Michael Leyton

A Generative Theory of Shape

New York: Springer-Verlag, 2001

Reviewed by Stephen R. Wassell

In *A Generative Theory of Shape* Michael Leyton presents an extremely rich and compelling theory with foundations in and applications to art and architecture, music, math, physics, psychology, computer science, mechanical design and manufacturing, and CAD. As is standard, shapes are viewed as originating from primitive shapes (e.g., circles, squares, cubes, cylinders, etc.), primitives themselves are ultimately defined from the fundamental building blocks (point, line, plane, etc.), and shapes are mathematically specified using group theory. What is different about Leyton's theory is that the information necessary to generate a given shape is recoverable from the mathematical definition of the shape. This generative approach is in contrast to the standard Klein geometry, where a shape is defined as an invariant under specified group transformations. Leyton describes this standard approach as memoryless, since the shape's generation is ignored rather than stored for later recovery. By contrast, his theory works by capturing the evolution of the shape from the building blocks, rather than merely specifying the end result. Leyton's approach has the additional benefit of allowing for the complete mathematical specification of quite complex shapes.

The hallmarks of Leyton's generative theory are transfer and recoverability. The ability to “*transfer* actions used in previous situations to new situations” (p. 1, original emphasis¹) results in the ability to carry over information throughout the generative sequence comprising the definition of the shape. Moreover, “the generative operations must be *recoverable* from the data set” (p. 2), i.e., from the end result of the generative process. Rather than merely specifying the end result of a creative process, Leyton's theory is able to capture the design process itself, by recording each step of the generative process via an appropriate symmetry group. Leyton's theory gives structure to this memory recording process.

If all of this sounds very mathematical, it most certainly is! Before we go into too many mathematical details, however, let's consider how Leyton's theory relates to art and architecture. This allows us to cite and discuss some additional tantalizing quotes—Leyton's captivating text is full of them.

“[T]he design process is one of successive symmetry-breaking” (p. 44); actually, we later learn that the more informative way to view this is as asymmetry-building.² Think of an architect designing a floor plan. He or she might start with a rectangle, which itself is a square having undergone a breaking of the 90-degree rotational symmetry. Each additional design element will break additional symmetries, and of course, the end result may not seem symmetric at all.

However, Leyton's constructive, generative method for recording the design process will capture each symmetry-breaking (or better yet, asymmetry-building), so that the whole design process can be recovered.

"Aesthetics is the maximization of transfer" (p. 33).³ The basic idea here is that the transfer of elements of a work of art onto each other (what Alberti or Palladio might call the correspondence of parts) is crucial to the design and the appreciation of works of art, music, etc. Aesthetics is also useful in physics, in the sense that many physical laws are uncovered as the result of applying mathematics—which here essentially means symmetry—to given empirical phenomena. This is especially true in fields like quantum mechanics, where some observed events are so utterly counterintuitive when viewed with principles from classical mechanics.

Leyton explains the aesthetic structure of science by his basic statement that "there is an *equivalence between the concept of geometry and the concept of memory-storage*" (p. 526). Though this concept appears a few times, this quote comes from the last chapter, in which Leyton outlines the difference between the standard Klein geometry and his generative geometry. The contrast is clearly summarized and is quite convincing. As for the quote under consideration, Leyton explains that with his approach, "a geometric object is defined as a structure from which one can recover the sequence of actions that created its current state." This gets back to the second hallmark of his theory, recoverability.

Chapter 15 is entitled "A Mathematical Theory of Architecture" and begins with the following text (p. 366):

Our mathematical theory of architecture is a means of describing the entire complex structure of a building (from the large-scale massing volumes down to the heating and ventilation systems) by a single symmetry group. This single group is required for all planning with respect to the building, because, by our theory of recoverability, plans come from the symmetries of a structure.

To explain further: Since almost all of the environment is artificially constructed, the rigorous study of architecture is essential not just for the understanding of design but for the study of perception, motor control, navigation, and robots.

While Leyton's theory may not inform the design process for most architects, it does have some important ramifications, not only for designers who appreciate being in tune with the mathematical underpinnings of their profession, but also for architectural theorists and specialists in virtual architecture.⁴ The ability to encapsulate a design in the programmable format made possible by Leyton's theory could open up new approaches and possibilities. While defining a single group to specify a whole architectural work may seem so complete as to be unwieldy, software could be developed to facilitate both the creation and the subsequent utilization of the encoded design process.

Leyton's theory is quite compelling as it applies to the mathematical specification of the primitives. Chapter 10 focuses on the surface primitives, e.g., plane, cube, sphere, cylinder, and torus. The representation of each, as specified by Leyton's theory, is intuitively clear, appropriately clean, and completely convincing. In Chapter 16 Leyton defines the solid primitives, i.e., the same sorts of objects as the surface primitives but viewed not just as the outer shells. The specifications of the solid primitives are more complicated than those of their surface counterparts, requiring some additional machinery that Leyton defines as part of his theory. He points out, in fact, that

the complexity of the solid symmetry groups “give crucial information not just for the physical sciences” but also other applications such as mechanical CAD and assembly planning (p. 418).

Let us consider the mathematics behind Leyton’s theory. It is based on a fairly specialized operation from group theory, namely the wreath product.⁵ The more common direct product of groups takes two groups, G and H , of size m and n , respectively, and creates a new group, $G \times H$, of size mn , whose elements are of the form (g, h) , where g is from G and h is from H . The wreath product of G and H , by contrast, has elements of the form $(g_1, g_2, \dots, g_n, h)$ where each g_i is from G and h is from H , and h has a certain action on all of the g_i .⁶ The details of this action are suppressed for simplicity, but in some sense the group H controls the n copies of the group G and transfers information between those copies; recall that transfer of information is a key aspect of Leyton’s generative theory of shape.

Realizing that mathematics, like any language, is designed to convey information, one might imagine that the wreath product allows for considerably more information storage (and therefore recoverability of information, the other key aspect of Leyton’s theory) than the direct product, simply due to the considerably larger size of the wreath product. More information storage in turn allows for the generative specification of more complex shapes. Of course, this also means that Leyton’s theory takes considerable mathematical savvy to understand.⁷

An example is in order. One primitive that Leyton uses instructively at various places in the book is the cylinder. He explains (p. 13) that the standard formulation of a cylinder is the direct product of a circle and the real number line, the circle being “swept” under the action of the line to create a cylinder.⁸ In Leyton’s theory the direct product is replaced by the wreath product. The result is that at each point on the sweeping line, the corresponding circle can be rotated independently from all of the other circles making up the cylinder; that is, by design, the group representation dictated by the theory has *built into it* the control of the individual circles (see pp. 94-95). One might imagine that the independent control of the circles could be exploited to deal with some of the complexities of real world cylinders, e.g., the type of coloring scheme found on barbershop signs or candy canes, the kinds of cylindrical machine parts that have increased and/or decreased diameters along their lengths as applicable to their function, perhaps even the sort of columns on the baldachin at St. Peter’s (though this would take some additional asymmetry-building, as well).

Leyton’s theory naturally inspires additional research. It represents, in fact, a brand new framework from which to explore a myriad of directions. Researchers in art and architecture, music, psychology, and mechanical design and manufacturing can develop applications of Leyton’s theory to their individual disciplines. CAD designers may be informed by exploring the relationships between Leyton’s generative theory and the approaches used in existing software.⁹ Mathematicians and computer scientists can work on the extension, refinement, codification, and facilitation of the theory itself.¹⁰

As for physics, Leyton informs us (p. 467) that he plans on a second volume that will focus on the conservation laws of physics. Chapter 20 is an intriguing preview, and it certainly whets the reader’s appetite. The standard approach towards developing conservation laws in quantum mechanics is to determine whether appropriate differential operators commute, and Leyton casts this into his theory by providing a correspondence between commutators and wreath products. It should be noted that this is one aspect of a very ambitious goal: “One of the purposes of our

generative theory of shape is to replace the different sets of laws in the different sciences by a single set of scientific laws that are universal across the sciences" (p. 52). Volume II should go a long way towards determining whether or not Leyton will ultimately be successful.

As one can imagine, while creating a theory capable of being used to define complex shapes is difficult enough, setting out such a theory in an understandable form is quite a task in and of itself. Leyton has done a fine job in this regard, incorporating many examples and figures throughout the text, dissecting large theoretical constructs into smaller, more understandable pieces, and restating aspects of the theory in various informative ways. Still, the book is not easy reading, and to be fair, it is certainly beneficial to have some exposure to group theory, which is usually not seen until advanced undergraduate courses for students majoring in mathematics. Moreover, examples from the other fields cited above may also be quite challenging for the non-specialist. Leyton is clearly a polymath, and the more the reader knows about these fairly diversified fields, the easier it will be to understand *A Generative Theory of Shape*. It is most certainly worth the effort.

Notes

1. Throughout this review all emphasis, both italics and boldface, within quotes is original.
2. From p. 242: "In asymmetry-building, one does not loose [sic] the initial symmetry ground-state, i.e., there is no actual symmetry-breaking."
3. This principle is restated on p. 75 with the addition of "**and recoverability**"; moreover, a new principle is given: "**Art works are maximal memory stores.**"
4. One example of virtual architecture is virtual reality, where users immerse themselves in simulated environments by wearing specialized sensory equipment that interfaces with computer systems. Even if this is not yet—or never becomes—mainstream, however, it should be realized that more and more people are spending more and more time "living" in virtual architecture simply via the popularity of first-person video games and the like. I would like to credit Jean Brangé, whom I met during preparations for the Nexus 2002 Round Table Discussion (see *Nexus Network Journal* vol. 3, no. 2, 2002), for conveying these ideas to me.
5. For those interested in a succinct general discussion of the wreath product, try the definition given under PlanetMath at:
<http://planetmath.org/encyclopedia/WreathProduct2.html>.
6. To be precise, the number n of the is the size of the set on which H acts, which may be different that the size of H itself. Moreover, the notation here is based on a finite value for n , so it is a bit deceiving if n is infinite. (While infinite—especially uncountably infinite—quantities of symmetry groups would necessarily lead to challenges in computer representation of Leyton's theory, most real-world applications inherently deal with, or can be restricted to, the finite.)
7. Leyton provides an appendix reviewing normal subgroups and semidirect products, which can be used as a litmus test for further reading, depending on whether or not the reader has the mathematical background to understand the appendix!
8. Actually, the direct product would be taken of the *symmetry groups* of the circle and of the real line, but that's a level of detail we can finesse at present.
9. At least for the foreseeable future, in fact, it seems that the theory is most applicable to areas such as CAD, robotics, mechanical design and manufacture, and human perception.
10. Michael Leyton is a very friendly and approachable individual who would welcome contacts from serious researchers.

About the Reviewer

Stephen Wassell received a B.S. in architecture in 1984, a Ph.D. in mathematics (mathematical physics) in 1990, and an M.C.S. in computer science in 1999, all from the University of Virginia. He is a Professor of Mathematical Sciences at Sweet Briar College, where he joined the faculty in 1990. Steve's primary research focus is on the relationships between architecture and mathematics. He has presented papers at Nexus 1998 and Nexus 2000, published articles in *The Mathematical Intelligencer* and the *Nexus Network Journal*. He is the author (with Kim Williams) of a book entitled *On Ratio and Proportion* (a translation and commentary of Silvio Belli, *Della proporzione et proporzionalità*). He has taught a course at Sweet Briar College entitled Architecture and Mathematics. He is currently working, with Branko Mitrovic, on a new book on Palladio's Villa Cornaro in Piombino Dese, to be published in 2005 by Aalto Books, Auckland. Steve's overall aim is to explore and extol the mathematics of beauty and the beauty of mathematics.