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The Holy Alignment: Geodesic and Astronomical Fundamentals for Calculating the Adjustment of Medieval Naves

Abstract. It is commonly believed that the longitudinal axes of churches extend exactly in an east-west direction. However, thorough investigations have shown that this is not always correct; rather, both southern and northern deviations of up to about 25° can occur. The angular deviation between the church axis and true east is called the *Holy Alignment*. This present study presents the possibility that the nave is oriented towards the direction of the sun-rising point on the name day of the patron saint of the church. If several saints share the patronage, the *Holy Alignment* equals the algebraic sum of the angular distances for each saint. The orientation of a nave can be analyzed by means of common mathematical relations used in geodesy, astronomy and gnomonics. In order to perform such an analysis, it is necessary to know the history of the patronage of the church; the Gauss-Krüger coordinates of the ground plan; and characteristic astronomical quantities at the time the church was built. A calculated example with the saints Andrew, James and Philip for the year 980 illustrates the analysis. An appendix deals with the influence of atmospheric refraction on the apparent altitude of the sun near the horizon.

The Holy Alignment

Research concerning the tendency towards a east-west orientation of naves has gone on for about 150 years. The findings thus so far, partly contradictory and partly based on some presumptions, can be found in detail in [Eckstein et al. 1995], especially with regard to the Holy Alignment differing by some 10° both to the north and to the south.

This paper is based on the assumption – supported by multiple and thorough examinations by Eckstein, Büll and Hörnig [Eckstein et al. 1995: 13] – that an unambiguous correlation between the saint or the saints of a church and the alignment of the nave can exist (see also [Dietrich and Mertens 1990: 129]). The reason for it is the meanings in parables that are attached to the horizon points changing daily where the sun rises.

Geographical east, the point the sun crosses the horizon on the days of the equinoxes, symbolizes Christ as the light of the world. Other sun-rising points symbolize the feast days of apostles, martyrs or other saints gathering around Christ. The angular distance of such a point from true east is called its *eastern amplitude*.

Deviations from the exact east-west direction are documented for numerous churches in the centuries before the first turn of the millennium and until about 1125. In relation to a change of the patronage during the reconstruction of a church, it is significant that the

Holy Alignment has been obviously modified. Its direction is the algebraic sum of the eastern amplitudes at the name days of the former and the more recent patron saints. The reason for discernable “bends” in the longitudinal direction of some buildings might lie, for example, in the inclusion of old foundation walls in the new or reorganized structure.

According to *Meyers Konversations Lexikon*, the Roman Catholic church teaches that saints are those who, having qualified themselves by their lives and deaths, are venerated and invoked by the people as advocates with God and Christ. Initially the choice of which saint or saints to commemorate with yearly feasts was left up to the individual Christian communities. These registered the names of their significant saints in local saint calendars (celebration calendars) with the accompanying date. For example, at the beginning of the fifth century the Armenian lectionarium of the church of Jerusalem mentions Tom, Philip, Andrew, James and John, in addition to, of course, Paul and Peter [Hoffmann 2000: 205].

Many local saint calendars were lost over the centuries, so that both the choice of the saints and their name days were subject to a constant change. At present the Roman general calendar of 1969 (second Vatican council) is obligatory; this, however, imposed some changes of calendar dates.

Overview of the mathematical procedure

Fig. 1. Flow chart for calculating wall declination d and eastern amplitude ω in dependence on geodesic (R_A, H_A, R_B, H_B, a, b) and astronomical (P_{2000}, Y, M, D) parameters

Fig. 1 shows the calculation scheme. The various quantities are defined as follows:

Geodesic input quantities

R, H R is the distance east (the German technical term is *Rechtswert*), and H is the distance north (the German technical term is *Hochwert*), from some reference point. Here we use the planar, right-angled coordinate system known as Gauss-Krüger coordinates. [Feustel 2003]

Indices A and B indicate two surveying points of the church ground plan, where the direction AB is parallel to the church axis.

a, b Major and minor semiaxis of a rotational ellipsoid approximating the earth's shape [Feustel 2003]; $a = 6377397.155$ m, $b = 6356078.963$ m.

Astronomical input quantities

P_{2000} Set of reference values for the parameters of the earth's orbit on 1 January 2000, 12:00 UT (the abbreviation of this so-called standard epoch is J2000.0) [Yallop and Hohenkerk 1992: 484]; the Julian Day number for this date is 2451545.0.

Y, M, D Year of the building of the church, month, and day for the name day of the patron saint. The Julian calendar is valid up to 4 October 1582; the Gregorian calendar is valid from 15 October 1582 [Meeus 2000: 59].

Intermediate quantities

JD Julian Day number: continuous count of days and fractions thereof from 1 January 4713 B.C., 12:00 UT [Meeus 2000: 59]; this method of counting, usual in astronomy, was suggested in 1581 by Joseph Justus Scaliger (1540-1609), a French scholar.

t Angle of direction: angle between line AB and a parallel to the north direction of the Cartesian-geodesic coordinate grid [Feustel 2003].

γ Meridian convergence: angle between geographical north and a parallel to the north direction of the Cartesian-geodesic coordinate grid [Feustel 2003].

ϕ Geographical latitude: angular distance between a location on the earth's surface and the earth equator. It is positive in the northern hemisphere, negative in the southern one [Feustel 2003].

δ Sun declination (coordinate in equator system): angular distance between the sun and the celestial equator. It is positive in the northern hemisphere, negative in the southern one [Yallop and Hohenkerk 1992: 484], [Meeus 2000: 91].

Output quantities

d The *Holy Alignment*: the angle between the church wall and the east-west direction. Looking from the zenith, a positive value (index p in formula (19)) means turning the wall clockwise away from the east-west direction; a negative value (index n in formula (20)) means turning it counterclockwise [Feustel 2003; 2005].

ω Eastern amplitude: the angle between the point where the sun rises on the saint's name day and the east point. By definition, northern angles have a positive sign and southern angles have a negative sign [Eckstein et al. 1995: 20].

First we shall see how one can calculate with Gauss-Krüger coordinates the geographical latitude of the church's location and the direction of a wall or building axis with respect to geographical east. The next calculation concerns the sun declination about a thousand years ago on the saint's day; for this, the relevant Julian Day number is required. Finally, geographical latitude and sun declination will be used to calculate the eastern amplitude for one saint or the resulting eastern amplitude for several saints.

The next three sections are extracted from [Feustel 2003].

Gauss-Krüger coordinates

The method devised by Carl Friedrich Gauss (1777-1855) and further developed by Johann Heinrich Louis Krüger (1857-1923) serves (according to mathematical rules of calculation) to project parts of the curved surface of the earth onto a plane or, vice versa, sectors of a plane onto the curved surface of the earth. For this reason the earth's surface (approximated by an ellipsoid surface) is divided into so-called meridian strips. Each meridian strip has a definite dimension in longitude and a main or middle meridian such that:

- the mutual mappings ellipsoid $\leftrightarrow$ plane shall be *conformal mappings*;
- the mutual mappings main meridian $\leftrightarrow$ abscissa at a two-dimensional right-angled reference system shall be *equal-length projections*.

Additional symbols and their meanings are:

- x Gaussian abscissa (equal-length projection of the main meridian): it is counted in meters and measured from the equator.
- y Gaussian ordinate: starting from the main meridian it is positive in an easterly direction and negative in a westerly direction.
- K Characteristic number for main meridians: it has the values 2, 3, 4 or 5 for Germany and is the first number of the input quantity R . This number tells how many strips to the east a particular main meridian is from the Greenwich meridian.
- λ_0 Geographical longitude of the main meridian.

The Gauss-Krüger system was officially introduced into German cartography in 1927. For the main meridians of the meridian strips with a width of 3° degrees the equation

$$(1) \quad \lambda_0 = 3 \cdot K$$

is valid, i.e., they are of 6° , 9° , 12° and 15° eastern longitude. The meridian strips have a dimension of width of about 200 km.

Within a 3° meridian strip system any surveying point listed in land-registers is fixed with the coordinates *Rechtswert* R and *Hochwert* H , giving the easterly and northerly location of the point.

$$(2) \quad R = K \cdot 10^6 + 500000 + y ,$$

$$(3) \quad H = x .$$

By the addition of 500000 negative R values are avoided. The quantities R and H have the dimension of meter, with seven digits before and three digits after the decimal point.

Geographical latitude, longitude

Gauss-Krüger coordinates of buildings can be acquired from the office responsible for ground management (in Germany *Amt für Bodenmanagement* formerly *Katasteramt* respectively *Landesvermessungsamt*).

We will need the latitude of a church in order to compute the eastern amplitude for a particular saint. It is possible to read the latitude and longitude off a good topographic map instead of computing it, but in my opinion it is more convenient to calculate the geographical coordinates with this procedure than to look for an adequate – and expensive – topographic map. Further, the Gauss-Krüger coordinates are required in any case for calculating the angle of direction.

We use power series to transform Gauss-Krüger coordinates into latitude and longitude. The following new symbols are used:

ϕ_f Base of latitude on the rotational ellipsoid;

e'^2, η_f^2, N_f^2 Auxiliary quantities;

N_f Normal curvature radius for ϕ_f .

Calculating the Geographical Latitude: We first use a power series to find the length of the arc B from the equator to the survey point, located at a latitude angle ϕ from the equator:

$$(4) \quad B = 6366742.521 \cdot \phi - 15988.63853 \cdot \sin(2 \cdot \phi) + 16.72995 \cdot \sin(4 \cdot \phi) - 0.02178 \cdot \sin(6 \cdot \phi) + 0.00003 \cdot \sin(8 \cdot \phi) + \dots$$

One gets with the condition

$$(5) \quad H - B = 0 \text{ for } \phi \rightarrow \phi_f$$

by iteration of the base of latitude. It will be used for the relations (7), (8), (10) and (12). An iteration method is indispensable because with the condition $H = B$ an explicit function for the sought-after angle of latitude $\phi_f = f(H = B)$ (measured in radians) cannot be derived from (4).

If one carries out the calculations with EXCEL, for example, a German version, the function "Zielwertsuche" is available. At first the adjusted "Maximale Iterationszahl" (maximum iteration number) and "Maximale Änderung" (maximum change) should be checked and corrected if necessary to achieve the required precision. One has to enter the appropriate values at "Zielzelle", "Zielwert" and "Veränderbare Zelle" in the dialog field "Zielwertsuche" to be found under "Extras".

The auxiliary quantities are derived from:

$$(6) \quad e'^2 = \frac{a^2 - b^2}{b^2},$$

$$(7) \quad \eta_f^2 = e'^2 \cdot \cos^2 \phi_f,$$

$$(8) \quad t_f = \tan \phi_f.$$

With (7) the normal curvature radius may be expressed as

$$(9) \quad N_f = \frac{a^2}{b \cdot \sqrt{1 + \eta_f^2}} .$$

With the inclusion of the quantities mentioned above the power series for the sought-after geographical latitude (measured in radians) is

$$(10) \quad \phi = \phi_f - \frac{t_f}{2 \cdot N_f^2} \cdot (1 + \eta_f^2) \cdot y^2 + \frac{t_f}{24 \cdot N_f^4} \cdot (5 + 3 \cdot t_f^2 + 6 \cdot \eta_f^2 - 6 \cdot \eta_f^2 \cdot t_f^2) \cdot y^4 + \dots ,$$

with the Gaussian ordinate following from remodeling (2)

$$(11) \quad y = R - K \cdot 10^6 - 500000 .$$

Calculating the Geographical Longitude: The power series for the geographical longitude λ (measured in radians) is indicated

$$(12) \quad \lambda = \lambda_0 + \frac{1}{N_f \cdot \cos \phi_f} \cdot y - \frac{1 + 2 \cdot t_f^2 + \eta_f^2}{6 \cdot N_f^3 \cdot \cos \phi_f} \cdot y^3 + \frac{5 + 28 \cdot t_f^2 + 24 \cdot t_f^4}{120 \cdot N_f^5 \cdot \cos \phi_f} \cdot y^5 + \dots$$

with the main meridian according to (1). In connection with the Holy Alignment it is only necessary in certain conditions, see equation (18).

Angle of direction, meridian convergence, wall declination

Definitions:

Local meridian	A great circle passing through the earth poles and through a surveying point. All points on this meridian have the same geographical longitude.
Geographical north	Northern direction of the local meridian through a surveying point.
Grid north	Northern direction of a parallel to the main meridian of a meridian strip system which passes through a surveying point.
Angle of direction	Arises if at given surveying points A (R_A , H_A) and B (R_B , H_B), a parallel, which is put through A to the direction grid north is turned clockwise until it crosses B.
Meridian convergence	Angle between the direction of geographical north and the direction of grid north. It is positive for points east of the main meridian, and negative for points west of the main meridian. Surveying points lying exactly on the main meridian have zero convergence.

The main meridian, located at the center of a meridian strip, points to true north. Other meridians within that strip, being parallel to the main meridian, cannot point north, since all meridians must converge at the north pole. The angle between grid north at a local meridian and true north is called *meridian convergence*.

Calculating the Angle of Direction: The angle of direction of the line connecting the two survey points (measured in degrees) is found using right-triangle trigonometry:

$$(13) \quad t = \arctan \frac{R_B - R_A}{H_B - H_A} .$$

Calculating the Meridian Convergence: A power series serves for calculating the meridian convergence γ (measured in radians)

$$(14) \quad \gamma = \frac{t_f}{N_f} \cdot y - \frac{t_f}{3 \cdot N_f^3} \cdot (1 + t_f^2 - \eta_f^2) \cdot y^3 + \dots$$

Using only the linear terms of the power series (10), (12) and (14) yields the following relations

$$(15) \quad \phi \approx \phi_f,$$

$$(16) \quad \lambda \approx \lambda_0 + \frac{1}{N_f \cdot \cos \phi_f} \cdot y,$$

$$(17) \quad \gamma \approx \frac{t_f}{N_f} \cdot y.$$

Combining the above three formulas with formula (8) gives an equation for approximating the convergence angle:

$$(18) \quad \gamma \approx (\lambda - \lambda_0) \cdot \sin \phi.$$

Calculating the Wall Declination: The wall declination, the angle between it and true north, is found from the angle of direction and the meridian convergence (expressed in degrees). It is necessary to distinguish whether the normal line of a south wall points westward, respectively the normal line of a north wall points eastward

$$(19) \quad d_p = t + \gamma + 90^\circ,$$

or the normal line of a south wall points eastward, respectively the normal line of a north wall points westward

$$(20) \quad d_n = t + \gamma - 90^\circ.$$

Julian Day number

For converting a date Y, M, D given in the Gregorian or in the Julian calendar into the corresponding Julian Day number, the following formula may be used:

$$(21) \quad JD = \text{floor}(365.25 \cdot (Y_Y + 4716)) + \text{floor}(30.6001 \cdot (M_M + 1)) + D + F - 1524.5$$

[Meeus 2000: 60].

Here we are considering the name day of a patron saint about one millennium ago, which means the Julian calendar applies.

The function $\text{floor}(x)$ or $\text{INT}(x)$ delivers the largest integer number $\leq x$.

Depending on the applicable calendar and number of the month, the variables Y_Y , M_M and F in equation (21) and an auxiliary quantity E can be calculated from the following expressions

		Gregorian calendar		Julian calendar	
		$M=1 \text{ or } 2$	$M>2$	$M=1 \text{ or } 2$	$M>2$
(22)	$Y_Y=$	$Y-1$	Y	$Y-1$	Y
(23)	$M_M=$	$M+12$	M	$M+12$	M
(24)	$E=$	$\text{floor}(\frac{Y-1}{100})$	$\text{floor}(\frac{Y}{100})$	not relevant	not relevant
(25)	$F=$	$2 - E + \text{floor}(\frac{E}{4})$	$2 - E + \text{floor}(\frac{E}{4})$	0	0

The Julian Day regularly begins at 12:00 UT. Hours, minutes and seconds are attached as decimals to the day of the month.

Sun declination, eastern amplitude

The following symbols will be used:

- T Time in Julian centuries (36525 days) from J2000.0.
- L Mean longitude: angular distance from the vernal equinox if the earth moved with a uniform velocity at a circular orbit in the celestial equator plane around the sun. At the vernal point the ecliptic intersects the celestial equator. The sun passes this point from north to south.
- G Mean anomaly: angular distance from the perihelion if the earth moved with a uniform velocity at a circular orbit in the celestial equator plane around the sun. The perihelion is the point of the earth's elliptical orbit which is nearest to the sun.
- λ Ecliptic longitude: angular distance from the vernal equinox at the ecliptic measured eastward in the direction of the annual circulation around the sun.
- ϵ Obliquity of the ecliptic: angle between ecliptic and celestial equator.
- h Sun altitude (this is a coordinate in the so-called *horizon system* of spherical coordinates): spherical distance from the horizon [Meeus 2000: 91].
- A Sun azimuth (coordinate in horizon system): spherical distance from the local meridian ($A=0^\circ$). In astronomy it is measured westward from the south (0°), through the west (90°), north (180°) and east (270°) [Meeus 2000: 91].

Calculating the Sun Declination: To calculate the sun's declination corresponding to a given Julian Day number it is necessary at first to find out the required values with the formulae (26) through (30) [Yallop and Hohenkerk 1992: 484-5]:

$$(26) \quad T = \frac{JD - 2451545.0}{36525},$$

$$(27) \quad L = 280.460^\circ + 36000.770^\circ \cdot T,$$

$$(28) \quad G = 357.528^\circ + 35999.050^\circ \cdot T,$$

$$(29) \quad \Lambda = L + 1.915^\circ \cdot \sin G + 0.020^\circ \cdot \sin 2 \cdot G ,$$

$$(30) \quad \varepsilon = 23.4393^\circ - 0.01300^\circ \cdot T .$$

Finally one gets the sun declination

$$(31) \quad \delta = \arcsin(\sin \varepsilon \cdot \sin \Lambda) ,$$

which is useful for calculating the eastern amplitude.

The symbol P_{2000} used at the beginning refers to all constants (angular values) in the formulae above.

Calculating the Eastern Amplitude: Among others, the relation

$$(32) \quad \sin \delta = \sin \phi \cdot \sinh - \cos \phi \cdot \cosh \cdot \cos A$$

exists between the coordinates of equator system and horizon system.

Setting $h = 0^\circ$ in equation (32) and disregarding for now the effects of refraction (see Appendix), the corresponding azimuth A_{h0} then will be found by

$$(33) \quad \cos A_{h0} = -\frac{\sin \delta}{\cos \phi} .$$

Because the sum of the sun's azimuth at the level of the horizon and the eastern amplitude ω is equal to 270° ,

$$(34) \quad A_{h0} = 270^\circ - \omega ,$$

combining (33) and (34) yields

$$(35) \quad \omega = \arcsin \frac{\sin \delta}{\cos \phi} .$$

If the patronage is shared by several saints, the resultant eastern amplitude can be obtained from:

$$(36) \quad \omega_{res} = \arcsin \frac{\sin \delta_1}{\cos \phi} + \arcsin \frac{\sin \delta_2}{\cos \phi} + \dots$$

This is valid for saints from different centuries, too.

Example

In 980 (Julian calendar!) a church is supposed to have been consecrated to the patron saints Andrew, James and Philip. According to *Meyers Konversations-Lexikon*, Saint Andrew's name day is 30 November; that of both Saints James and Philip is 1 May.

The southern wall of the church deviates to the west from east-west direction. At the corners *A* and *B* (fig. 2) it has the Gauss-Krüger coordinates:

$$\begin{aligned} R_A &= 3456064,640 \text{ m}, & H_A &= 5562773,360 \text{ m}, \\ R_B &= 3456046,600 \text{ m}, & H_B &= 5562777,110 \text{ m}. \end{aligned}$$

Fig. 2. Ground plan of a church with the surveying points A and B . Furthermore, with reference to geographical east, are shown the eastern amplitudes of the saints ω_A and ω_{PJ} , the resultant eastern amplitude ω_{APJ} (ω_{res}) as well as the direction of the symmetry axis of the nave (wall declination) d_p

It must be verified whether the directions of the nave's symmetry axis and the resultant eastern amplitude are identical.

In the following tabulation input values are in italics; intermediate values are in plain text; and output values are in bold.

formula	quantity	value	remark
	R_A	<i>345606.640 m</i>	<i>Rechtswert (eastern coordinate) of surveying point A</i>
	H_A	<i>556277.360 m</i>	<i>Hochwert (northern coordinate) of surveying point A</i>
	R_B	<i>345604.600 m</i>	<i>Rechtswert (eastern coordinate) of surveying B</i>
	H_B	<i>556277.110 m</i>	<i>Hochwert (northern coordinate) of surveying B</i>
	$R_{average}$	3456055.620 m	
	$H_{average}$	5562775.235 m	
	ϕ_f	0.876194833 rad	"veränderbare Zelle" (changeable cell)
(4)	B	5562775.235 m	
(5)	$H_{average} - B$	0,0000100 m	"Zielzelle"; "Zielwert": 0,00001 (target cell, target value)
(8)	t_f	1.200333804	
	t_f^2	1.440801241	
	t_f^4	2.075908216	
	a	<i>6377397.155 m</i>	<i>semiaxis of rotational ellipsoid</i>
	b	<i>6356078.963 m</i>	<i>semiaxis of rotational ellipsoid</i>
(6)	e^2	0.0067719219	
(7)	η_f^2	0.002752874	

(9)	N_f	6389997.463	
	N_f^2	4.08321E+13	
	N_f^3	2.60917E+20	
	N_f^4	1.66726E+27	
	N_f^5	1.06538E+34	
	K	3	
(11)	y	-43944.380 m	
	y^2	1931108534	
	y^3	-8.48614E+13	
	y^4	3.72918E+18	
	y^5	-1.63877E+23	
(10)	ϕ	0.876166372 rad	50,20063525°; 50° 12' 2,29"
(1)	λ_0	9°	
(12)	λ	0.146335891 rad	8,384428933°; 8° 23'3,94"
(13)	t	-4.810666667 rad	-78,25707946°; -78° 15'25,49"
(14)	γ	-0.008254447 rad	-0,472944994°; -28'22,60"
(19)	d_p	11.26997555°	11° 16'11,91": direction of nave's symmetry axis
	Y_{PJ}, M_{PJ}, D_{PJ}	980, 5, 1	name day Philip/James
(22)	Y_{YPJ}	980	
(23)	M_{MPJ}	5	
(25)	F_{PJ}	0	
(21)	JD_{PJ}	2.0791235E+6	
(26)	T_{PJ}	-10.196	
(27)	L_{PJ}	-315.810°	
(28)	G_{PJ}	-221.204°	
(29)	Λ_{PJ}	-314.568°	
(30)	ε_{PJ}	23.572°	
(31)	δ_{PJ}	16.552°	
(35)	ω_{PJ}	26.428°	eastern amplitude Philip/James
	Y_A, M_A, D_A	980, 11, 30	name day Andrew
(22)	Y_{YA}	980	
(23)	M_{MA}	11	
(25)	F_A	0	
(21)	JD_A	2.0793365E+6	

(26)	T_A	-10.191	
(27)	L_A	-105.867°	
(28)	G_A	-11.272°	
(29)	Λ_A	-106.249°	
(30)	ε_A	23.572°	
(31)	δ_A	-22.577°	
(35)	ω_A	-36.854°	eastern amplitude Andrew
(36)	ω_{res}	-10.426°	-10° 25' 33,6": resultant eastern amplitude according to saints

The absolute values of the angles of the nave's symmetry axis and the resultant eastern amplitude regarding geographical east differ only by 0.844° (also see fig. 2). And even so, it might be possible that the orientation of the church building correlates with the patronage of the saints Andrew, James and Philip.

Concluding remarks

The comparison of the values calculated with formulae (19) resp. (20) and (35) resp. (36) are surely helpful for analyzing (by calculation) the connection presumed to exist between the Holy Alignment and the eastern amplitude of the saint or saints. But for a final judgment, however, one is obliged to include further facts regarding history and site. This must also take into account how well (if at all) the measurements of these two angles harmonize. One has to consider the definitions of these two quantities with respect to their different signs.

With the mathematical procedure above it is also possible to calculate the (probable) year of the erection of the church, by knowing the day on which (according to the medieval calendar) the patron saint was commemorated. To do this, one must calculate the eastern amplitude ω for different years Y (equation (22)) with formula (35) and compare it with the present nave's direction d_p potentially d_n (formula (19) or (20)). That means the solution has to be found by trial and error. The same trial method is applicable if several saints share the patronage, and for construction phases from different centuries. In this case one has to evaluate the influence of the different years on the resultant eastern amplitude ω_{res} (formula (36)).

Appendix: Refraction

The refraction – bending of light while it passes through the earth's atmosphere – causes an apparent increase of a star's altitude, i.e., the apparent (observed respectively measured) altitude h_b is not identical with the true altitude h . The most marked example of this effect is the directly perceptible oval deformation of the sun's disk near the horizon: an apparent diminution of the vertical diameter. With the following approximation formula [Meeus 2000: 106] the refraction can be calculated with sufficient accuracy at all altitudes from 90° to 0°

$$(1) \quad R_{hb} = \frac{1}{\tan\left(h_b + \frac{7.31}{h_b + 4.4}\right)}.$$

The result is expressed in arc minutes. The amount of the refraction increases when the atmospheric pressure increases or when the air temperature decreases or when the wave length of the light decreases. Formula (1) is valid for observations at sea level and an air pressure $p=1010 \text{ mbar}$, an air temperature $\vartheta = 10^\circ\text{C}$ as well as yellow light. Air parameters differing from it can be taken into account with the correction factor:

$$(2) \quad k = \frac{p}{1010} \cdot \frac{283}{273 + \vartheta} \quad [\text{Meeus 2000: 107}].$$

If necessary one must multiply the refraction given by formula (1) with this factor.

The true altitude is defined by

$$(3) \quad h = h_b - R_{hb}.$$

Example: An observer is supposed to see the center of the sun disk at the horizon, i.e., $h_b=0^\circ$; the true diameter of the sun is about $32'$. Which one is the true position of the sun in this moment when $k=1$? Relation (1) of the appendix yields the refraction $R_{hb}=34.478'$ and with (3) one gets $h=0^\circ-34.478' = -34.478'$. Therefore the true position of the sun's center is $34.478'$ below the horizon or – in other words – the upper rim of the sun has an altitude of $16'-34.478' = -18.478'$.

When the true altitude of a star is known, the apparent altitude is defined by:

$$(4) \quad h_b = h + R_h.$$

Taking into consideration [Meeus 2000: 106],

$$(5) \quad R_h = \frac{1.02}{\tan\left(h + \frac{10.3}{h + 5.11}\right)}.$$

In this case the value of R_h should be multiplied by the correction factor k .

Example: Which altitude of the sun's lower rim would an observer measure if the true center of the sun is on the horizon and $k=1$? The true diameter of the sun is about $32'$. With $h = -16' = -0.267^\circ$ one gets from (5) $R_h=31.41'$ so that the lower rim of the sun could be observed at the altitude $h_b = -16'+31.41'=15.41'$ over the horizon.

The sun azimuth near the horizon

The two examples above have shown that due to the atmospheric refraction the sun disk can already be seen partly or completely over the horizon although it is completely or partly below the horizon. This effect happens in an altitude range of approximately a sun's radius or diameter. Therefore it is certainly of interest to find the sun azimuth values which correspond with these small altitude values.

Between the coordinates of the equator system (declination δ) and the horizon system (altitude h , azimuth A) as well as the geographical latitude ϕ , there exists, among others, the relation

$$(6) \quad \cos A = \frac{\sin \phi \cdot \sin h - \sin \delta}{\cos \phi \cdot \cosh}$$

The following table presents the difference $\Delta A = A(h=0) - A(h)$ for $\phi=50^\circ$ and some selected declination values. $A(h=0)$ symbolizes the sun azimuth at the true sunrise altitude $h=0^\circ$ and $A(h)$ refers to the true altitudes which equal one sun's radius, respectively diameter below the horizon.

	$\delta=23.44^\circ$	$\delta=15^\circ$	$\delta=0^\circ$	$\delta=-15^\circ$	$\delta=-23.44^\circ$
$h = -0.267^\circ$	$-0,407^\circ$	$-0,348^\circ$	$-0,318^\circ$	$-0,346^\circ$	$-0,403^\circ$
$h = -0.533^\circ$	$-0,815^\circ$	$-0,697^\circ$	$-0,635^\circ$	$-0,691^\circ$	$-0,802^\circ$

Consequently an observer can measure a value for the observed (apparent) eastern amplitude ω_b which differs from the true eastern amplitude ω – depending on his decision which sun position is just right for the moment of the sunrise. With reference to the exact east point one gets northwards $\omega_b \geq \omega$ and southwards $\omega_b \leq \omega$. Obviously differences can amount to $\pm 1^\circ$, at the worst.

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