of elevation shape of Chum-Sung-Dae by manipulating the cosine curve of a period

When the curve is tilted up and located on the Cartesian coordinate system in Step 3, the appropriate tilting angle can be figured by using the mirror image about the y -axis as shown in Fig. 6. The tilting angle can be determined by adjusting the angle of the diagonal to the one given in Eq. (2).

Fig. 7. Construction scenario with building up the probable soil hill

Engineering implications

The individual stone blocks and rods used in building Chum-Sung-Dae presumably weighed between 0.5 kN to 8.7 kN, and 3.6 kN on average [Lee 1986]. This range of weight is far beyond the laborers' ability to handle the stone blocks by hand; therefore, proper devices were required to place them into position precisely. However, there is no mention of such devices or instructions for how to build the stone masonry tower in the literature. In particular, the shape and dimension of Chum-Sung-Dae appear to make the use of any probable construction device difficult. That is, in order to place stone blocks on circular layers with outer diameters varying from 4.93 m to 2.85 m, the lifting device would have to have mobility in three dimensions to convey the stone blocks to the required position in the tower. In addition, the laborers would have to be able to work inside the circular body of the tower at a particular level above the ground, but the interior space of Chum-Sung-Dae (although there is soil fill up to the layer 12) is not spacious enough to install any temporary scaffolding structure on which the laborers could work. In conclusion, it is not reasonable to assume that a convenient and versatile lifting device with accessibility equivalent to the ones seen at modern construction sites was available at the time, about 1300 years ago.

On this basis, a construction scenario likely to have been applied to Chum-Sung-Dae is suggested in fig. 7.

In this scenario, the construction began with two layers of flat stone plate segments for the podium. Then stone blocks were placed on each layer of the circular body. In this step of construction, soil was piled up to the level of the current layer of stone blocks inside and outside of the body. The soil piled outside the body was leveled down outward in such a slope that the stone blocks and rods could be easily transported by dragging them to the position where they were supposed to be placed. In this way the soil layer grew higher as the number of layers of stone blocks increased and ultimately, at the completion of placement of all layers of stone blocks and rods, Chum-Sung-Dae was embedded in a mound of piled soil. The next step was to remove the soil around the completed tower. When the level of soil reached layer 12 and the opening at the middle of the body appeared, the soil inside the tower, from layer 13 to the top, could also be removed through the opening. It must have been hard to lift the soil fill up to the level of the lower edge of the opening, below layer 12, and they may have decided to just leave it. The removal of soil continued until the two-layered podium on the ground was revealed. Then, finally and literally, the construction was complete.

Although there is no evidence explicitly describing this scenario in the literature, there is some evidence to support it. The first evidence is the presence of soil fill up to layer 12. Otherwise there is no apparent reason to fill the soil up to the level just below the opening. Of layer, there have been other claims for the purpose of the soil fill: it might prevent the corbelled stone blocks from falling down during construction; and the overall structural stability could be improved by moving the center of gravity closer to the bottom of the tower [Lee 1986]. However, these other claims only point out partially probable aspects without taking into consideration the overall engineering insight. The claim of protecting the corbelled stone blocks from falling down was disproved by the experimental observation of a scale model [Son and Lee 2003]. The claim that overall structural stability is improved by lowering the center of gravity is hardly convincing, because the mass of the soil, in comparison with the heavy stone blocks surrounding it, is not large enough to influence the

overall structural stability. Moreover, in the event of lateral disturbances, such as earthquake-induced vibration, there is no positive mechanism tying the soil fill and stone blocks together, which means that the soil fill and the stone blocks are more likely to behave in their own modes, independently.

The second implication is the slightly distorted shape of the body. According to the field surveys, the body of Chum-Sung-Dae is neither perfectly axially symmetrical in elevation nor perfectly circular in plan. Such a skewed shape may have resulted from the pressure of the mound of soil piled for construction. It must not have been possible to maintain the exact alignment between adjacent layers when one layer of stone blocks had to be corbelled over the previous layer embedded in the piled soil. It is also claimed that the skewed shape is due to the result of differential settlement. This claim sounds quite persuasive. However, the distortion is discernable to such a degree that it can hardly be attributed to the 1° tilt due to differential settlement [Yoo, Jang and Kim 1995]. Furthermore, there is no noticeable discontinuity between adjacent layers of stone blocks over the surface of the body, as there probably would be if the distortion had been triggered by tilt-induced slippage between stone blocks.

The third evidence is the fact that even now in Kyung-Joo there are a number of large tombs equivalent in size to the soil hill required to cover Chum-Sung-Dae during construction. Therefore, to build the soil hill must not have been a strange or far-fetched idea in Silla at the time of construction.

In this type of construction, the #-shaped stone rod elements installed at layers 19, 20, 25 and 26 could provide the lateral resistance to maintain the curved shape of the circular body. However, with the existence of the #-shaped elements, the soil may not have been filled beyond layer 19, to make it easier to remove the soil fill after building the tower.

Conclusion

Chum-Sung-Dae, a thirteen hundred year old stone masonry structure, is rich in historical, structural, symbolical, mathematical and engineering implications. Although the lack of explicit evidence has led to some skepticism regarding its function as a celestial observatory, it has been shown that its various aspects imply one orientation for which the stone masonry tower must have been designed. It can be seen that the tower must have at least been related to the wide knowledge of the planetary movement, and ultimately the prayers of the queen and the people of Silla toward the God. It has been suggested that the three-dimensional shape of Chum-Sung-Dae with its double curvatures can be obtained by manipulating the cosine curve, whose values also have been embedded into the proportions between dimensions of the tower. A technique of construction for building the stone masonry tower has been proposed, in which soil piled to form a moderately sloped mound can be utilized as a platform to transport and place stone blocks at each layer. This explanation for the construction of the tower is quite new and probable. In particular, this construction method provides a plausible explanation for the existence of interior soil filled up to layer 12 of the body.

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