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***From Pentagon to Heptagon:
A Discovery on the Generation of the Regular Heptagon
from the Equilateral Triangle and Pentagon***

Geometer Marcus the Marinite presents a construction for the heptagon that is within an incredibly small percent deviation from the ideal. The relationship between the incircle and excircle of the regular pentagon is the key to this construction, and their ratio is $2 : \phi$. In other words, the golden section plays the critical role in the establishment of this extremely close-to-ideal heptagon construction.

The New Year always begins with increasing Light.

I mentioned in a previous column that we would visit the golden section, phi (from now on we will use the symbol ϕ to represent the golden section in our demonstration). Now before you move onto another article, I can assure you that we will be drawing our information from the regular pentagon, and using the mathematical standard:

$$\phi = (\sqrt{5} + 1) \div 2.$$

We won't be debating uses, approximations and the myriad of other difficulties ϕ sometimes elicits in the world of scholarship and postmodern thought. That said, I present the discovery of the first two regular, odd-sided polygons — equilateral triangle and pentagon — generating the next odd-sided polygon, the regular heptagon. This construction achieves a heptagon that is within an incredibly small percent deviation from the ideal. The heptagon, the side of which subtends a rational angle number having the repeating decimal expansion $51.428571428571...^\circ$, cannot be precisely rendered with compasses and straightedge. The present construction, however, comes about as close as possible. The key to this construction is the relationship between two circles related to the regular pentagon: the *incircle*, the circle inscribed within the pentagon and tangent to its sides internally, and the *excircle*, the circle that circumscribes the pentagon and is tangent to its sides externally; their ratio is $2 : \phi$. In other words, the golden section plays the critical role in the establishment of this construction.

We will begin with a construction and a brief analysis of the regular pentagon. This specific construction is credited to Albrecht Dürer. There are a variety of procedures, but I find this one to be of interest because of its utilization of the *vesica piscis* and the circle.

Drawing 1. The Regular Pentagon (Figure 1)

We will look at the pentagon to see some of the relationships within the five-sided figure, including the golden section. As relation to our discussion of the golden section, we find that where any two pentagonal chords (the straight lines connecting every other vertex) intersect one another divides those chords into the “extreme and mean ratio”, the golden section.

The pentagonal star — the pentagram — can be drawn inside the pentagon by drawing the five chords. These chords will intersect around the center of the pentagram and form a new, smaller pentagon. This progression continues both micro- and macro-cosmically to infinity. The sides of the pentagons and the arms of the pentagrams will be in a golden section progression based on $(\sqrt{5} + 1) \div 2 = \phi$, or approximately 1.618033.

The triangle formed from the center of the pentagon to the base (cathetus), the half-base (base), and the line from the center of the pentagon to the vertex (hypotenuse) is a right, scalene triangle with angles measuring 36° - 54° - 90° . (Very rarely, one hears the triangle called “The Sacred MR Triangle of the Egyptians”.)

The cathetus’s (vertical) length of this triangle forms the radius of the incircle of the pentagon.

The hypotenuse of the triangle forms the radius of the pentagon’s excircle.

These two circles are in a $2 : \phi$ (approximately 1.23606 : 1) relationship. This relationship is the key to the entire drawing, as the spacing between the two circles of the pentagon will be used for the smaller circle that will enclose the equilateral triangle, and this spacing doubled will enclose the heptagon. Hence, the spacing of the concentric rings based those of the pentagon will be in the ratio 1 : 2.

A beautiful relationship also exists within the pentagon in that the chord of the pentagon is the geometric mean between the extremes of the height of the pentagon and the diameter of the pentagon’s excircle. What follows is the procedure for establishing this relationship.

Drawing 2. The Pentagon’s Geometric Mean

1. Given regular pentagon, AKRMZ (Figure 2).
2. With center G, circumscribe a circle tangent to the vertexes of the pentagon.
3. Draw the diameter, PN, of the circle.
4. Draw the chord of the pentagon, KM.
5. Drop the vertical RB from the vertex R, passing through the center G, perpendicular to the chord KM, the base AZ, and (in this drawing) the diameter PN.
6. RB intersects AZ at B, and is not a diameter.
7. We can now use the height of the pentagon RB as the first extreme (A), and the diameter of the circumscribing circle PN as the second extreme (C). We now apply the formula for finding the geometric mean (B) between two extremes: $B = \sqrt{AC}$.
8. The geometric mean is KM, the chord of the pentagon.

Figure 1: The Regular Pentagon

Figure 2

compasses and straight edge for any geometric constructions. In this drawing, however, we are given a relatively simple and easy task, and with some care, the heptagon should be drawn without great effort. Careful measure with the compasses and accurate drafting will show just how close this drawing truly is.

Drawing 3. The Procedural Steps for the Generation of the Heptagon from the Pentagon and the Equilateral Triangle, and Starting with the Regular Pentagon

1. Draw a circle O with a radius of a few inches. When we do the proof, I will assign a specific scale of measures to the necessary geometric lengths for understanding the relationships. For now, the drawing need only be a comfortable size for the tools being used. (If you've done the other constructions, proceed to the two primary circles of the pentagon.) Given: pentagon KPJRM, Figure 3.1.
2. From the center of this circle, mark off 5 radii, OK, OP, OJ, OR, and OM, spaced at every 72° . (For convenience, clarity, and symmetry, place the radius, OJ, pointing up vertically to the "12 o'clock" position. The drawing is now bilaterally symmetrical).
3. Inscribe the sides of the pentagon within the circle where the radii intersect the circle at K, P, J, R, and M. This is the excircle. A radius of this circle, OM for example, is labeled r1.

While studying this construction, my attention was drawn to the small difference between the two extremes in our geometric mean relationship. Taking their half-lengths, GN and GB, we can generate two circles, that is, the excircle and the incircle. Curiosity led me to experiment with this length along with other concentric circles using this small difference between the two circles. I did not know the delight I was to find in my search!

It is important now to discuss the mechanism within the pentagon that will enable us to come extremely close to constructing a regular heptagon. The angle subtended by $1/7$ of a circle required in generating the heptagon is,

$$360^\circ \div 7 = 51.428571428571\dots^\circ.$$

As there is no construction known to generate this exact angle, we can only approximate the seven angles and sides to be "equal". The protractor's calibrations do not have decimals such as these, and programs such as AutoCad, like the protractor, cannot be accepted in the classical requirements of using only

4. Inscribe a circle with the radius r_2 , inside the pentagon and tangent to each of the five mid-sides. This is the incircle.
5. In Figure 3.2, note well the small length of line a , the difference between the two radii r_1 and r_2 . These two radii are in a ratio of $2 : \phi$. This length is extremely difficult to measure precisely with an inch rule or meter stick, so use compasses or dividers as this small line segment “ a ” is the key to the entire construction, and its measurement must be as precise as possible. Without it, we will not be able to draw either the circle for the equilateral triangle or the circle for the heptagon. (Figure 3.3 shows the two primary circles of the pentagon O_1 and O_2 , for clarity.)
6. Using “ a ”, subtract it from the radius r_2 , the incircle of the pentagon, to obtain r_3 . Using r_3 , inscribe the equilateral triangle FLE with one of the three vertexes in the vertical “12 o’clock” position. (Figure 3.4.)
7. From the radius r_1 of the excircle of the pentagon, now mark off two “ a ” lengths (labeled “ b ”) to make r_4 , the largest concentric circle, as in Figure 3.5.
8. Extend the base of the equilateral triangle FE to the left and right to intersect this largest circle at points I and H, also in Figure 3.5. (Note: That the base vertexes of the equilateral triangle align precisely with two of the seven vertexes of the heptagon is, by itself, an extraordinary factor in this extraordinary construction. Without the alignment, the construction fails.)
9. Bisect (by p_2) the angle q (BOH), subtended by this intersection at H and the vertex J. This angle is very nearly 102.8571° , and bisected, it yields, very nearly, 51.4285° , one of the seven angles of the heptagon. (We will discuss these angles in the proof.) Do the same bisection on the other side of the circle, at IOB. (Figure 3-6.)
10. With four of the seven sides of the heptagon completed by the two bisections, and using the lengths of each of the four completed sides of the heptagon to obtain the final three sides by measuring them with the compasses or dividers, cut the rest of the circle of the heptagon.
11. Draw the heptagon, UIXBWHY, as in Figure 3.7.

Drawing 4. Proof for the Construction (Figure 4)

AOM is the 36° - 54° - 90° right, scalene triangle in the regular pentagon.

OA (r_2) is the radius of the incircle (the cathetus) of the pentagon.

OM (r_1) is the radius of the excircle (the hypotenuse) of the pentagon.

It is a fact that the ratio of $OM : OA = 2 : \phi$. Without loss of generality, we may choose the scale such that $OM = 2$, and $OA = \phi$.

The smaller concentric circle for the equilateral triangle has the radius, OT (r_3). It has the same spacing “ a ” from the incircle as the spacing between the pentagon’s incircle and excircle. By our choice of scale, $a = 2 - \phi$. Therefore $OT = OM - 2a = 2 - 2(2 - \phi) = 2\phi - 2$.

In the equilateral triangle (FLE), it is a fact that the side of the equilateral triangle bisects the radius OT at Q. Therefore, $OQ = OT/2 = (2\phi - 2)/2 = \phi - 1 = 1/\phi$.

Now the radius for the heptagon, OH (r_4), is established by doubling the spacing a out from the excircle of the pentagon (this new spacing is labeled “ b ”). So, the radius of the heptagon, $r_4 = OH = 2 + 2(2 - \phi) = 6 - 2\phi$.

3-1

3-2

3-3

3-4

3-5

3-6

3-7

Figure 3

We can easily compute the difference between the actual and approximate values for the central angle of the regular heptagon:

- 51.4604832° Approximate
- 51.4285714° Actual
- $\approx 0.0319117^\circ$ Difference;

thus the percent deviation is:

- $0.0319117^\circ \div 51.4285714^\circ$
- ≈ 0.0006205
- $= 0.06205\%$ high.

It should be noted that this error is extremely small, imperceptible in even the most accurately executed pencil-and-paper constructions.

What should be pointed out is that ϕ has been used:

- 1) in its organic form — to grow something; and,
- 2) in its mathematical exactitude, as it maintains its integrity within the pentagon. There is no question of approximation or an issue of speculation.

This is an ideal situation. We have ϕ in both its major roles: mathematical and organic. In all the discussions of ϕ 's use / nonuse / misuse, I don't believe enough has been said in a positive mode to advance the application of the number organically in architecture and the arts. Too much can be made of its "cut", and not enough on its overall qualities in the growth of structures, systems, and patterns. It could be that ϕ is more ubiquitous than is commonly held, and that it does at times exist beneath the surface of things. Hopefully, this construction demonstrates that research can and should be continued on the subject.

Next column, we'll consider harmonic progressions, and a special presentation of a general formula for harmonic progressions. I will be showing how one harmonic progression can generate both 1- and 2- point linear perspective grids.

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