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Nexorades Based on Regular Polyhedra

Abstract. The objective of this paper is to discuss the characteristics of nexorades based on regular polyhedra. An important application of nexorades is for shelters of various sizes and shapes for temporary or permanent purposes. In such a shelter, the structural skeleton is provided by a nexorade and the cover is provided by a membrane material.

Keywords: mutually supporting elements, form finding, polyhedron, nexorades

1. What is a nexorade?

Consider the structure shown in fig. 1. This structure is made from wooden elements using an 'interwoven pattern' as shown. The wooden elements are about one metre long and have a 27mm×27mm square cross-section. The connections are made by passing pieces of wires through holes in the elements.

This structure was designed and constructed by Olivier Baverel and exhibited at the University of Nottingham, UK, during the third Colloquium of the IASS Working Group on Structural Morphology in August 1997.

Fig. 1. A nexorade made from wooden elements, exhibited in August 1997 at the University of Nottingham UK

The structure shown in fig. 1 is an example of a nexorade [Baverel, et. al 2000] or a 'multi-reciprocal grid' [Baverel and Saidani 1998]. Each one of the elements that constitute a nexorade is referred to as a 'nexor'. A nexor has four connection points, two of which are at the ends of the nexor and the other two are at two intermediate points along the nexor. The term 'nexor' is a Latin based word meaning a 'link' and the term 'nexorade' implies an 'assembly of nexors'.

2. Engagement windows and fans

The arrangement of nexors in a nexorade gives rise to a number of ‘openings’ that are referred to as ‘engagement windows’. A typical engagement window in the nexorade of fig. 1 is of the form shown in fig. 2. The assembly of the nexors that forms an engagement window is referred to as a ‘fan’ or a ‘reciprocal frame’ [Popovic 1996].

Fig. 2. A fan with a triangular engagement window

The length of each side of the engagement window is referred to as the ‘engagement length’. Also, the ratio of the length and the total length of a nexor is referred to as the ‘engagement ratio’. Typical values for the engagement ratio are normally in the range 0.1 to 0.4.

Nexors are usually tubular and are connected together using some sort of swivel connector. A view of a fan with scaffolding tubes as nexors and swivel couplers as connectors is shown in fig. 3.

Fig. 3. A fan with scaffolding tubes as nexors and swivel couplers as connectors

Another example of a nexorade is shown in fig. 4. Each of the fans in this nexorade involves four nexors. A typical fan of the nexorade of fig. 4 is shown in fig. 5. The number of nexors constituting a fan is referred to as the ‘valency’ of the fan. Thus, the fan of fig. 2 has valency 3 and the fan of fig. 5 has valency 4.

Fig. 4. A nexorade with 4-nexor fans

Fig. 5. 4-nexor fan

It is not necessary for all the fans in a nexorade to have the same valency. For example, the nexorade shown in fig. 6 has two fans with valency 3 and two fans with valency 4.

Fig. 6. A nexorade involving fans with different valencies

3. Method for creating nexorades based on regular polyhedra

A method referred to as the method of rotation has been used to generate nexorades based on regular polyhedra. Consider the dodecahedron shown in fig. 7.

Fig. 7. Dodecahedron

The dodecahedron is a polyhedron composed of 30 identical elements, where each vertex such as V1 joins three elements.

In order to explain the method of rotation for transforming the dodecahedron into a nexorade, some polyhedral particulars have to be introduced. Firstly, all the vertices of the dodecahedron are on a sphere that is referred to as the circumsphere. The radius of this sphere is denoted by R_c as shown in fig. 8. The midpoints of the edges of the dodecahedron are also on a sphere referred to as the intersphere. The radius of this sphere is denoted by R_i , as shown in fig. 8.

Fig. 8. Details of the polyhedron

Fig. 9 shows an arrangement of the edges of the dodecahedron joining at vertex V1. All the three edges A, B, and C shown in fig. 9 are tangent to the intersphere at their midpoint.

Fig. 9. Details of vertex V1

Fig. 10 shows a plane P1 that contains element A and is tangent to the intersphere. The direction of the normal vector of plane P1 at point M1 will pass through the centre of the dodecahedron which is taken as the origin of the coordinate system.

Fig. 10. Detail of element A

To use the method of rotation, element A is rotated by an angle θ around point M1 in plane P1, as shown in fig. 11. The same procedure is repeated for elements B and C that are connected to V1, as shown in fig. 12.

Fig. 11. Rotation of element A in plane P1

Fig. 12. Vertex V1

The V1 ends of the three elements are now in positions denoted by V1', V1'' and V1'''. In order to represent the eccentricity between elements A and B, a line perpendicular to both lines has to be found. This line is the common perpendicular which is also the shortest distance between the two lines. The method to find the common perpendicular is explained in the next section.

Calculation of the eccentricity. The lines denoted as 'line A' and 'line B' in fig. 13 are meant to indicate the directions of elements A and B in fig. 12, respectively. Consider the points Q1 and Q2 lying, respectively, on the lines A and B as shown in fig. 13.

The line Q1Q2 is meant to represent the eccentricity between line A and B. Line Q1Q2 has to be perpendicular to both lines A and B, thus the vector product between the vector V1'V2' and V1''V3' will give the direction of the line Q1Q2.

$$\vec{N} = \vec{V1'V2'} \times \vec{V1''V3'} \quad (1)$$

Fig. 13. Lines A and B

Now, consider the plane $Q_1Q_2V_1'V_2'$, which is denoted by Π_1 , fig. 14. Π_1 contains $V_1'V_2'$ and Q_1Q_2 , which is parallel to the vector N . Point Q_2 is the intersection of plane Π_1 and Line B, as shown in fig. 14.

Figure 14. Location of point Q_2

Consider now the plane $Q_1Q_2V_1''V_3'$, which is denoted by Π_2 . Π_2 contains $V_1''V_3'$ and Q_1Q_2 , which is parallel to the vector N . Point Q_1 is the intersection of plane Π_2 and line A, as shown in fig. 15.

Fig. 15. Location of point Q_1

The eccentricity is then equal to the distance between points Q1 and Q2, as shown in fig. 16.

Fig. 16. The eccentricity

Originally the ends of element A were V1' and V2'. The ends of this element are now V2' and Q1. If one repeats the above procedure for all the elements of the vertex, the fan shown in fig. 17 will then be generated. The engagement length for each element can now be calculated. For instance, the engagement length of element A is the distance Q1Q6.

Fig. 17. Resulting fan

Fig. 18. Rendered dodecahedric nexorade

The procedure for vertex V1, as described above, may be repeated for all the vertices of the polyhedron, a 'dodecahedric nexorade' would then be generated.

Fig. 18 shows a rendered view of the resulting dodecahedric nexorade. The particular nexorade shown corresponds to an angle θ equal to 5° and the nexors have identical length, identical eccentricity and identical engagement length.

Fig 19 shows the variations of the eccentricity, the engagement length and the nexor length with respect to the angle θ . The horizontal axis represents the angle θ and the vertical axis represents the value of the parameters.

Fig. 19. Variations of the eccentricity, engagement length and nexor length with respect to the angle θ for a dodecahedron with a circumradius of 10 units

Figs. 20 – 28 show nexorades with an angle θ equal to 5°, 10°, 20°, 30°, 45°, 60°, 70°, 80° and 85° degrees, respectively.

Fig. 20

Fig. 21

Fig. 22

Fig. 23

Fig. 24

Fig. 25

Fig. 26

Fig. 27

Fig. 28

The nexorade shown in fig. 20 is referred to as a ‘dodecahedric nexorade’ with angle θ equal to 5° . The nexorade shown in fig. 28 is also referred to as a ‘dodecahedric nexorade’ with angle θ equal to 85° , but this nexorade resembles more an ‘icosahedric nexorade’ than a ‘dodecahedric nexorade’. This phenomenon is a consequence of the fact that the dodecahedron is the dual of the icosahedron. To elaborate, the dodecahedron and the icosahedron have 30 elements. The dodecahedron has 20 vertices whereas the icosahedron has 20 faces. The dodecahedron has 12 faces whereas the icosahedron has 12 vertices. The dual of the dodecahedron is found by connecting all the points situated at the centre of each face with the centre of the neighbouring faces.

Consider the ‘dodecahedric nexorade’ shown in fig. 20. The 3-nexor fans that constitute the nexorade have engagement windows that are rather like triangles. This nexorade has also openings that are rather like pentagons, these correspond to the faces of the ‘mother’ dodecahedron. The nexorade is then neither exactly a dodecahedron nor an icosahedron but it is something in between. When angle θ is increased the size of the triangles increase and the size of the pentagons decrease. When angle θ is equal to zero, the engagement windows have an area equal to zero, thus the engagement windows represent the vertices of a dodecahedron. When angle θ is 90° , a side of the engagement window is equal to the length of a nexor and thus the nexorade has only triangular ‘faces’, that are the engagement windows. The nexorade in this case is an icosahedron. This example shows the use of the concept of duality between the dodecahedron and the icosahedron.

In order to name this type of nexorade, a dodecahedron transformed into a nexorade with an engagement length smaller than half of the length of the nexors will be referred to as a ‘dodecahedric nexorade’ ($\theta < 36.28^\circ$), a dodecahedron transformed into a nexorade with an engagement length larger than half of the length of the nexors will be referred to as an ‘icosahedric nexorade’.

A similar relationship exists between the cube and the octahedron, which are also duals of each other. Fig. 29 shows the variations of the eccentricity, engagement length and nexor length of a ‘cubic nexorade’ with respect to the angle θ . The horizontal axis represents the angle θ and the vertical axis represents the value of the other parameters.

Fig. 29. Variations of the eccentricity, engagement length and nexor length with respect to the angle θ for a cube with a circumradius length of 10 units

Figs. 30 to 33 show nexorades with an angle θ equal to 10° , 25° , 50° and 80° , respectively. The nexorades with θ equal to 10° and 25° should be referred to as 'cubic nexorades' ($\theta < 41.93^\circ$). The nexorades with angle θ equal to 50° and 80° should be referred to as 'octahedric nexorades', as their engagement lengths are larger than half of their nexors' length.

Fig. 30

Fig. 31

Fig. 32

Fig. 33

The remaining regular polyhedron, the tetrahedron, has four faces and four vertices. This means that the tetrahedron will be the dual of itself. Fig. 34 shows the variations of the eccentricity, engagement length and nexor length of a 'tetrahedric nexorade' with

respect to the angle θ . The horizontal axis represents the angle θ and the vertical axis represents the value of the other parameters.

Fig. 34. Variations of the eccentricity, engagement length and nexor length with respect to the angle θ for a tetrahedron with a circumradius length of 10 units

Figs. 35 – 38 show ‘tetrahedric nexorades’ with an angle θ equal to 10° , 30° , 45° and 80° , respectively. The nexorades with angle θ equal to 10° and 80° are identical except that the style of their fans are different, that is, the one with $\theta = 10^\circ$ is the mirror of the one with $\theta = 80^\circ$. The arrangements that look rather like faces and the engagement windows of the nexorade shown in fig. 37 have the particular property of being of identical size. Note also that in this figure some nexors are parallel to one another and the engagement length is equal to half of the nexor length.

Fig. 35

Fig. 36

Fig. 37

Fig. 38

In conclusion to this study of the method of rotation applied to regular polyhedra, one may state that there exist only three regular polyhedric nexorades. The first of these nexorades is the 'tetrahedric nexorade'. The second nexorade represents a cube or an octahedron depending on whether its angle θ is equal to 0° or 90° . The last nexorade represents a dodecahedron or an icosahedron depending on whether its angle θ is equal to 0° or 90° .

4. Aspect ratio

The ratio between the eccentricity and the engagement length for a regular fan is referred to as the 'aspect ratio' of a fan. This ratio may be equivalently defined as the ratio between the diameter of the cross-section of a nexor in a regular fan and the engagement length of the fan. For a nexor, the term 'aspect ratio' implies the ratio between the diameter of its cross-section and its engagement length.

Consider the elementary configuration (which is a configuration with zero eccentricity and zero engagement length) shown in fig. 39a, where the vertices V2, V3, V4 lie in the xy plane and where elements V2V1, V3V1 and V4V1 have the same length and the same angle with the xy plane. This angle is the base angle α . If one projects the elements V2V1, V3V1 and V4V1 on the xy plane, the angles between the projected elements, denoted by ϕ , will be equal to 120° , as shown in fig. 39b.

Figure 39 a (above) and b (below) Elementary configuration

If one wants to transform the configuration into a nexorade by translating the elements, one should proceed as follows:

$$\frac{N_z}{\|\vec{N}\|} \cdot e = \lambda \cdot \frac{V2V1_z}{\|V2V1\|} \quad (2)$$

where:

- N is the vector product between V2V1 and V3V1;
- $V3V1_z$ is the z component of vector N ;
- e is the eccentricity;
- λ is the engagement length.

The ratio between the eccentricity and the engagement length is, therefore, given by

$$\frac{e}{\lambda} = \frac{V2V1_z}{\|V2V1\|} \cdot \frac{\|\vec{N}\|}{N_z} \quad (3)$$

Note that the values of the scalars $V2V1_z$ and N_z depend on the orientation in space of elements $V2V1$ and $V3V1$. These orientations can be expressed in terms of two variables, namely, the base angle α and the angle ϕ . It is assumed that the length of element $V2V1$ and $V3V1$ are equal to one unit length. The components of vectors $V2V1$ and $V3V1$ are as follows:

$$V2V1 = \begin{pmatrix} 0 \\ \cos(\alpha) \\ \sin(\alpha) \end{pmatrix} \quad V3V1 = \begin{pmatrix} -\cos(\alpha) \cdot \sin(\phi) \\ \cos(\alpha) \cdot \cos(\phi) \\ \sin(\alpha) \end{pmatrix}$$

Vector N , that is, the vector product of $V2V1$ and $V3V1$ is as follows:

$$\vec{N} = \begin{pmatrix} \cos(\alpha) \cdot \sin(\alpha) - \cos(\alpha) \cdot \sin(\alpha) \cdot \cos(\phi) \\ -\cos(\alpha) \cdot \sin(\alpha) \cdot \sin(\phi) \\ \cos^2(\alpha) \cdot \sin(\phi) \end{pmatrix}$$

Now, the values of the vectors are substituted in Equation (3).

The final equation is:

$$\frac{e}{\lambda} = \tan(a) \cdot \tan(b) \cdot \left(\left(\frac{1}{\sin(\phi)} - \frac{1}{\tan(\phi)} \right)^2 + \frac{1}{\sin^2(\alpha)} \right)^{\frac{1}{2}}$$

For the configuration shown in fig. 39, the value of ϕ is 120° and therefore the values of $\cos(\phi)$ and $\sin(\phi)$ are, respectively, equal to $\sqrt{3}/2$ and -0.5 . The values of $\cos(\phi)$ and $\sin(\phi)$ are re-introduced in the general equation

$$\frac{e}{\lambda} = \left(4 \tan^2(a) - 3 \sin^2(\alpha) \right)^{\frac{1}{2}}$$

Now, suppose that one wants to use the method of rotation for the creation of a regular polyhedric nexorade. For any angle θ , one can calculate the aspect ratio of a typical fan of a nexorade, by dividing the eccentricity by the engagement length. The variations of the aspect ratios for 'tetrahedric', 'cubic' and 'dodecahedric' nexorades are shown in fig. 40, where the horizontal axis represents the angle θ and the vertical axis represents the aspect ratio.

Fig. 40. Variations of the aspect ratios for 'tetrahedric', 'cubic' and 'dodecahedric' nexorades

Consider for instance a fan with an aspect ratio of 0.4. From the curve in fig. 40, it is found that this fan will have a base angle of 19.3° . From the curve in fig. 40, it is found that this fan can be used to create a 'tetrahedric', 'cubic' or 'dodecahedric' nexorade. This means that the nexorades with the aspect ratio of 0.4 for tetrahedric, cubic and dodecahedric nexorades can have fans with identical base angles. The θ angles required to build the fans of 'tetrahedric', 'cubic' and 'dodecahedric' nexorades are 65.5° , 55.5° and 23° , respectively. The only difference between these fans will be the engagement ratio, that is, the ratio between the engagement length and the length. The variations of the engagement ratios for 'tetrahedric', 'cubic' and 'dodecahedric' nexorades are shown in fig. 41, where the vertical axis represents the engagement ratio and the horizontal axis represents the angle θ .

Fig. 41. Variations of the engagement ratios for 'tetrahedric', 'cubic' and 'dodecahedric' nexorades

Consider again, the example where the aspect ratio is 0.4. The θ angles required to build the fans of 'tetrahedric', 'cubic' and 'dodecahedric' nexorades are 65.5° , 55.5° and 23° , respectively. The fans of these nexorades must have engagement ratios of 0.81, 0.67 and 0.34, respectively, as may be obtained from fig. 41.

Fig. 42 shows the fan with an aspect ratio of 0.4 applied to a tetrahedric, ‘cubic’ and ‘dodecahedric’ nexorade:

Fig. 42. View of a ‘tetrahedric’, ‘cubic’ and ‘dodecahedric’ nexorades with the same fan

Summary. ‘Tetrahedric’, ‘cubic’ and ‘dodecahedric’ nexorades with aspect ratio of 0.4 have the following particulars:

- Base angle of 19.3° ;
- Engagement ratios of 0.81, 0.67 and 0.34, respectively.

Note that if one considers that the thickness of the elements is equal to zero, a tetrahedron cube and dodecahedron will require a specific value of angle α . In contrast, tetrahedric, cubic and dodecahedric nexorade can be built with a range of value of angle α .

5. History

Nexorades consisting of a single fan have been known for centuries. During the medieval ages, the French architect Villard de Honnecourt [Bowie 1959] provided a solution for the problem of covering a space with beams shorter than the span. He used flat 4-nexor fans with connections of the type shown in fig. 43.

Fig. 43. Connections between nexors in a flat fan

Leonardo da Vinci was also interested in the concept [Leonardo da Vinci 1956]. He made sketches for beam arrangements similar to the ‘flat fan’ of Villard de Honnecourt. Another architect of the Renaissance period, Sebastiano Serlio, proposed designs, similar to that of Villard de Honnecourt, for spanning spaces with short beams [Serlio 1970]. Less than a century later, John Wallis wrote his *Opera Mathematica* in Latin [1972], where he described a number of flat nexor assemblies.

In recent times, Graham Brown obtained a patent for a roof system consisting of a single fan. Also, D. Gat patented an interesting system consisting of 4-nexor fans [Gat 1978].

Artists are also interested by the concept, recently artist Rinus Roelofs has produced interesting sculptures (see <http://www.rinusroelofs.nl>).

The nexorades that are discussed in the present paper are the spatial generalisations of the flat beam arrangements of John Wallis. However, the idea of spatial nexorades is not entirely new. For instance, it is known that Leonardo da Vinci had a clear understanding of the workings of the spatial nexorade-like arrangements. This is evidenced by the sketch he produced for a temporary bridge, shown in fig. 44.

Fig. 44. A design for a temporary bridge, suggested by Leonardo da Vinci

Leonardo's bridge design does not quite fit into the family of the nexorades, as described in this paper. However, there are certainly intriguing similarities.

6. Conclusions

This paper has presented a method to generate nexorades based on regular polyhedra. It has been shown that 'tetrahedric', 'cubic' and 'dodecahedric' nexorades can be built with fans identical aspect ratio in other words with the same base angle α .

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