Martin Davis
Matt Insall

*Mathematics and Design:
Yes, But Will it Fly?*

Martin Davis and Matt Insall discuss a quote by Richard W. Hamming about the physical effect of Lebesgue and Riemann integrals and whether it made a difference whether one or the other was used, for example, in the design of an airplane. The gist of Hamming's quote was that the fine points of mathematical analysis are not relevant to engineering considerations.

The following discussion appeared as part of a thread about mathematical quotations on the listserv for Historia Mathematica [HM] forum (<http://www.chasque.net/jgc>) , a professional virtual environment for discussing history of mathematics, created and maintained by Julio Gonzalez Cabillon. This excerpt is reproduced here by courteous permission of the list's owner.

Does anyone believe that the difference between the Lebesgue and Riemann integrals can have physical significance, and that whether, say, an airplane would or would not fly could depend on this difference? If such were claimed, I should not care to fly in that plane. Richard W. Hamming

Matt Insall wrote:

Is there any historical record of anyone ever having scientifically tested whether the use of one mathematical concept, as opposed to another, hampered the design of something? In my experience, mathematical applications are not treated this way. Designers typically know little or no abstract mathematics, and so they typically do a series of tests. Frequently, they refer to them as statistical tests, but even the statisticians will at times shudder at what is being referred to as "statistical testing". Thus, for instance, I would say that this quote of Hamming's is a "straw man" attack on abstraction.

Personally, I think it is likely to make a difference whether Riemann or Lebesgue integration is used, but that the difference will only be noticeable provided that the designers rigorously adhere to the requirements that that particular integration theory be used for all the integrals. As soon as they revert to numerical approximations, instead of using calculus or analysis, they have muddied the waters so that no direct comparison between one integration theory and another can be made. Another way to test this might be to allow numerical approximations of all but one integral used in the design. For that integral, design the aircraft once using Riemann integration, and design it again using Lebesgue integration, but make sure that each of the other integrals approximated using numerical techniques uses the same techniques and has the same approximate value in the two different designs. Then try each aircraft out. An obvious problem with this "experimental design" is that scientists will be unwilling to part with the cost of an aircraft—let alone two aircraft—to test the usefulness of Lebesgue integration in this

way. Another obvious problem is more technical: When the one integral being tested for its effect on the design of the aircraft is computed, using Riemann integration, the value so obtained may be used as an input to some other portion of the design. This would skew the results of other numerically approximated integrals in the design. Similarly, when computing that particular integral using Lebesgue integration, if the value obtained is different from that obtained using Riemann integration, then the values elsewhere in the design will be skewed. Thus, the test may be inconclusive on the basis that the effects of the change in this single item is not isolated from the other. Finally, one may do statistical testing of the use of Riemann integration versus Lebesgue integration. This would be possible if there are some designers who typically use Lebesgue integration when the choice of which integral to use comes up, and when exact calculation of an integral is possible. (Of course, I expect that most designers of aircraft who use integration in the design actually use Riemann integration when they actually compute the integral in question. Part of the problem here is that the Lebesgue and Riemann integral agree with each other in most cases in which a designer would find that they are capable of computing the integral anyway, so the murkiness remains.) This type of statistical testing is likely, I believe, to not demonstrate much about the technical differences between the integration theories of Riemann and Lebesgue, but instead, it would point out whether it seems, at this time in history, to be more useful to a company to hire graduates who have learned the Lebesgue theory of integration, as opposed to hiring only those who only know the Riemann theory of integration.

Unfortunately, there are many complications involved in this issue: On the one hand, graduates who have studied Lebesgue integration are more likely to have taken some graduate mathematics or science or engineering courses, whereas many designers of various things—such as aircraft—typically, at least in the United States, have an undergraduate degree in some engineering discipline. Thus, in order to study the relevant statistical correlations in this instance, one needs to include correlations such as these, which involve, usually, a host of confounding variables: postgraduates are typically older, and so may be more mature, leading to better overall workmanship and greater levels of production overall, but also, being older, they may typically have less energy than their counterparts with no postgraduate training, leading to an attenuation of any productivity advantage they may have had over the others. On the other hand, those who are aware of the theoretical limitations of Riemann integration may have less practice with Riemann integration, having spent less time doing these integrals quickly than they have spent learning the theory of Lebesgue integration over the last few years of their training. This will in some cases attenuate their production levels even more than those for graduates who are oblivious to the distinctions between the two main theories of integration. In a sense, they are “overqualified”, meaning they know more, and know it better, but may not be practiced at doing things that the corporations need done quickly, in order to satisfy their stockholders’ need for dividend infusions.

Needless to say, I am not enamoured by such anti-mathematical quotations as this one attributed to Hamming.

Martin Davis replied:

At 06:39 AM 8/26/2002, Matt Insall wrote:

“Personally, I think it is likely to make a difference whether Riemann or Lebesgue integration is used, but that the difference will only be noticeable provided that the designers rigorously adhere to the requirements that that particular integration theory be used for all the integrals. As soon as they revert to numerical approximations, instead of using calculus or analysis, they have muddied the waters so that no direct comparison between one integration theory and another can be made.”

This is with reference to Hamming’s quip about this difference not being significant for engineering design.

Functions that are Riemann integrable are automatically Lebesgue integrable and the values of the integrals will be the same. There are two ways in which a function can be Lebesgue integrable but not Riemann integrable. Any bounded measurable function is L-integrable, but not all of them are R-integrable. Some unbounded measurable functions are also L-integrable, but not Riemann integrable, since all R-integrable functions are bounded. An example: the function $1/\sqrt{x}$ has the L-integral $2/3$ on the interval $[0,1]$. Being unbounded in that interval, it has no R-integral there.

Of course, this function does have an “improper integral” (sometimes called a Riemann-Cauchy integral) in that interval with the very same value. Only functions having absolutely convergent improper integrals have Lebesgue integrals. The classical example is $\sin x/x$ on the interval $[0,\infty]$ which has the value $\pi/2$ but no Lebesgue integral. This fact is sometimes confusingly stated as a case where something is R-integrable but not L-integrable. But one can perfectly well define “improper” L-integrals, so that this comparison is inappropriate. It’s just that the definition of L-integration automatically includes cases that in the Riemann case require this extra limiting step for which the inappropriate term “improper” is used.

Hamming of course hadn’t meant to be taken so literally. His aphorism was intended to say that the fine points of mathematical analysis are not relevant to engineering considerations. And, he was perfectly right.

In the long-ago days when I had occasion to be on committees administering an oral qualifying exam for the doctorate, I would often ask the hapless student why analysts prefer the L-integral. This was a trap question: students who fell for the trap would tell me that the function on $[0,1]$ that is 1 on the irrationals and 0 on the rationals is L-integrable, but not R-integrable. The right answer is that the L-integral has useful convergence properties not enjoyed by the R-integral.

Now whatever did Matt Insall have in mind? $\sin x/x$ on $[0, \infty]$? Surely not in any engineering analysis.

Matt Insall replied:

Of course, I have no specific examples in mind where one needs to integrate the sinc function on an interval including zero, or I would have written them in my previous off-the-cuff reply to the quote from Hamming. However, this function, and others like it, which are improperly R-integrable over some interval, but which are not (improperly) Lebesgue integrable on the same interval, are used reasonably often in various ways in engineering, and as we continue to need to make smaller objects with the properties expressible by the use of such functions, which fit in some significantly larger global structure, the tendency of certain engineers may well be to estimate certain values by computing integrals over intervals such as $(0, \infty)$. (This is especially the case if they have no introduction at all to higher analysis, and so do not know that the sinc function behaves so badly.) [The smallness of such an object would typically lead to a need to perform computations relating to nondegenerate intervals containing 0, whereas the remote location of the boundary of a region under consideration leads to models having lazy eight as the right endpoint of an interval over which one would like to integrate.]

Examples of articles I dug up (but have not read) by engineers, in which the sinc function and its relatives are used are below [in the bibliography]. I shall keep an eye open for other places where such functions are used. In the meantime, remember that the original question had to do with the dependency of the design on the use of Riemann versus Lebesgue integration. Thus although I focused on the version of this discussion in which a problem might involve a specific (improper, of course) integral for which the two theories diverge, my main point was that whether or not the choice of an integration theory is significant, in any specific way, to the design of an aircraft, it is unlikely that a reasonable answer to such a question will be found, and that factors that come into play in making the answering of such a question so unlikely include quite a few computational and societal difficulties that are not faced as often, for instance, in the fom community, where the question, for example, of whether the axiom of choice is a necessary ingredient in one's set-theoretic assumptions to obtain some specific proof in analysis from the axioms of Zermelo-Fraenkel set theory. (I certainly do not contend that one problem or the other is truly "easier". However, they are definitely different problems, and in each case, there are significant hurdles to overcome.)

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About the Authors

Matt Insall (1961; <http://web.umn.edu/~insall/>) is a mathematician who works mainly in applications of nonstandard methods to algebra and lattices, but works on a variety of interdisciplinary projects with other mathematicians and engineers. He has undergraduate degrees

in mathematics and chemical engineering, a masters degree in mathematics and a doctoral degree in mathematics. All of his degrees are from the University of Houston. Dr. Insall is currently an associate professor of mathematics in the Department of Mathematics and Statistics at the University of Missouri—Rolla, in Rolla Missouri, USA. He has been at the University of Missouri — Rolla since finishing graduate school in 1989. Dr. Insall organized the fifth meeting of the International Workshop in Analysis and Its Applications (IWAA) and was an editor for the Proceedings, published as a special issue of the journal *Mathematica Moravica*. At the University of Missouri — Rolla, Dr. Insall frequently interacts with engineers and students of engineering, and in fact, his first doctoral student studied applications of temporal logic to assembly and disassembly processes in manufacturing engineering, and had a coadvisor from the Department of Engineering Management.

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