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Keywords: curve fitting,
design analysis, Antoni Gaudí,
linear algebra, generalised
inverses, least squares,
hyperboloid, catenary, parabola,
golden number

Research

Curve Fitting in Architecture

Abstract. It used to be popular to draw geometric figures on images of paintings or buildings, and to propose them as an “analysis” of the observed work, but the tradition lost some credit due to exaggerated (golden section) interpretations. So, how sure can an art or mathematics teacher be when he wants to propose the profile of a nuclear plant as an example of a hyperboloid, or proportions in paintings as an illustration of the presence of number series? Or, if Gaudí intended to show chain curves in his work, can the naked eye actually notice the difference between these curves and parabolas? The present paper suggests applying the “least squares method”, developed in celestial mechanics and since applied in various fields, to art and architecture, especially since modern software makes computational difficulties nonexistent. Some prefer jumping immediately to modern computer machinery for visual recognition, but such mathematical overkill may turn artistic minds further away from the (beloved!) tradition of geometric interpretations.

Introduction

Until about twenty years ago, it was common to draw all kinds of geometric figures on images of artworks and buildings. Usually, simple triangles, rectangles, pentagons, or circles sufficed, but sometimes more general mathematical figures were used, especially after fractals became trendy. Recognizing well-known curves and polygons was seen as a part of the “interpretation” of an architectural edifice or painting. Eventually, segments of lines occurred in certain proportions, among which the golden section surely was the most (in)famous. Diehards continue this tradition, though curve drawing has lost some credit in recent times, in particular due to some exaggerated interpretations of the golden section.

Today, journals dealing with the nexus of mathematics and architecture tend to reject these “geometric readings in architecture”. Their point of view may be justified by several shortcomings of common research in the field:

- i. An architect may have had the intention of constructing a certain curve or surface, and even discuss these intentions in his plans, but for structural, technical or various practical reasons, the final realization may not conform to that intention.

Example: a nuclear plant is said to have the shape of a hyperboloid, but engineers modify its top to reduce wind resistance. So, how close to a hyperbola is the final silhouette of the structure?

- ii. An artist may have used a certain proportion, consciously or unconsciously, so that when such a “hidden” proportion or curve is discovered, even the author of the artwork may dispute its use.

Example: It happens that rational subdivisions are discovered in impressionist paintings. Some artists claim they intended only pure emotion, disliking art-school-like subdivisions such as the golden section. However, some critics can still demonstrate the presence (or absence) of this proportion using arguments similar to widely accepted interpretations of other artwork.

- iii. In the absence of accurate written sources about a work of art or an architectural design, how does one decide whether an interpretation is correct or not? Is it merely a matter of a subjective interpretation, a posteriori?

Example: in a discussion about pointed arches, how does one decide if an arch was pointed or not, without being misled by cultural penchants?

Despite this unenthusiastic introduction, this author still believes there could be positive reasons for continuing the study of geometric views on art and architecture:

- i. An engineer may want to reveal structural properties of the building.

Example: the difference between a parabola or chain curve when a cable supports a horizontal bridge, or only its own weight.

- ii. A mathematician may want to express what he finds beautiful in his preferred mathematical language and there is no reason why only literary minds should enjoy this privilege.

Example: Horta was inspired by spirals he found in plants. These spirals have a precise mathematical description, and though it is not certain that Horta intended to represent these curves, the mathematical interpretation may show a relationship with other sources of inspiration by other architects. If another architect was inspired by a nautilus shell, then both, in fact, had a similar inspiration, at least, as far as the mathematician is concerned.

- iii. An artist or architect may have been inspired by a curve or a proportion he finds elegant.

Example: well-defined proportions may have been used to subdivide a canvas or to construct buildings. This can help to collect information on how the work was done, and, for instance, to determine the unit length used (non-standard foot, cubit).

In the present paper, a similarity is proposed between these geometric studies in architecture and the history of (celestial) mechanics. It is suggested that the so-called “least squares method” developed in that field could be applied to examples in art as well. Of course, it can be argued that, unlike celestial mechanics with its involved applications, such a mathematical method would represent serious overkill with respect to the intended straightforward artistic applications. However, if this was true in the past, modern software allows slimming down computational aspects to some simple computer clicks.

Campbell and Meyer’s account on curve fitting

A well-known example in the history of curve-fitting is the story of how Carl Friedrich Gauss found a “lost planet”. Campbell and Meyer [1979] gave the following excellent account illustrating curve fitting, in the context of celestial mechanics and linear algebra.

In January 1801, astronomer G. Piazzi briefly observed a “new planet”, Ceres, and astronomers tried, in vain, to relocate it for the rest of 1801. Only Gauss could correctly

predict when and where to look for the lost planet but because he waited until 1809 to publish his theory, some accused him of sorcery.

For the sake of exposition, Campbell and Meyer simplified the problem to an elliptical orbit of which four observations were made: $(-1,1)$, $(-1,2)$, $(0,2)$, $(1,1)$. In this example, obviously, there must have been some errors, of whatever kind (inaccurate observations or instruments, or mere bad luck), since two data correspond to -1. Still, the question now is to find the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

approximating the four data as closely as possible.

Fig. 1. Campbell and Meyer's example of finding the ellipse closest to four data points

Putting $b_1 = 1/a^2$ and $b_2 = 1/b^2$, implies the four data points (x_i, y_i) , $i = 1, \dots, 4$, should make the error $x_i^2 b_1 + y_i^2 b_2 - 1$ minimal. In matrix notation, it means $\mathbf{X} \cdot \mathbf{b} - \mathbf{j}$ should be minimal, where:

$$\mathbf{X} = \begin{bmatrix} x_1^2 & y_1^2 \\ x_2^2 & y_2^2 \\ x_3^2 & y_3^2 \\ x_4^2 & y_4^2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \mathbf{j} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

In the general case, it turns out $\mathbf{X} \cdot \mathbf{b} - \mathbf{j}$ admits a unique least squares solution of minimal norm $\mathbf{b} = \mathbf{X}^+ \mathbf{j}$, while the precision of fit can be measured by comparing the image, $\mathbf{X} \mathbf{b}$, to the desired value $\mathbf{j}$. Thus, the fraction $R^2 = \|\mathbf{X} \mathbf{b}\|^2 / \|\mathbf{j}\|^2$ is used as a quantity giving an idea about the accuracy of the proposed approximation. The notation $\mathbf{X}^+$ represents the so-called "generalized inverse" in the sense of Moore-Penrose, which corresponds to the regular inverse if it exists (that is, when $\mathbf{X}$ is invertible, and thus, when an exact solution can be computed). We will not bother here about explaining computational aspects of this $\mathbf{X}^+$ when $\mathbf{X}$ is not invertible, since mathematical software such as MATHEMATICATM, allows computing this generalization of the notion of the inverse, called "pseudo inverse", without effort.

In the given example, $\mathbf{X} = \begin{bmatrix} 1 & 1 \\ 1 & 4 \\ 0 & 4 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \mathbf{X}^+ \mathbf{j} = \begin{bmatrix} 7 \\ 11 \\ 2 \\ 11 \end{bmatrix}$ and $\mathbf{R}^2 \approx 0.932$.

The set-up can be modified to find, for instance, the n^{th} degree polynomial

$$y = b_0 + b_1x + b_2x^2 + b_3x^3 + \dots + b_nx^n$$

which best fits a set of given points (x_i, y_i) . In this case, the expression $\mathbf{X} \cdot \mathbf{b} - \mathbf{j}$ should be minimal, and the involved matrices are:

$$\mathbf{X} = \begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^n \\ 1 & x_2 & x_2^2 & \dots & x_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & x_m^2 & \dots & x_m^n \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \mathbf{j} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Again, $\mathbf{X} \cdot \mathbf{b} - \mathbf{j}$ admits a unique “least squares solution of minimal norm” $\mathbf{b} = \mathbf{X}^+ \mathbf{j}$ while the precision of fit can be measured by $\mathbf{R}^2 = \|\mathbf{X} \mathbf{b}\|^2 / \|\mathbf{j}\|^2$.

Application 1: a chain curve approximates Gaudi’s Paelle Guell better than a parabola

Gaudi is known for his use of hyperbolic cosine functions. He used bags suspended by ropes which he inverted to get chain curves. However, the question remains whether or not a non-informed spectator can actually see this in Gaudi’s work, that is, if a spectator can tell that a Gaudi gate, entrance or window (as for example in the Paelle Guell, fig. 2a) has a hyperbolic cosine shape. Couldn’t he conclude that the observed curves are, for instance, parabolas?

Fig. 2a

I proposed this question during a course at the Sint-Lucas Institute for Architecture (Belgium), and student Sil Goossens came up with the following example, for which he

concluded the parabola with equation $y = -0.811x^2 + 5.704x + 2.643$ came pretty close, but not close enough. He claimed “the Gaudi entrance should be considered as a chain curve”.

Fig. 2b. Gaudi's Paelle Guell gate in Barcelona, some measurements made by a student, and a parabola fitting well, but not well enough

How can this idea be substantiated? First, checking the coordinates, they look fairly correct, but we make a change of coordinates to situate the top in $(0, 1)$, since this is a more straightforward choice when a hyperbolic cosine is expected. In addition, the x-axis is scaled down by 10% so that all $x < 1$, as this can make the use of series easier (though it will later turn out to be useless). A MATHEMATICA™ input provided the following :

In:

```
x1=-.4; x2=-.36; x3=-.31; x4=-.235; x5=-.185; x6=.105; x7=0; x8=.09; x9=.16;
x10=.2; x11=.29; x12=.34; x13=.4;
j={-8, -4, -2, -.25, .25, .7, 1, .7, .15, -.4, -2.55, -4.45, -8};
X={{1,x1,x1^2},{1,x2,x2^2},{1,x3,x3^2},{1,x4,x4^2},{1,x5,x5^2},{1,x6,x6^2},{1,x7,x
7^2},{1,x8,x8^2},{1,x9,x9^2},{1,x10,x10^2},{1,x11,x11^2},{1,x12,x12^2},{1,x13,x
13^2}};
B=PseudoInverse[X].j
Norm[X.B]^2/Norm[j]^2
```

Out:

```
{1.74011,-1.24714,-53.9541}
0.96035
```

that is, a parabola with equation $y = 1.74 - 0.125x - 54x^2$. It fits well, since $R^2 = 96\%$. We can try to fit a hyperbolic cosine, using:

```
X={{1,Cosh[x1] }, ..., {1,Cosh[x13]}};
B=PseudoInverse[X].j
Norm[X.B]^2/ Norm[j]^2
```

Now: $y = 107.5 - 105.8 \cdot \text{Cosh}[x]$, and it fits at 95.3%. Not only do we wonder if this 1% of difference could be noticed, but further, the result seems to contradict the student's conclusion. Looking carefully at the picture, we see the photo is slightly skew with respect to the observer, and because of this lack of symmetry, the proposed hyperbolic cosine will never give a nice fit, of course. Thus, we can try to take into account the imperfections in the picture, and propose a symmetric interpretation, estimating the second coordinates by modifying the given values slightly:

$x_{13} = -x_1; x_{12} = -x_2; x_{11} = -x_3; x_{10} = -x_4; x_9 = -x_5; x_8 = -x_6;$
 $j = \{-7.5, -4.4, -2.55, -4.25, .75, 1, .75, .25, -.4, -2.55, -4.4, -7.5\};$
 $X = \{ \{1, x_1, x_1^2, x_1^3, x_1^4, x_1^5, x_1^6\}, \dots, \{ \dots, x_{13}^6 \} \};$

This produced the following outcome:

$$y = 1.019 - 6.115 \cdot 10^{-16} x - 20.95 x^2 - 1.10 \cdot 10^{-13} x^3 - 60.7 x^4 - 9.9 \cdot 10^{-13} x^5 - 8.66.96 x^6$$

fitting at 99.875%. If we keep in mind the series for $\text{Cosh}(x)$:

$$2 - \text{Cosh}(x) = 2 - \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} \dots \right) = 1 - \frac{x^2}{2!} - \frac{x^4}{4!} - \frac{x^6}{6!} \dots$$

we see that:

$$y \approx 1 - 21x^2 - 61x^4 - 867x^6 \approx 1 - 21 \cdot \frac{x^2}{2!} - 1454 \cdot \frac{x^4}{4!} - 867 \cdot \frac{x^6}{6!} \approx 1 - \frac{(6.48x)^2}{2!} - \frac{(6.19x)^4}{4!} - \frac{(9.3x)^6}{6!}$$

Thus, we should try a different hyperbolic cosine, using a $\text{Cosh}[a \cdot x]$ expression, where the coefficient a is somewhere in the 6.19 or 9.3 range. Now $y = 1.34 - 0.36 \cdot \text{Cosh}[9.7x]$ fits at 99.88% and this indeed beats the closest possible parabola, which is $y = 1.84 - 52.12x^2$, fitting at only 96.75%. This 3% difference in closeness of fit can indeed be noticed, as the illustration shows.

Fig. 3. The catenary and parabola compared to the initial picture. Data were changed putting the picture in a frontal position: only the left half is to be considered.

Application 2: Gaudi's Collegio Teresiano can be seen either as a catenary or as a parabola

The previous result was nevertheless contradicted by another architecture student, Klaas Vandenberghe, who claimed some of Gaudi's work shows no hyperbolic cosines, but parabolas. His examples were the parallel arcs of the Collegio Teresiano. Again, the teacher was asked to set things straight and to decide who was right.

Fig. 4: The original picture and the obtained catenary and parabola, which are so close they cannot be distinguished

The procedure is similar: we measure the data as well as possible on the provided picture, and use the pseudo-inverse algorithm. We start with a 6th degree polynomial:

$$y = 1.026 - 3.33 \cdot 10^{-15} x + 5.3x^2 + 5.3 \cdot 10^{-14} x^3 + 29.5x^4 + 1.81 \cdot 10^{-13} x^5 - 100.9x^6,$$

for which the fit is $R^2 = 99.99\%$. Thus:

$$y \approx 1 + 5.3x^2 + 29.5x^4 + 100.9x^6 \approx 1 + 5.3 \cdot \frac{x^2}{2!} + 29.5 \cdot \frac{x^4}{4!} + 100.9 \cdot \frac{x^6}{6!} \dots$$

$$\approx 1 + \frac{(3.2x)^2}{2!} + \frac{(5.1x)^4}{4!} + \frac{(6.4x)^6}{6!} \dots$$

This again reminds us of the hyperbolic cosine series, and

$$y = -0.7468 + 1.75 \cdot \text{Cosh}[2.8x]$$

gives a 99.988% fit. A parabola can fit nicely too, as $y = 0.985 + 7.63x^2$, shows, with a 99.985% fit. This difference of 0.003% cannot be distinguished with the naked eye.

Application 3: The profile of a nuclear power plant is a hyperbola, or even, an ellipse

Fig. 5. Determining the profile of a nuclear plant

Consider fig. 5, from which the following values were drawn:

$$x_1 = 2.7; x_2 = 2.3; x_3 = 2; x_4 = 1.7; x_5 = 1.47; x_6 = 1.39; x_7 = 1.45;$$

$$y_1 = 0; y_2 = 1; y_3 = 2; y_4 = 3; y_5 = 4; y_6 = 5; y_7 = 6.$$

We compute the general conic section with equation

$$x^2b_1 + y^2b_2 + xb_3 + yb_4 + xyb_5 - 1 = 0$$

The minimal norm least squares solution follows from:

```
X={{x1^2,y1^2,x1,y1,x1*y1},{x2^2,y2^2,x2,y2,x2*y2},{x3^2,y3^2,x3,y3,x3*y3},{x4^2,y4^2,x4,y4,x4*y4},{x5^2,y5^2,x5,y5,x5*y5},{x6^2,y6^2,x6,y6,x6*y6},{x7^2,y7^2,x7,y7,x7*y7}};
j={1,1,1,1,1,1,1};
B=PseudoInverse[X].j
Norm[XB]^2/Norm[j]^2
```

It turns out to be

$$-0.11x^2 - 0.0105y^2 + 0.67x + 0.19y - 0.06xy = 1 \text{ at } 99.9996\%.$$

Surprisingly, this is an ellipse. Still, if we lift the x-axis over 5 units, we can propose the standard form equation for a hyperbola: $x^2b_1 + y^2b_2 = 1$, where b_2 should be negative. Now:

In:

```
x1=2.7; x2=2.3; x3=2; x4=1.7; x5=1.47; x6=1.39; x7=1.45;
y1=-5; y2=-4; y3=-3; y4=-2; y5=-1; y6=0; y7=1;
X={{x1^2,y1^2},{x2^2,y2^2},{x3^2,y3^2},{x4^2,y4^2},{x5^2,y5^2},{x6^2,y6^2},{x7^2,y7^2}};
j={1,1,1,1,1,1,1};
PseudoInverse[X].j
Norm[X.PseudoInverse[X].j]^2/Norm[j]^2
```

Out

```
{0.508687, -0.108291}
0.998771
```

This is indeed a hyperbola, which comes as close as 99.8%. Thus, we can faithfully claim that the shape of a nuclear plant is a hyperbola, and that the adaptations on the top of the building, for reasons of resistance to wind, can hardly be detected.

Fig. 6. The profile of a nuclear plant, and the approximating shapes, as an ellipse (middle) and a hyperbola (right)

Application 4: golden rectangles

Golden section “interpretations” of the Mona Lisa are popular and the influence of *The da Vinci Code* surely did not diminish its appeal. Even Wasler’s excellent book [2001] presents a strange inexplicable shift in the rectangles covering the body, so we’ll concentrate on the facial proportions, which seem more observable.

Fig. 7. The cover of Walser’s book (left), and a set of data points for the Mona Lisa (right)

Pointing the cursor on the scanned image, we get numbers that have rather useless and far too many decimals places, but it takes more time and energy to judge about their utility than to copy them from the computer output:

```
x1 = 17.496; y1 = 4.871; x2 = 19.129; y2 = 7.143; x3 = 19.200; y3 = 10.551;
x4 = 20.549; y4 = 13.107; x5 = 21.259; y5 = 16.160; x6 = 21.046; y6 = 19.426;
x7 = 20.549; y7 = 22.337; x8 = 20.052; y8 = 25.248; x9 = 18.064; y9 = 27.804;
x10 = 16.573; y10 = 30.644; x11 = 14.372; y11 = 32.490; x12 = 12.313; y12 = 33.413;
x13 = 10.396; y13 = 33.910; x14 = 8.195; y14 = 33.839; x15 = 5.852; y15 = 33.697;
x16 = 3.722; y16 = 33.058; x17 = 1.947; y17 = 31.993; x18 = .740; y18 = 29.792;
x19 = 0.456; y19 = 27.378; x20 = .243; y20 = 24.822; x21 = 0.243; y21 = 12.397;
x22 = 0.811; y22 = 9.912; x23 = 1.450; y23 = 7.427; x24 = 2.657; y24 = 5.013;
x25 = 4.361; y25 = 3.380; x26 = 5.710; y26 = 1.889; x27 = 7.688; y27 = 0.571;
x28 = 10.224; y28 = 0.571; x29 = 13.805; y29 = 1.541;
X = {{x1^2, y1^2, x1, y1, x1*y1}, ... {x29^2, y29^2, x29, y29, x29*y29}};
j = {1, ... 1};
```

PseudoInverse[X].j

Norm[X.PseudoInverse[X].j^2/Norm[j])^2

The result is the ellipse

$$- 0.0078x^2 - 0.0033y^2 + 0.1686x + 0.1241y - 0.0009xy = 1$$

fitting at $R^2 = 99.55\%$.

Now the minimum corresponds to $x=10.8697\dots$, while the maximum corresponds to $8.99537\dots$, and the difference in y -values is 34.014 . The (horizontal) width is $20.9757\dots$

$(-1.110\dots) = 22.09$, and the proportion is $N[34.014/22.09] = 1.54\dots$. This is closer to, say, 1.5, than to the golden number of 1.618...

Fig. 8. An ellipse that fits to 99.55%, showing that the vertical to horizontal ratio is 1.54

Conclusion

The least-squares method, known from celestial mechanics and since applied in the most various areas, could well be used for geometric interpretations in art as well, especially since modern software makes the computational difficulties nonexistent. One can only wonder why such a widespread and straightforward mathematical technique has not been applied before. Today, some prefer jumping immediately to modern computer machinery in visual recognition of patterns, but such mathematical overkill is not necessary and can even turn artistic minds away from the (beloved!) topic of geometric interpretations in architecture.

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About the author

Dirk Huylebrouck obtained his Ph.D. in algebra at the University of Ghent. He worked in Africa for about twelve years, at the Universities of Congo (Bukavu, Kinshasa) and Burundi, interrupted by assignments at the Portuguese University of Aveiro and at the Overseas Divisions of American universities in Europe (such as Maryland and Boston University). Since about ten years, he teaches at the Department for Architecture Sint-Lucas Brussels (Belgium). His research on generalized inverses was awarded by a quotation in the book "Current Trends in Matrix Theory 1987", and his study of $\zeta(3)$ the "Lester Ford Award 2002" of the Mathematical Association of America. As editor of the "Mathematical Tourist" column in *The Mathematical Intelligencer*, he paid attention to historical

and artistic aspects of mathematics from Tibet over Africa and Europe to Siberia. A mathematical artifact from Central Africa inspired him to write a book, from which he presented many talks, publications and TV-collaborations. He currently tours around Belgium in a multicultural mathematics show together with the percussion trio “Dakar Electric”.