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*In the Palm of Leonardo's Hand:
Modeling Polyhedra*

George W. Hart presents three examples of new computer-based “3D printing” techniques for recreating the historically important polyhedral models of Leonardo da Vinci and Luca Pacioli. It is hoped that such models will inspire students and the public to appreciate the history and beauty of polyhedra for architectural and other applications.

Introduction

As a sculptor and applied mathematician, I am very alert to the interactions throughout history between art and mathematics.¹ One set of Renaissance polyhedra was so beautiful and historically important that it deserves to be more widely known. By means of new “3D printing” technology, these forms are now easier than ever to reproduce and display. In this paper I present three examples of new computer-based 3D printing techniques for recreating the historically important polyhedral models of Leonardo da Vinci and Luca Pacioli. It is hoped that such models will inspire students and the public to appreciate the history and beauty of polyhedra for architectural and other applications.

Luca Pacioli, Leonardo da Vinci, and Polyhedra

In 1496, Leonardo da Vinci met the mathematician Luca Pacioli when they both worked for Ludovico Sforza, the Duke of Milan. Over the following five years, Leonardo and Pacioli worked, lived and traveled together. A Franciscan friar, Pacioli might be more accurately described as an encyclopaedist than a creative mathematician. During the period from 1496-1498, Pacioli wrote the text for a very influential book, *De divina proportione* (*On Divine Proportion*), about proportion in geometry and architecture.² Pacioli asked Leonardo to draw approximately sixty illustrations for the book, and never has a mathematician had a more able illustrator.

Figure 1, Figure 2, Figure 3 and Figure 4 are examples of Leonardo's illustrations for the manuscript copies of Pacioli's treatise. The polyhedra are drawn in an original style showing them with open rather than solid faces, which lucidly presents the entire form at once, front and back.³ The full set of illustrations includes the Platonic solids and various truncated and “elevated” forms derived from them (“elevated” refers to the result of erecting a pyramid on the outside of each face).⁴ In 1509, when the book was eventually published in print form, these beautiful manuscript illustrations were printed in the cruder form of woodcuts. An example is shown in Figure 5.

Fig. 1 (Facing page, upper left). “Elevated hexahedron” from the manuscript version of *De divina proportione*. The term “hexahedron” means “six-sided polyhedron”, i.e., the cube, and “elevated” refers to the result of erecting a pyramid on the outside of each face. Thus, the result has twenty-four triangular faces in a nonconvex arrangement. There are notes in the printed version of the book specifying that Pacioli invented this form as well as the “elevated tetrahedron”

Fig. 2 (Facing page, upper right). “72-sided sphere” from the manuscript version of *De divina proportione*. The polyhedron results from a construction in Euclid’s *Elements* and was drawn by many Renaissance artists, but this is the first representation of it with open rather than solid faces

Fig. 3 (Facing page, lower left). “Elevated icosidodecahedron” from the manuscript version of *De divina proportione*. It is a nonconvex construction of 120 equilateral triangles arranged with icosahedral symmetry. Three-sided and five-sided pyramids have been erected on the faces of the underlying icosidodecahedron, which consists of twenty triangles and twelve pentagons

Fig. 4 (Facing page, lower right). “Elevated rhombicuboctahedron” from the manuscript version of *De divina Proportione*. It is a nonconvex construction of ninety-six equilateral triangles arranged with octahedral symmetry. Three-sided and four-sided pyramids have been erected on the faces of the underlying rhombicuboctahedron, which consists of eight triangles and eighteen squares

Fig. 5 (This page, above). “Elevated icosidodecahedron” woodcut from the printed version of *De divina proportione*, cut by an anonymous woodcutter based on the drawing of Figure 3. The book text is in Italian, with figure captions in Greek, Italian and Latin. By means of these coarser images published in print and widely circulated, other scholars learned of these polyhedra

The Value of Polyhedra in Education and their Applications

These printed images of polyhedral models had a significant influence on other scholars and artists.⁵ Geometric models of this sort are valuable for an understanding of structure in three-dimensional space. In the Renaissance, they served as mathematical models for students studying Euclid's *Elements*—the essential mathematics text for every scholar, including architects. The widely-reproduced portrait of Pacioli by Jacopo de Barbari, now in the Capodimonte Museum in Naples, shows him lecturing from Euclid with a dodecahedron on the desk nearby. This gives a good indication of the probable size of the original polyhedron models.

Fig. 6 (left). Construction for Proposition 7 of Book 12 from the Heath edition of Euclid's *Elements*. When completed, the result is the structure underlying the form shown in Figure 2

Fig. 7 (right). Intarsia panel by Fra Giovanni da Verona, at the church of Santa Maria in Organo, Verona, circa 1520. This is a flat panel constructed of inlaid wood, designed to fool the eye by appearing three-dimensional. One of several polyhedral masterpieces by Giovanni, this art form peaked in the early 1500s. This example contains the polyhedra shown in Figures 1 and 3 (and the cuboctahedron from the same source); other examples contain the form shown in Figure 2 (photograph by Kim Williams)

The educational application of the 72-sided sphere shown in Figure 2 is clear if one studies Euclid. It illustrates the construction for a polyhedron inscribed in a given sphere (Book 12, Proposition 7) and approximates it arbitrarily closely, leading to the theorem that the volume of a sphere is proportional to the cube of its radius. Figure 6 is a traditional illustration detailing the construction,⁶ but the three-dimensional model shown Figure 2 gives a much better sense of the result.

Beyond this type of direct pedagogical use, these forms directly present aspects of symmetry, balance and design that are central to classical architecture. Pacioli discusses these issues in Chapter 53 of his book and also mentions how polyhedral forms can be used as a model for domes. *De divina proportione* was enormously influential, largely because of Leonardo's drawings. No one at that time had seen perspective illustrations of geometric forms. Many subsequent artists copied the designs or incorporated them into their work. Notable examples are the intarsia of Fra Giovanni da Verona, constructed as church ornamentation about 1520 (Figure 7).⁷

In the twentieth century, polyhedral geometry has been found to be the basis for a wide range of designs, such as Fuller's geodesic domes, space structures, deployable buildings and many other types of "nonstandard architecture".⁸ Today, progressive undergraduate architecture programs may include design labs that involve geometric model building, to develop students' understanding of space. However the wider culture is not very familiar with polyhedral models, because solid geometry is no longer featured in many high school curricula, and outside of school there are few opportunities to encounter polyhedra.⁹

To increase the general awareness about geometry, it is valuable to build and display such models. Printed figures and computer animations can be widely reproduced, but three-dimensional models, when available, have much more impact because they are real and tangible. Pacioli records that he carried wooden models with him to use as illustrations when he lectured. We know that the value of such models was officially recognized in the Renaissance, because there is an entry in the accounts for the building of the Council Hall in Florence indicating that a set of Pacioli's models was purchased by the City of Florence for public display.

Building Polyhedra

Two traditional model construction materials are paper and wood. Consider the paper models in Figure 8 and the wooden models in Figure 9 and Figure 10, all made by the author. The five Platonic solids in Figure 8 required a couple of hours to layout the faces, cut them (on the inside with a knife, the outside with scissors), and tape them together. The cherry icosahedron model of Figure 9 required about a day, and the elevated icosidodecahedron of Figure 10 required several days to cut, bevel, and assemble the 360 cherry pieces, then sand and finish it, and create the plaque and connecting brass wire.

Fig. 8. Paper models of the five Platonic solids. A popular, inexpensive way to make polyhedron models is by assembling paper polygons. Cutting out the faces' interiors gives an attractive result, in the style of Leonardo's drawings (models by the author)

Both paper and wood can lead to attractive results, but paper is of limited durability and woodworking requires more time and skill. I wish now to recommend a third technique: 3D printing. This is a new technology in which a computer-controlled robotic device automatically constructs a physical three-dimensional object from the description given in a computer file. While relatively expensive at present, there are several companies competing to develop these methods, so the cost will lower significantly in the future. Currently, manufacturing industries and larger universities are beginning to use this equipment. In the future, I expect all universities and high schools will have 3D printing equipment. There are many educational applications for 3D printing, such as mathematical, chemical, anatomical, and architectural models. But it is especially inspiring that such a state-of-the-art technology may be applied to recreate and disseminate copies of historically important artifacts.¹⁰

Fig. 9. Wooden model of an icosahedron. This model is 12 cm. in diameter and is assembled from sixty beveled trapezoids of cherry (model by the author)

Fig. 10. *Duodecedron Abscisus Elevatus Vacuus*, cherry, brass and aspen. This wooden reconstruction of the polyhedron illustrated in Figure 3 is 45 cm. in diameter (model constructed in 1999 by the author)

Three state-of-the-art 3D printing technologies are illustrated here. Each required several hours for a machine to construct, but only a few minutes of operator time to set up the job and start it. Of course there was a significant amount of work required for me to design the files that specify the forms, but that need not be repeated. As a computer scientist interested in making three-dimensional forms, it was natural that I wrote my own software for the purpose. However, many commercial 3D computer-aided-design tools would have been adequate for designing the files.

The plastic model in Figure 11 was constructed by a “fused deposition modeling” machine, which squirts melted plastic at just the desired locations, which then cools into the solid form. Figure 12 shows an example made in a machine where ink-jet technology is used to squirt water on layers of plaster powder. The plaster result is not as robust as plastic, but is less expensive. Figure 13 illustrates a translucent epoxy model. Made in a “stereolithography” machine, the surface of a rising liquid polymer is catalyzed by a computer-controlled laser that draws the calculated cross sectional slices, which solidify to form the result.

Fig. 11. 3D printing model of the polyhedron shown in Figure 3. This 8 cm model is built from ABS plastic, using a Stratasys FDM 3000 rapid prototyping machine, based on a file designed by the author

Fig. 12. 3D printing model of the polyhedron shown in Figure 2. This 8 cm model is built from plaster, using a Z Corporation Z402 rapid prototyping machine, based on a file designed by the author

Fig. 13. 3D printing model of the polyhedron shown in Figure 4. This 8 cm model is built from epoxy, using a 3D Systems SLA-3500 rapid prototyping machine, based on a file designed by the author

Modern 3D printing methods go beyond anything even Leonardo might have conceived, yet this technology can help his polyhedra to remain educationally relevant. Students now can create and display a mathematical “cabinet of curiosities”, or “instant museum” of these historically significant forms. I can provide the computer files that describe their structure, but it is an excellent pedagogical project to have students design the files. Constructing polyhedra is a computer-aided-design exercise that requires students to master precisely the sort of geometrical knowledge that Pacioli was teaching 500 years ago. However, instead of ruler and compass, the basic tools for manipulating polyhedra on a computer are their x -, y - and z -coordinates. The processing ability of computers then makes it straightforward to generate new models from old, e.g., to elevate or hollow out faces. Accessible models will then help others relive the experience of the physical models that Leonardo undoubtedly held in his hand, rotated, and studied as he drew them. In this way, these eternal forms can continue to be appreciated by new generations.

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Notes

1. This paper includes some material from my upcoming book, *Euclid's Kiss*. For photos of other Leonardo-inspired models I have made, and related geometric sculpture, see <http://www.georgehart.com>.
2. Written in the Italian vernacular, *De divina proportione* was translated into German in 1889, Spanish in 1946, and French in 1980, but has not yet been translated into English. At one time, there were at least three handwritten copies of the book, of which only two now survive, one at the Biblioteca Ambrosiana in Milan and one at the Bibliothèque de Genève. I have relied on an original 1509 Venice edition in the Leib Memorial Collection of the S.C. Williams Library at the Stevens Institute of Technology, a facsimile edition of the Ambrosiana manuscript (Milan: Silvana Editoriale, 1982), and a facsimile edition of the 1509 printed version, which includes its French translation (Paris: Compagnonnage, 1980). It is not certain whether the drawings shown as Figures 1-4 are the originals by Leonardo or copies traced by Pacioli.
3. Careful study of the manuscript drawings shows that a few of the polyhedra illustrated in Pacioli's treatise contain some geometrical inaccuracies. Portions of the rear surface visible through the openings of the front faces are occasionally misplaced. An analysis is beyond the scope of this paper, and could only be very speculative, but here are three possibilities one might consider: (a) Possibly these were approached as character sketches, and not intended to be as accurate as scientific illustrations; (b) possibly the extant drawings are Pacioli's inaccurate copies of lost Leonardo originals; and (c) possibly Leonardo did not have physical models for some of the drawings.

4. One can not help but be curious as to whether Pacioli and Leonardo had deeper reasons for presenting some of the particular forms that they chose—what might they have been models of? Several nonconvex forms assembled from equilateral triangles, e.g., Figures 1, 3 and 4 were original to them, and their book does not explain why they are being presented. Pacioli's text repeats the classical associations given in Plato's *Timaeus*: the tetrahedron represented Fire, the octahedron Air, the icosahedron Water, and the cube Earth. Leonardo, in his notebooks, makes clear that he was also familiar with this view. Plato had given a reasoned theory of nature founded on a "conservation of triangles" principle, which suggested how these classical elements transmute, e.g., "two and a half parts of air are condensed into one part of water." It is tempting to speculate that Pacioli and Leonardo explored new triangulated polyhedra because they were seeking structures that could be models of other substances, e.g., lead or gold, etc. However, this is a speculative conjecture and there is nothing in the text of *De divina proportione* to directly support any such proto-chemistry interpretation.
5. For a scholarly study of the wide influence which Leonardo's polyhedra drawings had and the connections between polyhedra and Leonardo's own designs for churches and cathedrals, see Kim H. Veltman, with Kenneth D. Keele, *Linear Perspective and the Visual Dimensions of Science and Art* (Munich: Kunstverlag, 1986).
6. Euclid, *Elements*, T.L. Heath, ed. (Cambridge: Cambridge University Press, 1908).
7. Alan Tormey and Judith Farr Tormey, "Renaissance Intarsia: The Art of Geometry," *Scientific American* **247** (July 1982): 136-143.
8. For an excellent recent survey of how polyhedral forms have been applied in twentieth-century architecture, see J. Francois Gabriel, ed., *Beyond the Cube: The Architecture of Space Frames and Polyhedra* (New York: John Wiley, 1997).
9. For interesting background on how models may be used in an architecture course, see Pierangela Rinaldi, "The Renaissance, Geometry and Architecture", *Nexus Network Journal* **2** (2000): 155-158. For other examples of polyhedron reconstructions, see the Leonardo Museum in Vinci (Pi): (<http://www.leonet.it/comuni/vincimus/inp2fram.html%20>).
10. The technology of 3D printing is advancing rapidly, so the best source of current information is to do a search for "3D print" on the Internet.

About the Author

George W. Hart is a research professor in the computer science department at Stony Brook University, NY. He holds a B.S. in Mathematics and a Ph.D. in Electrical Engineering and Computer Science, both from MIT, and was previously a professor at Columbia University. He is the author of a linear algebra text *Multidimensional Analysis* (Springer Verlag, 1995), a geometry text, *Zome Geometry*, (coauthored with Henri Picciotto, Key Curriculum Press, 2001), and over fifty scholarly articles, conference papers, and technical reports. Hart is also a sculptor, and his works have received praise and awards in numerous exhibitions. His web site (<http://www.georgehart.com>) illustrates the range of his work.