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Research

# *Milankovitch's* Theorie der Druckkurven*:* *Good mechanics for masonry architecture*

**Abstract.** During the nineteenth century many studies on the theory of the thrust line were written in connection with the stability of masonry structures. However, a general treatment of the theory of the thrust line from both a mechanical and mathematical point of view may be found only in the contributions of the Serbian scholar Milutin Milankovitch, published between 1904 and 1910 and substantially unknown to the historians of mechanics applied to architecture. This paper aims at presenting Milankovitch's theory and discussing its improvements with respect to the previous literature on the subject.

## Why good mechanics for masonry architecture?

There are different reasons for putting this question with respect to Milutin Milankovitch's contribution on the theory of the line of thrust applied to masonry vaulted structures. From our modern point of view, the result of Heyman's lesson, the first reason concerns his methodological approach. When Milankovitch discussed his doctoral thesis, *Beitrag zur Theorie der Druckkurven* [1904], (Fig. 1) at the Technische Hochschule in Vienna in 1904 and then published the two papers in the *Zeitschrift für Mathematik und Physik* [1907a, 1910],[1] the application of elastic theory to masonry and stone structures was a well-established trend. Starting from the 1870s the old tradition of studies based on the model of the arch as a system of rigid and infinitely resistant voussoirs was rapidly abandoned and the "new theory" of the elastic arch became the official tool of the structural engineer, as well as for systems and materials that only with difficulty obeyed the classical hypotheses of the mathematical theory of elasticity. This new attitude, a product of the times as well, was the inevitable result of nineteenth-century progress in structural mechanics related to the great development of the general methods for the analysis of hyperstatic elastic systems.

Milankovitch takes his distance from this official trend. He obviously knows that the only rational way for solving a statically indeterminate system requires adopting the elastic approach, that is, using the complete set of equilibrium, compatibility and stress-strain equations in order to determine the actual state of the system. His purpose, however, is not to search for the actual state of the arch. His interest is focussed on a general theory of the equilibrium of masonry structures and for this aim there is no need to take the elastic approach. It is sufficient to assume some basic hypotheses concerning the mechanical behaviour of masonry materials and to state the equilibrium equations in accordance with them. Now, these hypotheses are essentially three: 1) masonry has no tensile strength; 2) its compressive strength is practically infinite in comparison with the stresses occurring in real structures; 3) friction is large enough to prevent sliding between masonry elements.


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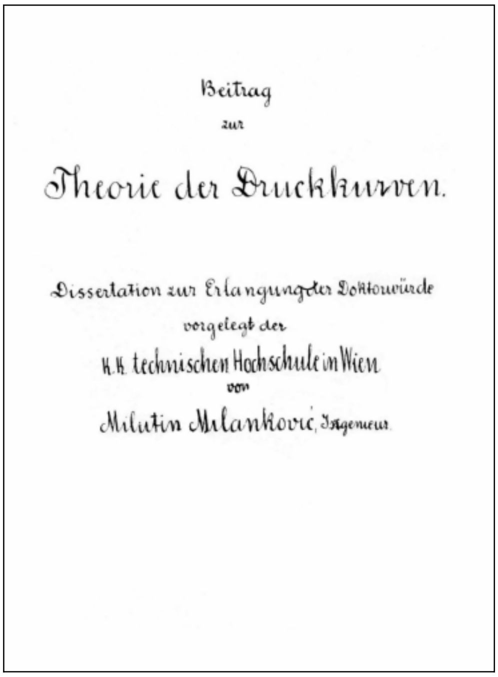

Beitrag
zur
Theorie der Druckkurven.

Dissertation zur Erlangung der Doktorwürde
vorgelegt der
K.K. technischen Hochschule in Wien
von
Milutin Milanković, Ingenieur

Fig. 1. Frontispiece of the manuscript of Milankovitch's doctoral thesis [1904]

Milankovitch tacitly adopts these assumptions, nowadays called Heyman's hypotheses after the first essay that the great Cambridge scholar devoted to the stone skeleton [1966], and for this reason his work on the theory of the line of thrust is of particular value from a methodological standpoint. He recognizes that the main point of masonry structures concerns the global stability, not the local stresses. In this sense, and in spite of the times, his contribution ideally belongs to the pre-elastic tradition[2] and anticipates the present view on the matter.

A second reason comes from the scientific quality of Milankovitch's analysis. As far as we know, his theory of the line of thrust is probably the most general discussion in the technical literature on the subject. Among the previous and later studies on this topic it is be difficult to find a similarly high standard from both a mathematical and mechanical point of view, as we shall see in the following sections. The author himself seems to be conscious of that and his criticisms towards some previous contributions are particularly noteworthy in this sense.

Finally, a third reason concerns the role of the historical research in the field of the mechanical sciences. In spite of the remarkable level of Milankovitch's contribution, it must be said that for many decades it remained totally unknown to the historians of structural mechanics.[3] There are reasons also for that, and the main one is probably connected with the singular scientific career of the Serbian scholar. It is true that Milankovitch attended the Technische Hochschule in Vienna (today Vienna University of Technology) where he graduated in Civil Engineering in 1902 and earned his doctorate in 1904. After discussing his thesis and publishing the two papers cited above, he also worked in the then-famous firm of Adolf Baron Pittel Betonbau-Unternehmung in Vienna, wrote some technical texts on reinforced concrete [1905, 1907b, 1908] and built dams, bridges, viaducts, aqueducts and other structures in reinforced concrete throughout the Austria-Hungary of the time. However, this initial career as a structural engineer came to an end in 1909 when he was offered the chair of applied mathematics (rational mechanics, celestial mechanics, theoretical physics) in Belgrade and decided to concentrate on fundamental research in geophysics. As a matter of fact, Milankovitch's name is best known for his theory of ice ages, relating variations of the Earth's orbit and long-term climate change, now known as Milankovitch cycles.[4] It is clear that, in comparison with his main scientific production, the first studies on the line of thrust are forgotten episodes. The task of the historian, whatever his field of research, is to rediscover what time forgets and to select, among the many mediocre episodes, the few good ones. This is the case with Milankovitch's forgotten *Theorie der Druckkurven*.

## *The* mathematische Stilisierung *of the mechanical problem of the arch*

We have already underlined the general character of Milankovitch's analysis. This character comes out of his correct *mathematische Stilisierung* – we could say mathematical modelling – of the mechanical problem concerning the equilibrium of an infinitesimal voussoir of the arch. The main steps of this analysis concern:

### 1) Location of the centre of mass of an infinitesimal voussoir

A first important point for a correct *Stilisierung* concerns the position of the centre of mass of an infinitesimal voussoir with respect to the middle point of the joints. In order to determine this position Milankovitch considers a finite voussoir $NN'N_nN_n'$, bounded by the joints $NN'$ and $N_nN_n'$, with centre of mass at point $S$ (fig. 2).

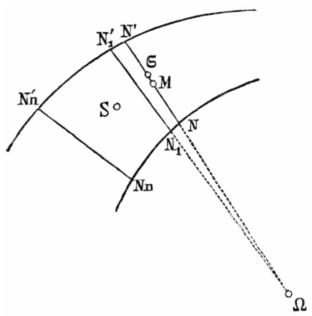


Fig. 2.

By moving the joint $N_nN_n'$ to the position $N_1N_1'$ infinitely close to the joint $NN'$, the point $S$ moves to a limit position $G$, intentionally drawn by Milankovitch on the line of the joint $NN'$ and in this sense called *Scherwpunkt der Fuge* $NN'$ (centre of mass of the joint).[5] The distance of this point $G$ with respect to the middle point $M$ of the joint $NN'$ can be easily determined. Let $\overline{\Omega M} = \rho$ be the distance between $M$ and the point of intersection $\Omega$ of the two joint lines $NN'$ and $N_1N_1'$, $\overline{NN'} = \delta$ the thickness of the joint $NN'$, so that $\overline{NM} = \delta/2$. Moreover, let $df_1$ and $df_2$ be the areas of the infinitesimal triangles $\Omega NN'$ and $\Omega NN_1'$, whose centres of mass have the distances $\frac{2}{3}(\rho + \delta/2)$ and $\frac{2}{3}(\rho - \delta/2)$ from the point $\Omega$, respectively. Then, the following equality holds:

$$(df_1 - df_2)\overline{\Omega G} = \tfrac{2}{3}(\rho + \delta/2)\,df_1 - \tfrac{2}{3}(\rho - \delta/2)\,df_2 \ .$$

Moreover, the ratio between the areas $df_1$ and $df_2$ is proportional to the ratio of the squares of the distances $\frac{2}{3}(\rho+\delta/2)$ and $\frac{2}{3}(\rho-\delta/2)$, so that

$$df_1 : df_2 = (\rho+\delta/2)^2 : (\rho-\delta/2)^2 .$$

From the two previous relations, it follows that

$$\overline{\Omega G} = \rho + \frac{\delta^2}{12\rho} \tag{1}$$

that is, the centre of mass $G$ of the infinitesimal voussoir has the finite distance $\overline{MG} = \dfrac{\delta^2}{12\rho}$ from the middle point $M$ of the joint $NN'$. This is the general case. The particular condition $\overline{MG} = 0$ is true only in the case of parallel joints ($\rho = \infty$) or in the case of vanishing thickness ($\delta = 0$).

## 2) General typology of geometry and load condition

The general feature of Milankovitch's approach concerns also the geometry of the arch and the load condition. As shown in fig. 3, the intrados and extrados lines are generic continuous (regular) functions, so that his analysis takes into account the case of variable thickness of the arch ring.

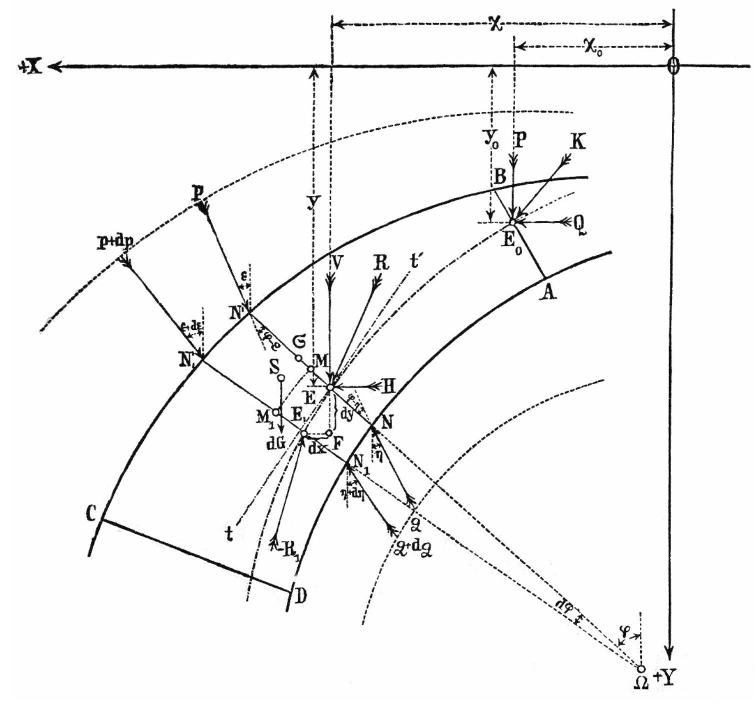


Fig. 3

Further, besides the self weight the load condition is represented by generic continuous (regular) functions $p$ at the extrados and $q$ at the intrados, with variable inclination measured by the angles $\varepsilon$ and $\eta$ with the $y$-axis. Finally, the direction of the joints is not necessarily perpendicular to the axis of the vault.

3) **Correct formulation of the rotational equilibrium equation of the infinitesimal voussoir**

On the basis of the statement concerning the position of the centre of mass $G$, Milankovitch derives the rotational equilibrium equation of the infinitesimal voussoir $NN' N_1N_1'$. By supposing that the line of thrust goes through the points $E$ and $E_1$ at the joints $NN'$ and $N_1N_1'$, and taking into account the self weight $dG$ of the voussoir and the forces $pde$, $qdi$ acting on the extrados and intrados elements $de$ and $di$ respectively, the equilibrium equation about $E_1$ has the form:

$$Vdx - Hdy + M_g + M_e + M_i = 0 \tag{2}$$

where $V$ and $H$ are the finite vertical and horizontal components of the resultant force $R$ at the joint $NN'$ with infinitesimal lever arms $dx$ and $dy$ about $E_1$, so that the first two terms are infinitesimal quantities of first order; the other terms $M_g$, $M_e$, $M_i$ represent the moments of the infinitesimal forces $dG$, $pde$, $qdi$ about the same point $E_1$, respectively. As far as their values Milankovitch correctly observes that, because of the infinitesimal distance between $E$ and $E_1$ and of the finite distance of the application points of $dG$, $pde$, $qdi$ from both $E$ and $E_1$, the three moments $M_g$, $M_e$, $M_i$ are also infinitesimal quantities of first order and the lever arms with respect to $E_1$ may be taken equal to the lever arms with respect to $E$. Thus the moment $M_g$ is given by:

$$M_g = -dG\left(\overline{MG} + \overline{ME}\right)\sin\varphi = -g\beta\delta\left(\frac{\delta^2}{12\rho} + \xi\right)\sin\varphi\,\rho\,d\varphi \ , \tag{3}$$

where $\overline{ME} = \xi$ represents the eccentricity of the resultant $R$ at the joint $NN'$, $g$ is the specific weight of the masonry, $\beta$ is the depth of the vault and $\rho\,d\varphi = d\sigma$ is the length of the arc $MM_1$, so that $dG = g\beta\delta\,\rho\,d\varphi$. The other two moments $M_e$ and $M_i$ are, respectively

$$M_e = -(pde)\overline{N'E}\sin(\varphi - \varepsilon) = -p\left(\frac{\delta}{2} + \xi\right)\sin(\varphi - \varepsilon)de \tag{4}$$

$$M_i = -(qdi)\overline{NE}\sin(\varphi - \eta) = -q\left(\frac{\delta}{2} - \xi\right)\sin(\varphi - \eta)di \ , \tag{5}$$

where $\varepsilon$ and $\eta$ are the angles formed by the direction of the load p at $N'$ and the load $q$ at $N$ with the vertical line, as shown in fig. 3.

By introducing these expressions in the equilibrium equation Milankovitch finds

$$(6)\quad \begin{aligned} &Vdx - Hdy - g\beta\delta\,\rho\left(\frac{\delta^2}{12\rho} + \xi\right)\sin\varphi\,d\varphi - p\left(\frac{\delta}{2} + \xi\right)\sin(\varphi - \varepsilon)de \\ &- q\left(\frac{\delta}{2} - \xi\right)\sin(\varphi - \eta)di = 0 \end{aligned}$$

Given certain boundary conditions, for instance that for $x = x_0$ it is $y = y_0$, $V = P$ and $H = Q$ , then the values of the components $V$ and $H$ at the joint $NN'$ become

$$(7)\quad V = P + g\beta \int_{x_0}^{x} \delta\,\rho\,d\varphi + \int_{x_0}^{x} p\cos\varepsilon\,de - \int_{x_0}^{x} q\cos\eta\,di\,,$$

$$(8)\quad H = Q - \int_{x_0}^{x} p\sin\varepsilon\,de + \int_{x_0}^{x} q\sin\eta\,di$$

Finally, by differentiating with respect to $x$ and dividing by $H$, Milankovitch obtains the general equation

$$(9)\quad \frac{V}{H} - \frac{dy}{dx} = \frac{1}{H}\left[g\beta\delta\rho\left(\frac{\delta^2}{12\rho} + \xi\right)\sin\varphi\frac{d\varphi}{dx} + p\left(\frac{\delta}{2} + \xi\right)\sin(\varphi - \varepsilon)\frac{de}{dx} + q\left(\frac{\delta}{2} - \xi\right)\sin(\varphi - \eta)\frac{di}{dx}\right].$$

4) Clear distinction between the direction of the resultant and the tangent to the line of thrust

From the previous equation (9) Milankovitch derives the important conclusion that the direction of the resultant force $R$ at point $E$ does not coincide with the direction of the tangent straight line $tt'$ to the line of thrust at the same point $E$. As a matter of fact, the ratio $\frac{V}{H} = \tan\psi$ is the trigonometric tangent of the angle $\psi$ between the force $R$ and the $x$-axis, while the ratio $\frac{dy}{dx} = \tan\alpha$ is the trigonometric tangent of the angle $\alpha$ between the line $tt'$ and the same $x$-axis. In the general case these angles are different as the second member of the equation (9) is different from zero and Milankovitch observes that at different joints it may be $\psi > \alpha$ , $\psi = \alpha$ , $\psi < \alpha$ , that is the direction of the resultant force may intersect the tangent to the line of thrust.

This result is not completely original. In Milankovitch's analysis, however, it takes a new light and becomes the basis for a critical review of previous studies on the line of thrust where the equation

$$(10)\quad \frac{V}{H} - \frac{dy}{dx} = 0$$

has been wrongly assumed as invariably true.

## *From the general to the particular: Application of the theory to particular cases of masonry vaults*

Starting from the general equation (9) Milankovitch discusses the following particular cases of geometry and load condition.

### 1) *Die Gewölbe* (the vault)

The term *Gewölbe* is used by Milankovitch when the system is subject only to its own weight and a vertical load at the extrados. Thus, for the *Gewölbe* he takes $q$=0 and $\varepsilon$=0. Moreover he assumes that the joints are perpendicular to the axis of the vault, so that $\overline{\Omega M} = \rho$ represents the curvature radius of the axis at point $M$ and $\varphi$ is equal to the angle between the tangent line to the axis of the vault and the $x$-axis. Under these assumptions the equations (9), (7) and (8) become

(11) $$\frac{V}{H} - \frac{dy}{dx} = \frac{1}{H}\left\{ g\beta\delta\left(\frac{\delta^2}{12\rho} + \xi\right)\sin\varphi\,\frac{d\sigma}{dx} + p\left(\frac{\delta}{2} + \xi\right)\sin\varphi\,\frac{de}{dx}\right\},$$

(12) $$V = P + g\beta\int_{x_0}^{x}\delta\,d\sigma + \int_{x_0}^{x} p\,de,$$

(13) $$H = Q\,.$$

### 2) *Die Gewölbe gleichen Widerstandes* (the vault of equal resistance)

The vaults of equal resistance fulfil the following two requirements for a given load condition: 1) The equation of the axis of the vault is a particular integral of the differential equation of the line of thrust; 2) The thickness $\delta$ of the joints is proportional to the normal component $\overline{N}$ of the resultant force $R$.

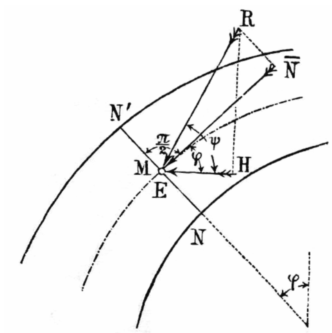


Fig. 4

From the first requirement it follows that the line of thrust may be taken as the one coinciding with the vault axis, that is $\xi = 0$. As a consequence, each joint is subject to constant normal stresses at any point and the following equalities hold (fig. 4)

$$\tan\varphi = \frac{dy}{dx} = \tan\alpha \qquad \varphi = \alpha \qquad d\sigma = ds = \frac{dx}{\cos\varphi} \ ,$$

so that equation (11) becomes

$$(14) \qquad \frac{V}{H} - \frac{dy}{dx} = \frac{1}{H}\left\{\frac{1}{12} g\beta \frac{\delta^3}{\rho} \tan\varphi + \frac{1}{2} p\delta \sin\varphi \frac{de}{dx}\right\}$$

while (12) and (13) do not change.

The second requirement implies $\delta = k\overline{N}$, where $k$ is a constant. As a consequence, the normal stresses are equal at any joint. As $R = \frac{H}{\cos\psi}$ and $\overline{N} = H\frac{\cos(\psi - \varphi)}{\cos\psi}$, the thickness varies in accordance with the following law

$$(15) \qquad \delta = kH\frac{\cos(\psi - \varphi)}{\cos\psi} \ .$$

3) ***Die Stützlinie***

The term *Stützlinie* is used by Milankovitch when both the loads and the joints are vertical, that is when $\varepsilon = 0$, $\eta = 0$ and $\varphi = 0$. Under these assumptions the equation (9) becomes

$$(16) \qquad \frac{V}{H} - \frac{dy}{dx} = 0$$

that is,

$$(17) \qquad \psi = \alpha \ .$$

Thus the *Stützlinie* is the line of thrust for which the direction of the resultant force $R$ coincides with the direction of the tangent $tt'$. As the loads are assumed to act vertically, a single function $\omega = f(x)$ may be chosen to represent both the self weight and the extrados and intrados load, so that the result is:

$$(18) \qquad V = P + \int_{x_0}^{x} \omega \, dx$$

$$(19) \qquad H = Q$$

and then

$$(20) \qquad \frac{dy}{dx} = \frac{1}{Q}\left\{P + \int_{x_0}^{x} \omega \, dx\right\} .$$

By differentiating with respect to $x$ it follows

$$(21) \qquad \frac{d^2 y}{dx^2} = \frac{\omega}{Q} \ ,$$

which is the second order differential equation of the *Stützlinie*.

By integrating the equation (20) and considering that for $x = x_0$ it is $y = y_0$, there follows the explicit equation of the *Stützlinie* :

$$(22) \qquad y = y_0 + \frac{1}{Q}\int_{x_0}^{x} dx \int_{x_0}^{x} \omega\, dx + \frac{P}{Q}(x - x_0).$$

This equation shows that, where $\omega = 0$ , the *Stützlinie* becomes the straight line

$$(23) \qquad y = y_0 + \frac{P}{Q}(x - x_0)$$

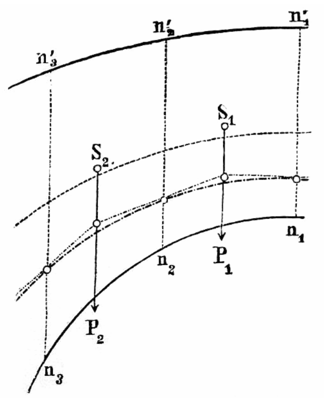


Fig. 5

Then, if the actual load distribution is divided in vertical stripes by the lines $n_1n_1'$, $n_2n_2'$... and substituted with the resultant forces $P_1$, $P_2$ ... applied at the centres of mass $S_1$, $S_2$ ... of each stripe (fig. 5), it follows that the *Stützlinie* becomes a *Stützpolygon* whose vertices lie on the direction of the loads $P_1$, $P_2$, ... . At the vertical lines $n_1n_1'$, $n_2n_2'$, ... the *Stützpolygon* is tangent to the actual *Stützlinie* so that it circumscribes the line of thrust.

4) ***Die Kettenlinie***

The term *Kettenlinie* is adopted by Milankovitch for the case of a vault with vanishing thickness, that is $\delta = 0$ . In this case the vault becomes a perfectly flexible chain and the equilibrium is possible only if the line of thrust coincides with the chain itself, that is if $\xi = 0$ . Thus equation (9) becomes, taking into account that for this limit case it is $de = di = d\sigma = ds$ ,

$$(24) \qquad \frac{V}{H} - \frac{dy}{dx} = 0.$$

If $g_x$ represents the specific weight of the chain, it follows:

$$(25) \qquad V = P + \int_{x_0}^{x} g_x ds + \int_{x_0}^{x} p \cos\varepsilon\, ds - \int_{x_0}^{x} q \cos\eta\, ds$$

(26) $$H = Q - \int_{x_0}^{x} p \sin \varepsilon \, ds + \int_{x_0}^{x} q \sin \eta \, ds \, .$$

If all the loads act vertically, then the *Stützlinie* coincides with the *Kettenlinie*.

5) Searching for the *mathematische Stärke* (mathematical thickness) in the case of the semicircular arch under its own weight

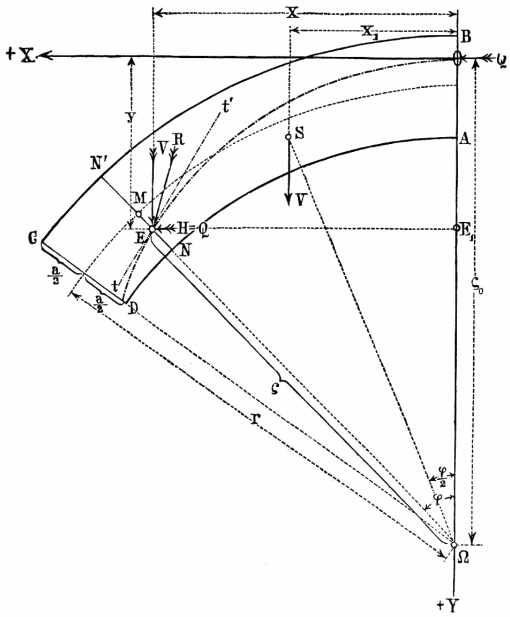

Fig. 6

A detailed application of the general theory is given by Milankovitch for the case of a circular arch of constant thickness $\delta = a$ subject to its own weight (fig. 6). In this case the *Gewölbe* has an axis with constant radius of curvature $\rho = r$ .

For the symmetry of the system the resultant force at the crown joint *AB* has only the horizontal component *Q* applied at a certain point *O* with distance $\rho_0$ from the centre of curvature $\Omega$ . Thus, by taking the origin of the reference axes at the same point *O*, it follows that, for $x = 0$, it is $y = 0$, $V = P = 0$ and $H = Q$.

The equations (11), (12) and (13) for the *Gewölbe* become, taking $p = 0$ and assuming $\beta = 1$ and $g = 1$ for the sake of simplicity,

(27) $$\frac{V}{H} - \frac{dy}{dx} = \frac{1}{H} ar \left( \frac{1}{12} \frac{a^2}{r} + \xi \right) \sin \varphi \frac{d\varphi}{dx} \quad ,$$

(28) $$V = ar\int_0^{\varphi} d\varphi = ar\varphi\ ,$$[6]

(29) $$H = Q$$

By introducing the polar coordinates $\overline{\Omega E} = \rho$ and $O\hat{\Omega}E = \varphi,$ then $\xi = r - \rho$, $y = \rho_0 - \rho\cos\varphi$ and $x = \rho\sin\varphi$, so that

(30) $$dy = -\cos\varphi\, d\rho + \rho\sin\varphi\, d\varphi,$$

(31) $$dx = \sin\varphi\, d\rho + \rho\cos\varphi\, d\varphi.$$

Thus equation (1) becomes

(32) $$\rho(ar\varphi\cos\varphi\, d\varphi + ar\sin\varphi d\varphi - Q\sin\varphi\, d\varphi) + (ar\varphi\sin\varphi + Q\cos\varphi)d\rho\frac{1}{12}a\left(a^2 + 12r^2\right)\sin\varphi\, d\varphi\ .$$

Milankovitch shows that this is an exact differential equation. Its integral is

(33) $$\rho(ar\varphi\sin\varphi + Q\cos\varphi) = -\frac{1}{12}a\left(a^2 + 12r^2\right)\cos\varphi + C$$

where $C$ is a constant. As for $\varphi = 0$ it is $\rho = \rho_0$, the constant $C$ results

(34) $$C = \rho_0 Q + \frac{1}{12}a\left(a^2 + 12r^2\right).$$

Finally, by substituting this value in (33) and observing that $(1-\cos\varphi) = \sin^2\frac{\varphi}{2}$, Milankovitch finds the explicit equation of the line of thrust

(35) $$\rho = \frac{\rho_0 Q + \frac{1}{6}a\left(a^2 + 12r^2\right)\sin^2\frac{\varphi}{2}}{Q\cos\varphi + ar\varphi\sin\varphi},$$

As Milankovitch observes at the end of the first paper, this equation could be directly obtained without deducing the differential equation and then integrating it. This occurs when it is possible to find the analytical expression of the resultant load and its points of application for a finite portion of the system. For instance, if $S$ is the centre of mass of the finite portion $ABNN'$, $x_1$ its abscissa and $V$ the weight of $ABNN'$, from the rotational equilibrium about $E$, it follows

$$V(x - x_1) - Qy = 0\ .$$

For a circular arch of constant thickness $\delta = a$ it is known that

$$\overline{\Omega S} = \tfrac{1}{6}\frac{a^2 + 12r^2}{r}\frac{\sin\frac{\varphi}{2}}{\varphi}$$

and then

$$x_1 = \overline{\Omega S} \sin \tfrac{\varphi}{2} = \tfrac{1}{6} \frac{a^2 + 12r^2}{r} \frac{\sin^2 \frac{\varphi}{2}}{\varphi} \quad .$$

Moreover, it is also known that $V = ar\varphi$ , $x = \rho \sin \varphi$ and $y = \rho_0 - \rho \cos \varphi$ , so that by introducing the previous formulas in the equilibrium equation it follows again (35)

$$\rho = \frac{\rho_0 Q + \frac{1}{6} a\left(a^2 + 12r^2\right)\sin^2 \frac{\varphi}{2}}{Q \cos \varphi + ar\varphi \sin \varphi} \quad .$$

This explicit equation of the line of thrust still contains two constants to be determined, that is the distance $\rho_0$ and the horizontal thrust *Q*. Milankovitch applies it to search for the mathematical thickness of a semicircular arch under its own weight, that is, the value of the thickness corresponding to the unique admissible line of thrust wholly lying within the arch ring, as shown in fig. 7 .

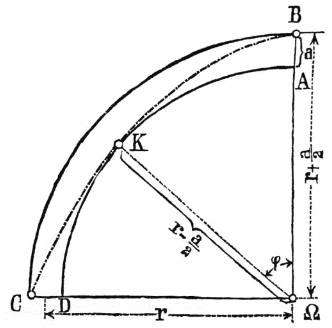


Fig. 7

This is a case already studied in the literature[7] for which it is known that the line of thrust goes through the extrados points *B* at the crown and *C* at the springing. Thus the following two boundary conditions must be fulfilled:

(36) $\rho(0) = r + \dfrac{a}{2}$ , which gives $\rho_0 = r + \dfrac{a}{2}$

(37) $\rho\left(\frac{\pi}{2}\right) = r + \dfrac{a}{2}$ , which gives $Q = \dfrac{3\pi ar\,(a + 2r) - a\left(a^2 + 12r^2\right)}{3a + 12r}$

so that the equation (35) becomes

(38) $$\rho(\varphi) = \frac{3\pi\, r(a + 2r)^2 - (a + 2r)\left(a^2 + 12r^2\right)\cos \varphi}{6\pi\, r(a + 2r)\cos \varphi - 2(a^2 + 12r^2)\cos \varphi + 12r(a + 2r)\varphi \sin \varphi} .$$

Now, the search for the minimum thickness requires that the line of thrust touches the intrados at a certain point *K* corresponding to the unknown rupture angle $A\hat{\Omega}K$ . This means that the minimum value of the function $\rho(\varphi)$ must be equal to the intrados radius. Thus, where $\frac{\partial \rho}{\partial \varphi} = 0$ it must be $\rho = r - \frac{a}{2}$.

After some mathematical elaborations Milankovitch finds that the rupture angle is $A\hat{\Omega}K = 54°29'$ [8] and the minimum thickness is $a = 0{,}1075r$.

### 6) The vault of equal resistance with vertical joints under its own weight

Another special application of the general theory concerns the search for the line of thrust of a vault of equal resistance with vertical joints, subject to its own weight. This case requires that the *Stützlinie* of case 3 fulfills the two requirements of case 2. By taking the origin of the axes at the middle point of the crown joint, for $x = 0$ it must be $y = 0$, $P = 0$, $H = Q$. From the equations (16), (18) and (19) it results

(39) $$\frac{V}{H} - \frac{dy}{dx} = 0$$

(40) $$V = \int_{x_0}^{x} \delta \, dx$$

(41) $$H = Q = N \quad ,$$

so that

(42) $$\delta = kN = kQ$$

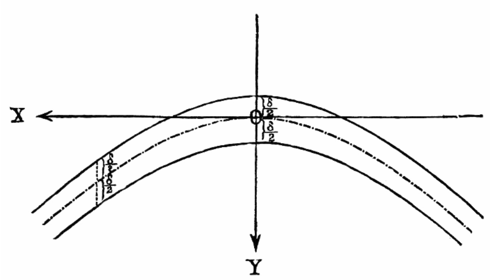

Fig. 8

where $k$ is a constant. By substituting the values of $V$ and $H$ in the differential equation it follows

(43) $$k \int_{x_0}^{x} dx = kx = \frac{dy}{dx} \quad .$$

Then, by integrating and taking into account that for $x = 0$ it is $y = 0$, the result is:

(44) $$y = \frac{1}{2} kx^2 .$$ [9]

This equation shows that the axis of the vault is a parabola. Then the intrados and extrados lines are also parabolas running at a vertical distance $\pm \delta/2$ from the axis, as shown in fig. 8.

### 7) Design of the external profile of a retaining wall

Another interesting application of the theory is developed by Milankovitch at the end of his thesis for the design of a retaining wall subject to its own weight, a load $K$ at the top with components $P$ and $Q$ and the lateral pressure of water (fig. 9). Given the internal vertical wall, the purpose is to determine the external profile $BN'C$ in order that the line of thrust is the locus of the external points of the middle third of each horizontal joint, as in this limit case the joints are still subject only to compressive stresses.

By taking the origin of the reference axes at point $A$ and assuming that the line of thrust has equation of the type

(45) $$x = f(y),$$

then the equation of the external profile $BN'C$ is

(46) $$x' = \tfrac{3}{2} f(y) \ .$$

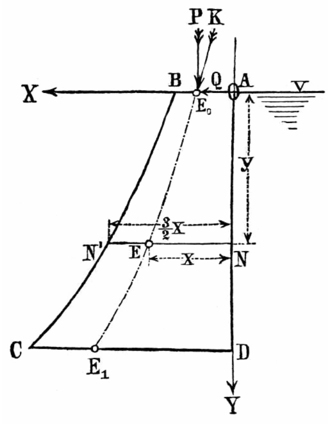


Fig. 9

The general equations (5), (7) and (8) must now be written for the special geometry of the system and the particular load condition under the requirement stated above. This means that $\delta = \frac{3}{2}x$, $\rho = \infty$ (parallel joints), $\rho d\varphi = dy$, $\overline{ME} = \xi = \frac{3}{4}x - x = -\frac{1}{4}x$ (eccentricity of the centre of pressure), $\varphi = \frac{\pi}{2}$ (horizontal joints), $p = 0$ (no extrados load), $q = y$ (internal water pressure, for specific weight equal to one), $\eta = \frac{\pi}{2}$ (horizontal internal load, perpendicular to the internal wall), $di = dy$, $y_0 = 0$. Moreover, by indicating with $g$ the specific weight of masonry and taking the depth $\beta = 1$, the equations (5), (7) and (8) become

(47) $$Vdx - Hdy + \tfrac{3}{8} g x^2 dy = 0$$

(48) $$V = P + \tfrac{3}{2} g \int_0^y x dy$$

(49) $$H = Q + \int_0^y y dy = Q + \tfrac{1}{2} y^2$$

By introducing the formula of $V$ and $H$ and differentiating with respect to x, it follows the differential equation of the line of thrust

$$(50)\qquad \left(8Q-3gx^2+4y^2\right)\frac{d^2y}{dx^2}+8y\left(\frac{dy}{dx}\right)^2-18gx\frac{dy}{dx}=0\ .$$

Milankovitch observes that the general integral of this equation cannot be determined. However, a particular integral can be found for $x_0=0$, $P=0$ and $Q=0$, that is when the thickness and the load at the top are zero. In this case it results

$$(51)\qquad y=\tfrac{3}{2}\sqrt{g}x\ ,$$

so that the external profile of the wall is

$$(52)\qquad y=\sqrt{g}x\ .$$

This equation shows that the wall becomes the right-angled triangle $ADC$ (fig. 10) for which results $\tan C\hat{A}D=1/\sqrt{g}$ .

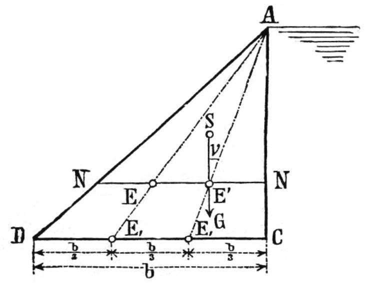


Fig. 10

The line of thrust is the straight line $AE_1$, where $E_1$ is the left point of the middle third at the base $DC$ of the wall. In the absence of the internal pressure of water, the centre of pressure at each joint coincides with the right point of the middle third so that the line of thrust is the straight line $AE_1^{'}$.

## *Milankovitch's critical review of previous studies*

The previous report shows the mastery of the young Milankovitch in the mathematical treatment of the equilibrium analysis of the arch. On this theoretical basis he also develops some critical remarks that show his mature scientific approach.

To put the problems of applied mechanics in mathematical terms, he says, the scientist has to introduce some hypotheses concerning, for instance, the mechanical behaviour of the materials and the action of the loads. These hypotheses do not fit completely with the reality but they are necessary if the scientist does not want to renounce the use of mathematical tools. Obviously, the assumption of these hypotheses produces a solution whose reliability is directly connected with the reliability of the hypotheses. In this sense this solution is surely approximate but not wrong from a theoretical point of view.

Besides this sort of approximation, a different and more serious type may derive from theoretical mistakes in the *mathematische Stilisierung* of the problem. In particular, Milankovitch recognizes two sources of mistakes invalidating the correctness of a theory: 1) when, by putting the problem in mathematical terms, some circumstances are not correctly taken into account; 2) when the mathematical treatment of a problem correctly modelled is unconsciously wrong.

Having in mind a discussion of some previous contributions to the theory of the line of thrust, Milankovitch observes that the first source of mistakes occurs when the weight *dG* of the infinitesimal voussoir is supposed to be applied at the middle point *M* of the joint instead of at the centre of mass *G*, that is when it is wrongly assumed $\overline{MG}=0$ instead of $\overline{MG}=\dfrac{\delta^2}{12\rho}$. Now, the condition $\overline{MG}=0$ holds only if $\rho=\infty$ (case of parallel joints) or if $\delta=0$ (case of vanishing thickness). Even though the distance $\overline{MG}$ is usually very little, the assumption $\overline{MG}=0$ cannot be accepted in a good mechanical analysis.

The second source of mistake occurs when the three moments $M_g$, $M_e$ and $M_i$ are considered as infinitesimals of second order and then neglected in the rotational equilibrium equation. The consequence of this mistake is that the equilibrium equation becomes invariably $\dfrac{V}{H}-\dfrac{dy}{dx}=0$, so that it would always be $\psi=\alpha$. Now, the condition $\psi=\alpha$, which states the coincidence of the direction of the resultant force R with the tangent straight line $tt'$ to the thrust line, holds only if both the loads and the joints are vertical (case of the *Stützlinie*) or if $\delta=0$ and the loads acts vertically (case of the *Kettenlinie* under vertical loads). Milankovitch quotes some authors who have erroneously assumed this particular condition even when it is not valid. For instance, Hagen [1846, 1862] takes it for granted in the case of the vault of equal resistance. Similarly, J. Résal [1901] affirms that the coincidence of the line of thrust with the axis of the vault implies the relation $\dfrac{V}{dy}=\dfrac{H}{dx}$. In his deduction of the equation of the line of thrust for an arch subject to its own weight, H.A. Résal [1889, vol. 6, § 238] explicitly writes that the moment $M_g$ is infinitesimal of second order and obtains the wrong result $\dfrac{dy}{dx}=\dfrac{V}{H}$.

Milankovitch also remarks that the first authors who have dealt with the theory of the line of thrust do not committed this sort of mistake. In this sense he quotes Moseley,[10] who clearly states that the direction of the resultant force at a generic joint cuts the line of thrust (called by Moseley line of resistance) and introduces the notion of envelope of the directions of the resultant forces (called by Moseley line of pressure). Milankovitch also cites Dupuit [1870], who gives a correct analytical discussion of the theory of the line of thrust for the special case of a vault under its own weight. In this list we can add also the name of Méry [1840], who gives an analytical proof of the difference between the angular coefficients of the direction of the resultant force and the direction of the tangent to the line of thrust.

### *Application of the theory to the design of masonry buttresses*

In the second paper of 1910 Milankovitch devotes particular attention to the search for the *theoretisch günstigen Formen* of masonry buttresses, that is, for the optimal design under given geometric and static requirements. He considers the general case of a buttress subject to its own weight and a load $K$ applied at a given point $E_0$ of the top section (fig. 11).

Supposing that the joints are horizontal, as usually occurs for masonry buttresses, and that the line of thrust is the curve $E_0EE_1$, he follows the reasoning already used for the vault and derives the rotational equilibrium equation of the infinitesimal element $NN'N_1N_1'$ about the point $E_1$. If $V$ and $H$ are the vertical and horizontal components of the resultant $R$ at the joint $NN'$ and $dG$ is the weight of the voussoir $NN'N_1N_1'$, this equation has the form

$$V\,dx - H\,dy + M_g = 0 \tag{53}$$

where $dx$ and $dy$ are the infinitesimal lever arms of the finite forces $V$ and $H$ about $E_1$, and $M_g$ represents the moment of the infinitesimal weight $dG$ whose finite arm with respect to $E_1$ may be taken equal to the arm with respect to $E$ because of the infinitesimal distance between $E$ and $E_1$. Thus the three moments in equation (53) are all infinitesimal quantities of first order.

Starting from equation (53), Milankovitch searches for the optimal shape of the buttress in order to assure that, for given geometry and load condition, the joints are subject only to compressive stresses, as it should be in masonry structures. In particular, he studies three cases of specific interest.

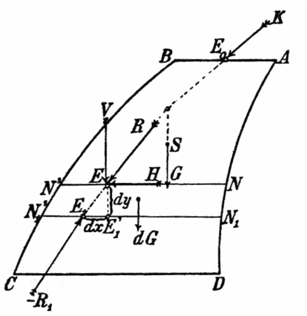

Fig. 11

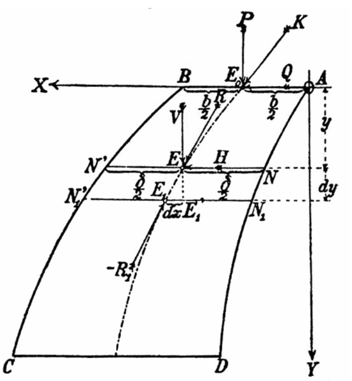

Fig. 12. Profile of a buttress of equal resistance subject to its own weight and a load applied at the top section

#### 1) Design of the buttress of equal resistance

As in the case of the vault, the design of the buttress of equal resistance (fig. 12) fulfils two requirements: 1) the line of thrust must coincide with the locus of the middle points at each horizontal joint; 2) The thickness of the joints must be proportional to the normal component $V$ of the resultant force $R$.

Thus equation (2) becomes

(54) $$Vdx - Hdy = 0$$

as the moment $M_g$ is an infinitesimal of second order and may be neglected. Moreover, for the second requirement it must be

(55) $$\delta = kV$$

where $k$ is a constant.

If $P$ and $Q$ are the vertical and horizontal components of the force $K$ applied at the middle point $E_0$ of the top section, it results, taking into account that $Q$ is the only horizontal force on the buttress and recalling equation (55),

(56) $$H = Q$$

(57) $$V = \frac{\delta}{k} = P + g\int_0^y \delta\, dy$$

where g is the specific weight of the masonry.

Differentiating equation (5) with respect to y and separating the variables, it follows that

(58) $$\frac{d\delta}{\delta} = kgdy \quad .$$

Thus, by integrating under the boundary condition that for $y = 0$ it is $\delta = b$, the following law for the thickness is obtained

(59) $$\delta = be^{kgy} \quad .$$

Now from equations (54), (56), (57) and (59) it comes

(60) $$Q\frac{dy}{dx} = P + gb\int_0^y e^{kgy}\, dy = P + \frac{b}{k}e^{kgy} \quad ,$$

so that

(61) $$x = kQ\int \frac{dy}{kP + be^{kgy}} + C \ .$$

Milankovitch shows that the function under the integral sign may be put in the form

(62) $$\frac{1}{kP + be^{kgy}} = \frac{1}{kP}\frac{kP + be^{kgy} - be^{kgy}}{kP + be^{kgy}} = \frac{1}{kP}\left\{1 - \frac{be^{kgy}}{kP + be^{kgy}}\right\} = \frac{1}{k^2 gP}\left\{kg - \frac{kgbe^{kgy}}{kP + be^{kgy}}\right\}$$

and obtains

(63) $$x = \frac{Q}{kgP}\left\{kgy - \log_{nat}\left(kP + be^{kgy}\right)\right\} + C$$

As for $y = 0$ it must be $x = \frac{b}{2}$, the constant $C$ becomes

$$C = \frac{b}{2} + \frac{Q}{kgP}\log_{nat}\left(kP + b\right)$$

so that the middle line of the buttress of equal resistance, which coincides with the line of thrust, has the equation

(64) $$x = \frac{b}{2} + \frac{Q}{kgP}\left\{kgy - \log_{nat}\frac{kP + be^{kgy}}{kP - b}\right\} .$$

In the particular case for which $Q = 0$, from equation (64) it follows that the middle line becomes the vertical line $x = \frac{b}{2}$, while the law of the thickness is always given by (59).

### 2) Design of a buttress with vertical internal line, when the line of thrust goes through the middle point of the joints

A second application concerns the design of a buttress with vertical internal line, under the requirement that the line of thrust coincide with the locus of the middle points of the horizontal joints (fig. 13).

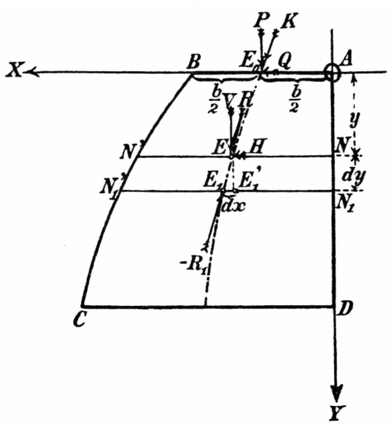


Fig. 13. Profile of a buttress with vertical internal line, when the line of thrust coincides with the axis of the buttress

Also in this case the rotational equilibrium equation results

(65) $$Vdx - Qdy = 0$$

as the term $M_g$ is infinitesimal of second order. The vertical component $V$ at the generic joint is

(66) $$V = P + 2g\int_0^y x\,dy$$

so that

(67) $$\frac{dy}{dx} = \frac{V}{Q} = \frac{P}{Q} + \frac{2g}{Q}\int_0^y x\,dy\,.$$

Differentiating with respect to x it follows

(68) $$\frac{d^2 y}{dx^2} = \frac{2g}{Q}x\frac{dy}{dx}$$

Thus, by integrating Milankovitch finds

(69) $$\frac{dy}{dx} = Ce^{\frac{g}{Q}x^2}$$

where $C$ is a constant to be determined. Now, as the equation $\frac{dy}{dx} = \frac{V}{Q}$ states that the resultant force at each joint is tangent to the line of thrust, it must be for the top section:

(70) $$\left.\frac{dy}{dx}\right\}_{x=\frac{b}{2};\; y=0} = \frac{P}{Q} = Ce^{\frac{g}{Q}\frac{b^2}{4}}\,,$$

so that

(71) $$C = \frac{P}{Q}e^{-\frac{g}{Q}\frac{b^2}{4}}$$

Thus equation (69) becomes

(72) $$\frac{dy}{dx} = \frac{P}{Q}e^{-\frac{g}{Q}\frac{b^2}{4}}e^{\frac{g}{Q}x^2}$$

and its integral results

(73) $$y = \frac{P}{Q}e^{\frac{g}{Q}\frac{b^2}{4}}e^{\frac{g}{Q}x^2}\int e^{\frac{g}{Q}x^2}\,dx$$

This is the equation of the line of thrust. To calculate the integral contained in this equation Milankovitch refers to the method suggested by Stieltjes [1886].

### 3) Design of a buttress with a vertical internal line, when the line of thrust goes through the external points of the middle third

A third application deals with the design of a buttress with a vertical internal line, under the requirement that the line of thrust coincide with the locus of the external points of the middle third of the horizontal joints (fig. 14).

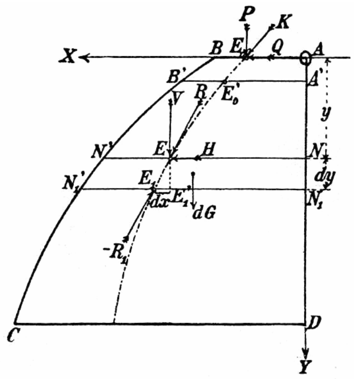


Fig. 14. Profile of a buttress with a vertical internal line, when the line of thrust goes through the external points of the middle third

In this case the term $M_g$ is infinitesimal of first order as the moment of the weight $dG$ has a finite lever arm with respect to $E_1$, and this arm may be taken equal to the distance of $dG$ from $E$, that is $\frac{x}{4}$.

Thus the moment $M_g$ is

(74) $$M_g = \frac{x}{4} dG = \frac{x}{4}\frac{3}{2} xgdy = \frac{3}{8} gx^2 dy \quad ,$$

and the rotational equilibrium equation results

(75) $$Vdx - Qdy + \frac{3}{8} gx^2 dy = 0 \ .$$

The vertical component $V$ at the generic joint is

(76) $$V = P + \frac{3}{2} g \int_0^x xdy \ ,$$

so that the equilibrium equation becomes

(77) $$P + \frac{3}{2} g \int_0^x xdy - Q \frac{dy}{dx} + \frac{3}{8} gx^2 \frac{dy}{dx} = 0 \ .$$

Differentiating with respect to $x$ it follows

(78) $$\left(\frac{3}{8}gx^2 - Q\right)\frac{d^2y}{dx^2} + \frac{9}{4}gx\frac{dy}{dx} = 0$$

and then

(79) $$\frac{\frac{d^2y}{dx^2}dx}{\frac{dy}{dx}} = -\frac{18gx}{3gx^2 - 8Q}dx \ .$$

Now, by integrating Milankovitch finds

(80) $$\frac{dy}{dx} = C\left(3gx^2 - 8Q\right)^{-3}$$

The equation (75) for the rotational equilibrium of the infinitesimal voussoir about $E'_0$ gives

(81) $$Pdx - Qdy + \frac{3}{8}gx_0^2dy = 0 \ .$$

Thus, taking into account (80), for the top section it must be

(82) $$\left.\frac{dy}{dx}\right\}_{x=x_0;y=0} = C\left(3gx_0^2 - 8Q\right)^{-3} = -\frac{8P}{3gx_0^2 - 8Q}$$

so that

(83) $$C = -8P\left(3gx_0^2 - 8Q\right)^2 \ .$$

By replacing the value of $C$ equation (80) becomes

(84) $$\frac{dy}{dx} = -8P\ \left(3gx_0^2 - 8Q\right)^2\left(3gx^2 - 8Q\right)^{-3}$$

and its integral results, by putting $\frac{8Q}{3g} = k^2$,

(85) $$y = -\frac{8}{3}\frac{P}{g}\left(x_0^2 - k^2\right)^2\int\frac{dx}{\left(x^2 - k^2\right)^3} \ .$$

Milankovitch observes that the function under the integral sign can be written as the following sum:

(86)
$$\frac{1}{\left(x^2 - k^2\right)^3} = \frac{1}{8k^3}\frac{1}{(x-k)^3} - \frac{3}{16k^4}\frac{1}{(x-k)^2} + \frac{3}{16k^5}\frac{1}{x-k} - \frac{1}{8k^3}\frac{1}{(x+k)^3} - \frac{3}{16k^4}\frac{1}{(x+k)^2} - \frac{3}{16k^5}\frac{1}{x+k} \ .$$

Thus, taking into account that if $x = x_0$, then $y = 0$, Milankovitch obtains the equation of the line of thrust $E_0EE_1E_1^{'}$

$$(87) \qquad y=\frac{2}{3}\frac{P}{k^2g}\left(x_0^2-k^2\right)^2\left\{\begin{array}{l}\dfrac{x}{\left(x^2-k^2\right)^2}-\dfrac{x_0}{\left(x_0^2-k^2\right)^2}-\dfrac{3}{2k^2}\left[\dfrac{x}{x^2-k^2}-\dfrac{x_0}{x_0^2-k^2}\right]-\\ -\dfrac{3}{4k^3}\log_{nat}\dfrac{(x-k)(x_0+k)}{(x+k)(x_0-k)}\end{array}\right\}$$

If $P = 0$, that is if at the top section acts only the horizontal component $Q$ (fig.15), then the function $y$ is defined only for $x_0 = k = \sqrt{\frac{8Q}{3g}}$. In this case the line of thrust becomes the vertical line with equation $x = \overline{AE_0} = \sqrt{\frac{8Q}{3g}}$ and consequently the external profile $BC$ is also vertical and has equation $x = \overline{AB} = \frac{3}{2}\overline{AE_0} = \sqrt{\frac{6Q}{g}}$.

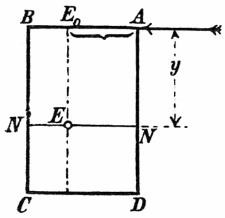


Fig. 15

As Milankovitch observes, a remarkable property of this case is that the resultant force at the top section is perpendicular to the line of thrust.

## *Notes*

1. [Milankovitch 1907] reproduces the content of the doctoral thesis, with few revisions and some different notations. [Milankovitch 1910] is an application of the general theory of the thrust line for the optimal design of masonry buttresses.
2. The rediscovery of this mechanical tradition starts with the first historical investigation by Jacques Heyman [1972]. Significant historical studies on the theory of the arch have been given by Edoardo Benvenuto [1981, 1991]. Special attention to the role of geometry and equilibrium in the history of masonry vaulted structures is given by Santiago Huerta [2001, 2004]. A historical analysis of the pre-elastic methods is contained in Federico Foce [2002, 2005].
3. As far as we know, the first discussion of Milankovitch's two published papers appears in [Huerta 1990]. Another discussion has been given by the author of the present paper [Ageno, et. al, 2004].
4. For more details on Milankovitch (b. 28 May 1879 in Dalj near Osijek, (Austria-Hungary) – d. 12 December 1958 in Belgrade), see the many websites on his scientific work.
5. [Milankovitch 1904, 2; 1907, 3]. Obviously, the centre of mass $G$ of the infinitesimal voussoir does not lie on the joint line and in this sense the term *Schwerpunkt der Fuge* is rather improper. However, as we shall see in the following discussion where the weight $dG$ of the voussoir is correctly applied at the true centre of mass $S$ (see fig. 3), this choice is taken by

Milankovitch just to point out that *G*, projection of *S* on the joint line, has the same finite distance from the middle point *M*.

6. In the first printed paper [1907a], but not in the manuscript of the thesis, this formula is erroneously written $V = ar\int_0^{\varphi} d\varphi = \mathrm{arc}\varphi$ .
7. On this point Milankovitch quotes Ritter [1899] and Pilgrim [1877].
8. Milankovitch adds that for the rupture angle Ritter has given 54°10′ and Pilgrim 54°27′. Other authors, not quoted by Milankovitch, have obtained quantitative results on the minimum thickness of a semicircular arch under its own weight, even though by different methods of analysis. For instance: Couplet, who for the minimum thickness gives 0.1061 of the intrados radius by taking the a priori value of 45° for the rupture angle [Couplet 1732]; Monasterio, who for the thickness gives the range 0.111-0.125 of the intrados radius and for the rupture angle the range 54-56° [Monasterio, ca.1800]; Petit, who obtains the good approximation 0.114 of the intrados radius with rupture angle at 54° [Petit 1835].
9. Milankovitch writes $y = kx^2$ , perhaps including the factor ½ in the constant *k*.
10. Milankovitch refers to the German translation by Scheffler of [Moseley 1845].

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## About the author

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