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Four Approximations for Finding the Golden Section of a Circle's Circumference from the Square Root Two Rectangle

Geometer Mark Reynolds discusses four approximations for finding the Golden Section of the circumference of a circle from the root-2 rectangle.

Introduction

The vertical spiral of the distribution of leaves around the stems of certain plants ("the architecture of upright plants"¹) exhibit what is sometimes called the "Ideal" (or "Fibonacci") Angle² as viewed from the top of the plant looking down. The inverse angle of $(\sqrt{5}+1) \div 2$ of 360° is $137^\circ 30' 27.95''$, and if the leaves of a plant were set round a straight stem at a divergence angle of about $137^\circ 30' 28''$ (137.5°), no two leaves would ever be exactly one over the other, which is generally the ideal condition. From this we deduce the fact that plants which utilize this angle to give the minimum superposition and maximum light exposure to their assimilating members are employing a leaf arrangement which may be expressed in terms of Fibonacci numbers.³

Simply stated, the golden section of the circumference of a circle is approximately 137.5° ($137.50776\dots^\circ$). There are no exact constructions for this angle (most notably because, like π , it is irrational) that I'm aware of. However, I had the delight and good fortune to have found four drawings that come exceedingly close to it and that are generated from a master grid for the $\sqrt{2}$ rectangle. I have not done the calculations for the percent deviations, but the measures are virtually accurate. As a result, I am certain that the proofs would validate the careful constructions within an acceptable deviation that demonstrate that the relationships exist.⁴

The Ideal Angle, ϕ or ϕ of the circle, $360^\circ/\phi$ (not to be confused with ϕ of π , a totally different issue), angle AOZ, is shown in Figure 1. We can find the golden section of almost anything having to do with time and space. The golden section of the circle seen here is quite unique and striking, and can sometimes even be recognized.⁵

Figure 2 is the Square Root Two ($\sqrt{2}$) Rectangle. This master grid contains:

- Square AKMZ, where $AK = 1$;
- Diagonal of the square, AM; $AM = AR = \sqrt{2}$;
- $\sqrt{2}$ rectangle, AKPR;
- θ rectangle ZMPR (θ (theta) = $\sqrt{2} + 1$), which contains square CMPT and $\sqrt{2}$ rectangle ZCTR.

Fig. 1

Fig. 2

Figures 3, 4, 5, and 6 will illustrate the four methods for creating the approximations of the Ideal Angle in the $\sqrt{2}$ rectangle. Before we begin, it is important to note that the demonstrations made here do not happen frequently. Although it is true that if one develops a sufficiently dense enough grid, possibilities can develop that could lead to many discoveries, what is being shown here is well outside the norm, especially because: a) only the basic elements of the grid are being used; and b) because the systems that develop are incommensurable. What is found here is unexpected in the extreme, as we find not one, but four, relationships with these two incommensurables, the Ideal Angle related to ϕ , and $\sqrt{2}$.

Method I

This series of drawings actually began with an exploration of the centers of the circle/square progression within the rectangle (manifested by the square portion of the ever reducing theta rectangles within the field of the master rectangle). I was curious about what the straight line spiral looks like that develops from the centers of the squares, and to see what the advanced grid would look like as well.

Fig. 3

Method II

A geometric diagram illustrating the construction of a circle tangent to two perpendicular lines and a larger circle. The diagram features a large square frame with vertices labeled K (top-left), I (top-center), P (top-right), A (bottom-left), Z (bottom-center), and R (bottom-right). A large circle is inscribed within the square, centered at point O. A vertical line segment KI passes through the center O. A horizontal line segment AZ passes through the bottom of the circle. A diagonal line KR connects the top-left corner K to the bottom-right corner R. Another diagonal line PR connects the top-right corner P to the bottom-left corner A. A smaller circle is shown in the upper right quadrant, tangent to the vertical line KI, the horizontal line AZ, and the arc of the large circle. This smaller circle's center is at point G. A line segment MG connects the top-right corner M to the center G. A line segment EG connects the center G to the point E on the vertical line KI. A line segment FG connects the center G to the point F on the horizontal line AZ. A line segment PG connects the top-right corner P to the center G. A line segment RG connects the bottom-right corner R to the center G. A line segment AG connects the bottom-left corner A to the center G. A line segment KG connects the top-left corner K to the center G. A line segment IG connects the top-center I to the center G. A line segment JG connects the bottom-center Z to the center G. A line segment LG connects the left side of the square to the center G. A line segment NG connects the right side of the square to the center G. A line segment OG connects the center O to the center G. A line segment HG connects the top edge of the square to the center G. A line segment DG connects the bottom edge of the square to the center G. A line segment CG connects the left edge of the square to the center G. A line segment BG connects the right edge of the square to the center G. A line segment AG connects the top-left corner A to the center G. A line segment BG connects the top-right corner B to the center G. A line segment CG connects the bottom-left corner C to the center G. A line segment DG connects the bottom-right corner D to the center G. A line segment EG connects the top edge of the square to the center G. A line segment FG connects the bottom edge of the square to the center G. A line segment HG connects the left edge of the square to the center G. A line segment IG connects the right edge of the square to the center G.

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In Figure 4, I used the construction arc ZP and where it cuts the diagonal of the square ZEFR at point G. This point yields line NO by drawing through the center of the Master Circle O_1 . Again, from point I an arc can be swung from N through O to S to obtain this same very close approximation of the Ideal Angle.

Method III

I next looked to the master circle O_1 within the master (generating) square, and what I call its “satellite”, circle O_2 , in the accompanying theta (q) rectangle ZMPR in Figure 5. By drawing the diagonals CP and MT of the square CMPT to generate the point G where the diagonal MT cuts the circle, I was able to swing the arc GN with its center at I. Again, the angle NS closely approximates the Ideal Angle, and is very nearly the same angle generated in the first two methods.⁶ The same thing can be said for Method IV.

Method IV

Figure 6 was done as a curious venture to see if any further constructions existed to generate the angle. I did not think that there could possibly be any other points in the grid; I was already surprised to find three, let alone one! As can be seen in Figure 6, another possibility does indeed exist. Square VUTF in the reciprocal h rectangle, ECTF, (to theta rectangle, ZMPR) was drawn and the midpoint J of its side, VF was found. A

line was then drawn to the mid point I of the side KM of the master square AKMZ. Line IJ cuts the circle O_1 at S and generates the Ideal Angle NO_1S once again.

Perhaps if I were to continue, there would be other connections as well. I leave those possibilities to other adventuresome spirits while I go back to the drawing board to see what else geometry might be kind enough to teach me.

Notes

1. Michael Schneider in *The Beginner's Guide to Constructing the Universe* (New York: Harper Collins, 1995), p. 168.
2. Theodore Andrea Cook, *The Curves of Life* (New York: Dover Books, 1979), p.418.
3. Ibid., p.418.
4. I always invite associates, scholars, mathematicians, artists, designers, and architects to get in touch with me to do collaborative work on the constructions and principles I present in the column.
5. We can find a close approximation to this angle when we look at advertising showing the hands of traditional watches, which are almost always in this position.
6. As these drawings are done on tracing vellum, as overlays on the original drawing in Figure 2, all four methods virtually lie any one over the other three.