Christopher Stone

AFEC
University of Derby
Kedleston Road
Derby DE22 1GB
UNITED KINGDOM
c.p.stone@derby.ac.uk

Keywords: algebra, algorithms,
circles, design theory, fractals,
geometry, infinity, iteration,
lines, mappings, morphology,
ratio, rectangles, shape
grammars, topology,
transformations

Didactics

The Use of Linear Fractional Transformations to Produce Building Plans

Abstract. Linear fractional transformations are mappings on the complex plane that can be used to form building plans. Linking topological and geometrical considerations to meet certain criteria is one way of generating plans. This present study seeks any mathematical structures underlying basic plan forms which would link topological and geometrical maps. It concludes with an algorithm for plan generation.

“The plan is the generator” said Le Corbusier; but how are plans generated? We usually start with a topological map (bubble diagram) and morph this into a geometrical map (plan). I was interested to find a mathematical structure underlying this process which was, perhaps, more elegant than the “chopping block” techniques of most shape grammar methods.

Linear fractional transformations (LFTs) are mappings on the complex plane and have been used to produce fractal patterns. I have used the same transformations to generate rectangular configurations to form building plans.

Fig. 1.

A particular rectangle can be generated by inscribing it in a particular circle starting at a specific point on the circumference. LFTs map circles to circles and if the circles overlap this will generate further adjacent rectangles which form architectonic patterns. To ensure that circles overlap the transformations have to be elliptic. This ensures that there are two

fixed points and the position of these determines the way the layout progresses. By varying the parameters a , b , c and d a range of plan types is produced using mathematical software.

This is a first small step in a new method of plan generation.

Introduction

The main problem of space layout planning is linking topological considerations with geometrical ones to meet certain criteria [Jo and Gero 1998]. Topological considerations of connectedness and adjacency of spaces can be represented by a graph comprising a set of vertices and edges, familiar to architects as a bubble diagram. To transform this into a building plan involves turning the vertices into geometrical shapes (usually rectangles) while considering spatial, access and aesthetic requirements. Formal enumeration of these transformations is fraught with problems, not least the ‘combinatorial explosion’ of numbers of possible solutions [Steadman 1989: 112]. In practice architects and designers make tentative arrangements of shapes sometimes using circular or elliptical forms to approximate the rectangles before the layout is firmed up. Many approximations may be made and at each stage a selection is made based on intuitive skill.

I wanted to investigate the possibility of finding any mathematical structures underlying basic plan forms which would link topological and geometrical maps.

There is currently much mathematical interest in packings of circles into geometric shapes [Stephenson 2005] and the exploration of soap bubbles [Oprea 2000: 43-50], both of which make extensive use of mathematics in the complex plane. Complex analysis has also been used to map the surface of the brain, explain Escher’s drawing “Hand with Reflecting Sphere”, analyse Celtic art and generate fractal patterns (see [Hurdal 2003]; [Blair 2000: iv]; [Ellacott 2000] and [Mumford et al, 2002]). With such fertile links with geometry I wanted to see if this area of maths had applications in architectural design.

A final inspiration was the picture “Several Circles” by Vasily Kandinsky which consists of a pattern of overlapping circles (this is reproduced on various websites; see for instance <http://guggenheim.stores.yahoo.net/kanpossevcir.html>).

Architectural framework

Buildings consist of a collection of adjacent spaces. Adjacency is the key to the production of building plans. If spaces are adjacent then they can be directly connected. Plans are built up by grouping connected spaces with circulation spaces. Spaces are generally polygonal to ensure adjacency and most usually rectangular or made up of rectangles. Rectangles are placed next to each other to form an overall plan layout. There are several categories of overall plan layouts. I wanted to start by looking at linear configurations and cyclic configurations (arrangements of spaces around a common space).

Mathematical framework

A rectangle of any size, proportion and position can be generated by inscribing horizontal and vertical lines from a point on a circle in the Cartesian plane. An adjacent rectangle can be generated by inscribing it in another circle that overlaps the first and passes through at least one vertex of the first rectangle. The vertex chosen and the size of the second circle determine the size, proportion and configuration of the second rectangle. A plan can be thus generated by a series of overlapping circles. The process can be pictured as

a circle flipping over about two points on its perimeter while changing size. At each flip a rectangle will be left.

On the complex plane points are represented by numbers composed of two parts; a real number that appears on the conventional horizontal number line and an imaginary number¹ that is plotted on a vertical scale. The imaginary number is a multiple of $\sqrt{-1}$ (the square root of minus one denoted as i). A complex number $(x + iy)$ is the sum of a real number and an imaginary one and is both a number and a point on a plane. Points on the plane can be moved around by adding, subtracting, multiplying or dividing these numbers. This property means that geometrical operations are generally simpler compared with the matrix operations required on a real Cartesian plane.

Conformal mapping on the complex plane preserve certain geometrical properties. Linear fractional transformations (or Moebius transformations) map circles to circles

[Needham 2000: 148].² LTRs are of the form $w = \frac{az + b}{cz + d}$ where a, b, c and d are complex parameters and z and w are complex variables. A circle on the z plane can be transformed to a circle on the w plane and these planes can be displayed together. The extended complex plane includes the point at infinity and part of the underlying structure of these operations involves the poles of the transformations. There are two poles. z_∞ is the point that is taken to the point at infinity and Z_∞ is the point that the point at infinity is transformed back to [Needham 2000: 170]. They are defined by:

$$z_\infty = -\frac{d}{c} \quad \text{and} \quad Z_\infty = \frac{a}{c}$$

The poles are included for completeness and have been plotted and tabulated with other results.

There are four categories of LFTs depending on the values of a, b, c and d [Needham 2000: 150]. In two cases the circles do not overlap (hyperbolic and loxodromic). In one case the circles touch at one point (parabolic) and in one case the circles overlap (elliptic). A transformation is elliptical iff $(a+d)$ is real and $|a+d| < 2$ [Complex Analysis 1995: 23]. This can be written as $0 \leq (a + d)^2 < 4$. This condition applies to normalised transformations. For a transformation to be normalised a, b, c and d have to be multiplied by $\frac{\pm 1}{\sqrt{ad - bc}}$. To ensure an LFT is elliptical it would have to be normalised and the condition checked but the LFT does not have to be used in its normalised form [Schwerdtfeger 1979: 41].

If two rectangles are adjacent their circumscribed circles will overlap. The LFT transforming one circle to the other can be found from:

$$\frac{(z - \alpha)(\beta - \gamma)}{(z - \gamma)(\beta - \alpha)} = \frac{(w - \alpha')(\beta' - \gamma')}{(w - \gamma')(\beta' - \alpha')}$$
 this maps $\alpha \rightarrow \alpha', \beta \rightarrow \beta'$ and $\gamma \rightarrow \gamma'$ where these are any three points on the first circle transformed to three points on the other.

A circle is defined uniquely by three points and the LFT of the three points will uniquely transform one circle to the other.

Investigations

I have arranged rectangles in six linear configurations (Transformations A-F) and three cyclic configurations (Transformations G, H and J). Transformations A and B show lines of squares and C and D show 2x1 rectangles. Points that are fixed and that are to be transformed were selected (three in total) and substituted in the above formula and the parameter values were generated using *DERIVE*. The direction of transformations is shown and the path of the z_0 pole is indicated as are the overlapping circles and fixed points. In this case both fixed points are also vertices of the rectangles:

Transformations A-B are shown in fig. 2. Transformations C-D are shown in fig. 3. Results are shown in **Table 1 (A-D)**.

Fig. 1. Transformations A and B

Fig. 2. Transformations C and D

	a	b	c	d	Pole z_{∞}	Pole Z_{∞}	Fixed Points
A ↑	$-(2n+)+i$	$2n-(2n^2)i$	$2i$	$(2n-1)-i$	$\frac{1}{2}+\frac{(2n-1)i}{2}$	$-\frac{1}{2}+\frac{(2n+1)i}{2}$	$n, 1+ni$
B →	$-1+(2n+1)i$	$2n-(2n^2)i$	$2i$	$1-(2n-1)i$	$\frac{(2n-1)}{2}+\frac{i}{2}$	$\frac{(2n+1)}{2}+\frac{i}{2}$	$n, n+i$
C ↑	$-(2n+1)+2i$	$4n+(2n^2)i$	$2i$	$(2n-1)-2i$	$1+\frac{(2n-1)i}{2}$	$-1+\frac{(2n+1)i}{2}$	$ni, 2+ni$
D →	$-1+(4n+2)i$	$4n-(8n^2)i$	$2i$	$1-(4n-2)i$	$(2n-1)+\frac{i}{2}$	$(2n+1)+\frac{i}{2}$	$2n, 2n+i$

Table 1 (A-D). Results are summarised by creating formulae where n represents the order number of the transformation starting at 1

Transformations E show a vertical stack of rectangles of increasing aspect ratio. In this case only the fixed point which is also a vertex is shown. Transformation F show a stack of squares of increasing size and again only one fixed point is shown. Transformations E and F are shown in fig. 4 and fig. 5 respectively. Results are shown in **Table 2 (E,F)**.

Fig. 4. Transformations E

Fig. 5. Transformations F

	a	b	c	d
E ↑	$(2n^2 + 2n + 1) + (n^2 + n)i$	$(2n^2 + 2n + 1) + (n^2 + n)i$	$(2n + 1)i$	$2n^2 - (n^2 + n)i$
F ↑	$-(n + 1)^2 + (n + 1)i$	$(n^3 + \frac{3n^2}{2} + \frac{n}{2}) + (\frac{4n^3}{3} + \frac{7n^2}{2} - \frac{11n}{6})i$	$2i$	$n^2 - ni$

	Pole z_{∞}	Pole Z_{∞}	Fixed Point
E ↑	$\frac{n(n+1)}{2n+1} + \frac{2n^2i}{2n+1}$	$\frac{n(n+1)}{2n+1} + \frac{(2n^2 + 2n + 1)i}{2n+1}$	ni
F ↑	$\frac{n}{2} + \frac{n^2i}{2}$	$\frac{n+1}{2} + \frac{(n+1)^2i}{2}$	$\frac{n(n+1)i}{2}$

Table 2 (E,F). Results are summarised by creating formulae where n represents the number of the transformation

Fig. 6. Transformations G

Transformations G show a cyclic arrangement of squares and both fixed points are shown (fig. 6). Results are shown in **Table 3 (G)**.

	a	b	c	d	Pole z_{∞}	Pole Z_{∞}	Fixed Points
↑	$-3+i$	$2+2i$	$2i$	$1-i$	$\frac{1}{2} + \frac{i}{2}$	$\frac{1}{2} + \frac{3i}{2}$	$i, 1+i$
→	$-3+3i$	$6+2i$	$2i$	$3-i$	$\frac{1}{2} + \frac{3i}{2}$	$\frac{3}{2} + \frac{3i}{2}$	$1+2i, 1+i$
↓	$-1+3i$	$6-2i$	$2i$	$3-3i$	$\frac{3}{2} + \frac{3i}{2}$	$\frac{3}{2} + \frac{i}{2}$	$2+i, 1+i$
←	$-1+i$	$2-2i$	$2i$	$1-3i$	$\frac{3}{2} + \frac{i}{2}$	$\frac{1}{2} + \frac{i}{2}$	$1, 1+i$

Table 3 (G). (Results cannot be easily summarised by creating formulae)

Fig. 7. Transformations H

Transformations H show a cyclic arrangement of 2x1 rectangles around a square. One fixed point per transformation is shown (fig. 7). Results are shown in **Table 4 (H)**.

	a	b	c	d	Pole z_{∞}	Pole Z_{∞}	Fixed Point
↑	$-14 + 2i$	$12 + 24i$	$5i$	$8 - 4i$	$\frac{4}{5} + \frac{8i}{5}$	$\frac{2}{5} + \frac{14i}{5}$	$2i$
→	$-13 + 14i$	$54 + 3i$	$5i$	$11 - 8i$	$\frac{8}{5} + \frac{11i}{5}$	$\frac{14}{5} + \frac{13i}{5}$	$2 + 3i$
↓	$-1 + 13i$	$18 - 24i$	$5i$	$7 - 11i$	$\frac{11}{5} + \frac{7i}{5}$	$\frac{13}{5} + \frac{i}{5}$	$3 + i$
←	$-2 + i$	$6 - 3i$	$5i$	$4 - 7i$	$\frac{7}{5} + \frac{4i}{5}$	$\frac{1}{5} + \frac{2i}{5}$	1

Table 4 (H). (Results cannot be easily summarised by creating formulae)

Transformations J show a cyclic arrangement of squares and rectangles around a rectangular and looks more like a building plan. One fixed point per transformation is shown (fig 8). Results are shown in **Table 5 (J)**.

Fig. 8. Transformations J

	a	b	c	d	Pole z_{∞}	Pole Z_{∞}	Fixed Point
↑	$-18 + 4i$	$30 + 45i$	$4i$	$9 - 6i$	$\frac{3}{2} + \frac{9i}{4}$	$1 + \frac{9i}{2}$	$3i$
→	$-13 + 16i$	$96 - 3i$	$3i$	$11 - 8i$	$\frac{8}{3} + \frac{11i}{3}$	$\frac{16}{3} + \frac{13i}{3}$	$4 + 5i$
→	$-19 + 36i$	$238 + 127i$	$5i$	$19 - 26i$	$\frac{26}{5} + \frac{19i}{5}$	$\frac{36}{5} + \frac{19i}{5}$	$6 + 5i$
↓	$-4 + 36i$	$146 - 198i$	$5i$	$18 - 31i$	$\frac{31}{5} + \frac{18i}{5}$	$\frac{36}{5} + \frac{4i}{5}$	$8 + 2i$
←	$-2 + 8i$	$24 + 48i$	$3i$	$4 - 16i$	$\frac{16}{3} + \frac{4i}{3}$	$\frac{8}{3} + \frac{2i}{3}$	4

Table 5 (J). (Results cannot be easily summarised by creating formulae)

Observations

- (i) There is a consistency about parameter c which only appears to vary with the proportions of successive rectangles;
- (ii) There is some interchange of real and imaginary values with opposite signs for a and d for successive transformations;
- (iii) Clearly the lack of pattern in the parameters grows as the layouts become more complicated particularly with parameter b (in this case it may not be a problem);³
- (iv) The poles shown follow a less consistent path as the layout becomes more complicated.

Conclusions

There is sufficient information to enable a selection of parameters to be made to feed into an algorithm to generate plan forms something like this:

Notes

1. A slightly pejorative term reflecting the awkward history of this branch of mathematics. When devised it was clear that these numbers could not be used for counting or measuring so were not considered proper numbers. They are now used in quantum physics to do just that and complex analysis is regarded as a basic skill in higher mathematics)
2. Strictly straight lines through the point at infinity appear as straight lines but are regarded as generalised circles. Angles of intersection of circles are preserved)
3. The fixed points are at $\xi_{\pm} = \frac{(a-d) \pm \sqrt{(a+d)^2 - 4}}{2c}$ and so are independent of b).

References

- BLAIR, David E. 2000. *Inversion Theory and Conformal Mapping*. Rhode Island: The American Mathematical Society.
- COMPLEX ANALYSIS. 1995. *Complex Analysis, Unit D1, Conformal Mappings*. Milton Keynes: The Open University, 1995.
- ELLACOTT, Stephen. 2000. The Arithmetical Structure of Celtic Key Patterns. *Mathematics Today* (October 2000): 142-145.
- HURDAL, Monica A. 2003. Imaging the Brain. <http://www.math.fsu.edu/~mhurdal/>. Last updated on 21.07.2007.
- JO, Jun H. John S. Gero. 1998. Space Layout Planning using an Evolutionary Approach. *Artificial Intelligence in Engineering* **12** (3): 149-162.
- MUMFORD, David, et al. 2002. *Indra's Pearls*. Cambridge: Cambridge University Press.
- NEEDHAM, Tristan. 2000. *Visual Complex Analysis*. Oxford: Clarendon Press.
- OPREA, John. 2000. *The Mathematics of Soap Films*. Rhode Island: The American Mathematical Society.
- SCHWERTDFEGER, Hans. 1979. *Geometry of Complex Numbers*. New York: Dover Publications.
- STEADMAN, J.P. 1989. *Architectural Morphology*. London: Pion.
- STEPHENSON, Kenneth. 2005. *Introduction to Circle Packing*. Cambridge: Cambridge University Press.

About the author

Christopher Stone studied architecture at the Bartlett School, University College London and qualified as an architect in 1973. He worked in private practice and in the transport sector until 1994 when he began working for myself. At about the same time he began studying for a degree in mathematics and combining teaching with my architectural work. He now has an established architectural practice and a part-time post as a mathematics lecturer at the University of Derby, where he runs an entry level programme. See: <http://www.derby.ac.uk/staff-search/mr-christopher-stone>.