Leonard K. Eaton

Hardy Cross and the “Moment Distribution Method”

Leonard K. Eaton resurrects the reputation of Hardy Cross, developer of the “moment distribution method” and one of America’s most brilliant engineers. The structural calculation of a large reinforced concrete building in the nineteen fifties was a complicated affair. It is a tribute to the engineering profession, and to Hardy Cross, that there were so few failures. When architects and engineers had to figure out what was happening in a statically indeterminate frame, they inevitably turned to what was generally known as the “moment distribution” or “Hardy Cross” method. Although the Cross method has been superseded by more powerful procedures such as the Finite Element Method, the “moment distribution method” made possible the efficient and safe design of many reinforced concrete buildings during an entire generation.

Introduction

In his paper “The Influence of Mathematics on the Development of Structural Form” in *Nexus II: Architecture and Mathematics*, Holger Falter comments on the difficulty of calculating statically indeterminate frames of reinforced concrete. He remarks, “Forming hinges was simple with iron, but difficult with reinforced concrete. It created statically indeterminate frames continuing through several spans, a difficulty surmountable only by using analytical methods” [Falter 1998:61]. He emphasizes the change from graphic to analytical statics and concludes that, “Only a limited number of statically indeterminates could be solved with these new procedures, which is probably the reason for the mostly simple structures, causing as little calculating effort as possible” [Falter 1998: 61]. While I am in general agreement with Falter’s account, I would like to offer an emendation and in so doing resurrect the reputation of Hardy Cross, one of America’s most brilliant engineers.

When I started teaching at the University of Michigan, Ann Arbor, in 1950, architects were understandably cautious about designing large buildings with reinforced concrete frames, because they were uncertain about the location of moments (movements) within a frame of reinforced concrete. Nobody likes building failures. They lead to lawsuits and shattered careers. It happened that I was recruited by an architectural school which has always taken the structural side of the profession seriously. Unlike many places, which have traditionally “farmed out” this end of architecture to schools of civil engineering, the architectural school at University of Michigan has always taken structures as part of its domain. I knew some of my colleagues in the structural program rather well, and occasionally asked them for help in understanding problems of structure in

Gothic, Renaissance, and Baroque buildings. Sometimes I heard them talking about the behavior of reinforced concrete structures. The name of Hardy Cross would be invoked with awe.

The attitude of my fellow faculty members has to be understood in the context of the decade. In the nineteen fifties the post-World War II building boom was well under way, and a number of American architects were confronted with demands for multistory structures which had, by their very nature, statically indeterminate frames. Many of these were done in reinforced concrete, especially in the years of the Korean war when steel was generally unavailable. Even Mies Van Der Rohe, the apostle of steel construction, did one reinforced concrete building; it was far from his best performance. The old column and slab system, developed by C. A. P. Turner, and the more refined version of Maillart, were alright for industrial buildings but deemed unsuitable for first class office buildings and apartment houses.[1] But even in the fifties concrete was seen as a tricky material. I had one colleague who was always talking about "the dreaded creep," a movement of concrete after it has hardened. Over in Jackson, Michigan a building framed in concrete collapsed and four or five men were killed. An investigation showed that the material had not been given sufficient time to cure. The workmen pulled away the forms too quickly, and a failure at the joints was the result. The structural calculation of a large reinforced concrete building in the nineteen fifties was a complicated affair. It is a tribute to the engineering profession, and to Hardy Cross, that them were so few failures (This may have been one of the reasons for the widespread interest in the unusual reinforced concrete systems developed by Frank Lloyd Wright for his Johnson Wax office building in 1939). When architects and engineers had to figure out what was happening in a statically indeterminate frame, they inevitably turned to what was generally known as the "moment distribution" or "Hardy Cross" method.

The "Hardy Cross Method" or "Moment Distribution Method"

The greatest contribution of Hardy Cross has an amusing background. His dean, Milo Ketchum, resented Cross's reputation as a great teacher. During his term of office he prevented Cross from getting salary increases, and several times suggested that the professor might do better working elsewhere. One of Ketchum's charges against Cross was that he failed to publish enough papers, although in 1930 he published his important work on "The Column Analogy". It appeared in Bulletin 215 of the University of Illinois Experiment Station. Undoubtedly Cross benefited greatly by his contact with this structural laboratory. In any event, his response to Ketchum's threats was the publication of a ten-page paper in the May, 1930 Proceedings of the American Society of Civil Engineers.

The paper was entitled "Analysis of Continuous Frames by Distributing Fixed-End Moments", and it set forth an entirely new method of analyzing building frames. Discussion was closed in September 1930; the paper was not published in the Transactions until September 1932. At that time space was afforded for 38 commentators who took up 146 pages. This may be a record for comment on a single paper. Cross was immediately hailed as a man who had solved one of the knottiest problems in structural analysis. And he did so in a way that could be adopted by any engineer working in the field. Although

Cross was known as “philosophical”, his approach here was extremely practical. His view was that engineers lived in a real world with real problems and that it was their job to come up with answers to questions in design even if approximations were involved. In his first paragraph he wrote, “The essential idea which the writer wishes to present involves no mathematical relations except the simplest arithmetic” [Cross 1949:1].[2] This statement is not quite accurate. The Cross method depended on the solution of three problems for beam constants: the determination of fixed-end moments, of the stiffness at each end of a beam, and of the carry-over factor at each end for every member of the frame under consideration. The author remarks that the determination of these values is not a part of his method. In point of fact, these moments are often determined by calculus. In 1932, however, that branch of mathematics was part of the intellectual equipment of any well-trained engineer. Its application required no contrivance more complicated than a slide rule.

In order to understand the achievement of Hardy Cross, it will be helpful to review developments of the theory of elasticity as it applies to statically indeterminate structures. While there were important additions to the theory of elasticity in the late eighteenth century, major interest in the analysis of indeterminate structures started in the early nineteenth century. Most of the attention was directed to the solutions of trusses in timber and iron having redundant (i.e. indeterminate) members. Clebsch and Castigliano come to mind as pioneers. Indeterminate frame structures, like continuous beams, were of little interest, because in the available materials it was difficult (and perhaps undesirable) to create continuity at the nodes between members. The developed methodology in solving indeterminate trusses was cumbersome, since it required the solution of as many simultaneous equations as there were redundancies (indeterminacies) in the structure. Tedious longhand calculations were required, with an accuracy to many decimals, carried forward at each step. Anyone who has tried to solve only five equations with that many unknowns in them will appreciate the problem. And with each higher order of indeterminacy the amount of longhand calculation work increased geometrically.

Clapeyron was the first to offer a practical solution to the problem of continuous beams over supports. His Three Moment Method was widely used well into the twentieth century [Timeoshenko 1953]. He still had as many equations as there were indeterminacies (number of supports beyond two) but in any of these there were only a maximum of three unknowns.

The development of a new material, reinforced concrete, made it imperative to find solutions for statically indeterminate frames. Monolithic reinforced concrete structures are highly indeterminate. Hence methods which gave reasonably accurate results and did not require an horrendous amount of calculation were a necessity. In this, Professor Falter is quite correct. Axel Bendixen in 1914 offered a procedure known as the slope deflection method [Timoshenko 1983]. It was the first readily practiced way to solve rigid frame structures. His method leads to an easily written series of simultaneous equations. The initial writing of the equations required little work. Each equation is rather “sparse” (i.e. it contains only a few of the unknowns). Thus the effort required for the solution of these simultaneous equations became less as compared to the methods developed earlier. The results from the solution of the simultaneous equations yielded rotations and displacements at the ends of individual members that in turn could be used to find moments and shears.

In 1922, K. A. Calisev, writing in Croatian, offered a method of solving the slope deflection equations by successive approximations [Timoshenko 1983; Bulletin 108]. Probably because of the linguistic difficulty, Hardy Cross seems not to have been aware of Calisev's contribution. Though cumbersome, it was a pioneering work. The problem with it was that Calisev still used successively adjusted rotations to establish moment balances at the nodes.

It was Hardy Cross's genius that he recognized he could bypass adjusting rotations to get to the moment balance at each and every node. He found that he could accomplish the same task by distributing the unbalanced moments while unlocking one joint at a time and keeping all the others temporarily fixed. By going around from joint to joint, the method converged very fast (at least in most cases) and it had enormous psychological advantages. Many practicing engineers had dubious mathematical skills in handling simultaneous equations, and many had difficulties in visualizing rotations and displacements. Moments are much "friendlier" for the average engineer and therefore easier to deal with. Thus the Moment Distribution Method (also known as the Cross Method) became the preferred calculation technique for reinforced concrete structures. The description of the moment distribution method by Hardy Cross is a little masterpiece. He wrote: "Moment Distribution. The method of moment distribution is this: (a) Imagine all joints in the structure held so that they cannot rotate and compute the moments at the ends of the members for this condition; (b) at each joint distribute the unbalanced fixed-end moment among the connecting members in proportion to the constant for each member defined as "stiffness"; (c) multiply the moment distributed to each member at a joint by the carry-over factor at the end of the member and set this product at the other end of the member; (d) distribute these moments just "carried over"; (e) repeat the process until the moments to be carried over are small enough to be neglected; and (f) add all moments — fixed-end moments, distributed moments, moments carried over — at each end of each member to obtain the true moment at the end" [Cross 1949:2].

In the next paragraph Cross observed that for the mathematically inclined his method would appear "... as one of solving a series of normal simultaneous equations by successive approximation" [Cross 1949:2]. Indeed it was.

Cross supplied an illustration shown in Figure 1. He assumed that the members would be straight and of uniform section. Siffnesses were proportional to the moments of inertia (I), divided by the lengths (L), but the relative volume given for I/L in the problems might as well be the relative stiffnesses of a series of beams of varying section. In that case the carry-over factors would not be –1/2. Figure 1 is entirely academic. It does not represent any particular type of structure or any probable conditions of loading. It has the advantage that it involves all the conditions that can occur in a frame that is made up of straight members and in which the points are not displaced.

The table in Figure 2 shows the remarkable convergence that occurs after the moments have been distributed according to the Cross method.

Cross's willingness to accept approximation troubled that minority in the engineering profession who insisted on exactitude. Cross himself readily admitted that the values of moments and shears could not be found exactly. He concluded that there was then no point in trying to find them exactly. He was after a method of analysis which combined

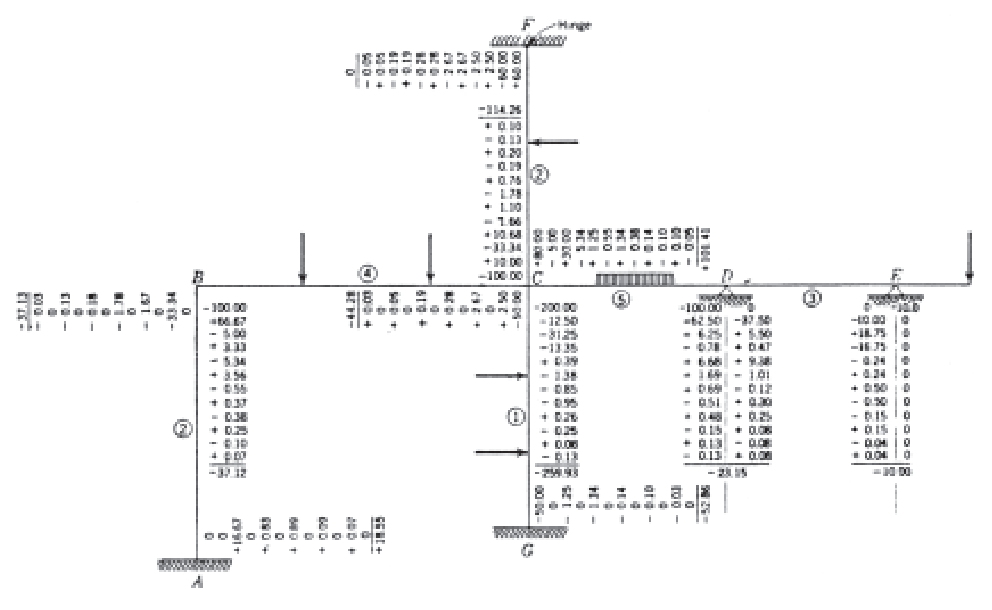

Figure 1

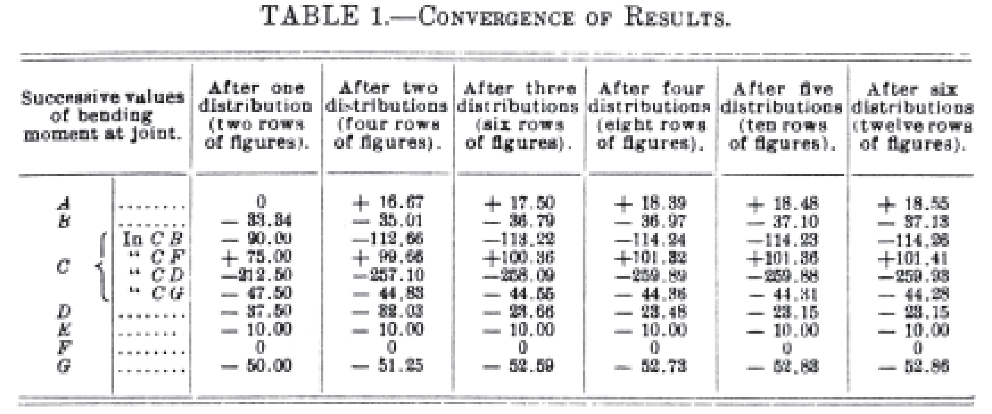

TABLE 1.—CONVERGENCE OF RESULTS.

Successive values of bending moment at joint.		After one distribution (two rows of figures).	After two distributions (four rows of figures).	After three distributions (six rows of figures).	After four distributions (eight rows of figures).	After five distributions (ten rows of figures).	After six distributions (twelve rows of figures).
A	....	0	+ 16.67	+ 17.50	+ 18.39	+ 18.48	+ 18.55
B	....	− 33.34	− 35.01	− 36.79	− 36.97	− 37.10	− 37.13
C	In *C B*	− 90.00	−112.66	−113.22	−114.24	−114.23	−114.26
	" *C F*	+ 75.00	+ 99.66	+100.36	+101.32	+101.36	+101.41
	" *C D*	−212.50	−257.10	−258.09	−259.89	−259.88	−259.93
	" *C G*	− 47.50	− 44.83	− 44.55	− 44.36	− 44.31	− 44.28
D	....	− 37.50	− 22.03	− 23.66	− 23.48	− 23.15	− 23.15
E	....	− 10.00	− 10.00	− 10.00	− 10.00	− 10.00	− 10.00
F	....	0	0	0	0	0	0
G	....	− 50.00	− 51.25	− 52.59	− 52.73	− 52.83	− 52.86

Figure 2

reasonable precision with speed. That he achieved his end is clear from a survey of the thirty eight commentators to his brief article. These individuals, partly academics, partly practicing engineers, generally saw that his method was a major contribution to structural analysis. It quickly passed into the curricula of the best American schools of engineering and by 1935 was generally taught. It continued to be received doctrine until the nineteen-sixties.

In the light of subsequent developments in twentieth century architecture and engineering, it is worth noting that Hardy Cross included a list of seventeen problems for which his method of analysis would not work. Chief among these were (1) Methods of constructing curves of maximum moments and (2) Methods of constructing curves of maximum shears. The Cross system would not therefore be of any assistance to an engineer computing the stresses in an elegant bridge by Maillart or one of the superb hyperbolic paraboloids of Felix Candela. For this kind of building other tools of analysis were required. And since much late twentieth-century architecture involves curved forms (one thinks immediately of Frank Gehry), there is no possible application of the moment distribution method. But for anyone building a structure with a reinforced concrete frame between 1932 and 1960, the method of Hardy Cross was a blessing indeed. And there were many of these, perhaps more in Europe than in the United States, which has always had a building economy where multistory structures have been framed in steel. Professor Robert Darvas, an eminent structural engineer who was educated in Budapest after World War II, learned the moment distribution method of Hardy Cross. Bob remarked in conversation that “His reputation was absolutely world wide”. Hence it seems to me that Professor Falter overlooked an important mathematical contribution to structural theory in the years 1932-1969.

The life of Hardy Cross

Hardy Cross was a Virginian, born in 1885 in Nansemond County. His first degree was a B.A. in English, which he took at Hampden-Sydney College in 1902. In his final year he began his teaching career in the English department at that college. Hampden-Sydney is a small institution, long known for its strong traditions in the liberal arts. It emphasizes Greek, Latin, the English classics, mathematics and science, philosophy, and religious history. One of my teachers in secondary school was a Hampden-Sydney graduate. He knew his field thoroughly and was a demanding master. So I have often wondered whether the excellence of this early training may be at least partly responsible for the clarity and forcefulness of the published papers of Hardy Cross. In 1903 Cross received a B.S. from Hampden-Sydney, and continued teaching for three years at Norfolk Academy. He took a B.S. in civil engineering from the Massachusetts Institute of Technology in 1908, and then joined the bridge department of the Missouri Pacific Railroad in St. Louis, where he remained for a year, after which he returned to Norfolk Academy in 1909-10. After a year of graduate study at Harvard he was awarded the M.C.E. degree in 1911. He next became an assistant professor of civil engineering at Brown University, where he taught for seven years. After a brief return to general engineering practice, he accepted a position as professor of structural engineering at the University of Illinois, Champaign-Urbana, in 1921.

The most creative years of Hardy Cross were spent at the University of Illinois. There he developed a reputation as a brilliant, if forbidding, classroom teacher. Like his eminent colleague, Harald Westergard, Cross suffered from deafness, and he used this handicap to his own advantage, both in and out of the classroom. Students soon found out that it was difficult to improvise answers at the tops of their voices. And they learned either to be explicit or to admit that they couldn't answer his questions. He lectured without notes, and his performance was always calculated to produce an atmosphere. Occasionally he would stomp out of the class early because no one had attempted a particular problem, and then ask some one who had observed his exit, "how do you think they took it?" Cross believed that the classroom was, above all else, the place to develop ingenuity and self confidence. He held that a university was a place to make intellectual mistakes, many mistakes, and learn to rectify them. This is not a bad definition of a university.

A chronicle of his years at Illinois remarks that he sometimes chose to play the devil's advocate. Once a student named Alford told Cross that he thought one of the problem solutions in their text was wrong. Cross paced back and forth, staring hard at the student, and pointing at him fiercely.

"Can you, a graduate student, actually have the temerity to accuse the internationally known engineer who wrote this book of MAKING A MISTAKE? Can you really believe that the publishers would allow such an alleged error to be printed? Can you show us the error?" Alford seemed unable to answer.
Still pacing, Cross said, "Can anyone help Mr. Alford? Do any of you see a mistake in problem four?" The class was silent.
"Well, Mr. Alford," Cross said sternly, "would you care to retract your accusation?"
"It's just that I can't..."
"Speak up!" Cross thundered.
"I still believe it's wrong!" Alford shouted, his face red with embarrassment.
"Then kindly come to the board and prove it to us," Cross taunted. "We shall be pleased to see the proof of your unfounded allegation."
Alford labored at the board without success for the rest of the period.
Cross began his next lecture by saying, "In our last meeting Mr. Alford raised a serious and unfounded charge against the author of our text." Staring at Alford, he said, "Have you reconsidered your accusation?"
"No, sir," Alford replied. "I still believe he is wrong."
"To the board, then. We still await your proof."
Alford's labors were again unsuccessful.
The third time the class met, Cross said, "Mr. Alford, are you ready to withdraw your ill-considered accusation about problem four?"
Moments later Alford was at the board. Within a few minutes he managed to show the solution to the problem in the book was incorrect, and he returned to his seat. Cross's pleasure was evident from his expression.
"You must always have the courage of your convictions," he said. "Mr. Alford does; apparently the rest of you do not, or you are not yet sufficiently well educated to realize that authority — the authority of a reputation or the authority of a printed page — means very little. All of you should hope to someday develop as much insight and persistence as Mr. Alford."[3]

Hardy Cross's later career

A word is in order on the later career of Hardy Cross. With N. D. Morgan he amplified his paper and published a book on Continuous Frames of Reinforced Concrete (Wiley, 1932). And he extended his geometrical methods to the solution of pipe network problems that arise in municipal water supply design. These methods have been used to solve similar networks such as gas pipelines. He received numerous honors. Among these were the Lamme Medal of the American Society for Engineering Education(1944), the Wason Medal of the American Concrete Institute (1935), and the Gold Medal of the Institution of Structural Engineers of Great Britain shortly before his death in 1959. In 1937 he moved to Yale to become Strathcona Professor and Chairman of the Department of Civil Engineering. A year after retirement from Yale in 1951 he published *Engineers and Ivory Towers*, a short book of his papers edited and arranged by Robert C. Goodpasture. This little volume is as lively and loaded with good sense today as it was when it came out in 1952 from McGraw-Hill. The reader will learn to enjoy its irony its blunt statements, and its good sense. An excellent example is author's discussion of standardization. He wrote, "Standardization, as a check on fools and rascals or set up as an intellectual assembly line, has served well in the engineering world." [Cross 1952:22] One has the feeling that Cross might have been a memorable chairman. We wonder how he worked with other administrators and with the president of the university.

Conclusion

So Hardy Cross died full of years and honor, having achieved an international reputation. Yet today his name is almost unknown. He does not even rate an entry in the *Dictionary of American Biography*. What is the reason for this neglect? I think it is that the advent of the computer rendered his moment distribution method old fashioned and unnecessary. With a computer there is no longer any such thing as a statically indeterminate structure. One can punch in the numbers and get a result to any desired accuracy. The Cross method has been superseded by more powerful procedures such as the Finite Element Method mentioned by Professor Falter. Still, the moment distribution method made possible the efficient and safe design of many reinforced concrete buildings during an entire generation. Perhaps Professor Falter will accept this emendation to his paper.

Notes:

1. To some extent I have discussed the early history of the concrete slab in [Eaton 1989: 113-128] and in [Eaton 1998: 315-325]. my article, It is worth noting that the flat slab system was also used in fashionable hotels such as the Benson in Portland (1915) and the Leamington in Minneapolis (1923). I have not found any instances of its use in first class business buildings.

2. In addition to the publications noted in the text, this classic paper made two other appearances. It was included in the report of a symposium at Illinois Institute Technology in Chicago. The symposium was dedicated to Hardy Cross, "whose simple demonstration of the power of numerical analysis brought these methods within the horizon of practicing engineers". It was also published in [Cross 1963]. This extremely valuable volume contains two of Cross's additional papers on moment design.

3. This account of Cross at the University of Illinois is taken from *Men and Ideas in Engineering: Twelve Histories from Illinois* (Urbana, 1967). I am much indebted to Professor Emory Kemp of West Virginia for this reference and for his encouragement.

Acknowledgments

The writer is much indebted to Professors Robert Darvas and Emory Kemp for help with this paper.

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About the author

Leonard K. Eaton is Emil Lorch Professor of Architecture Emeritus, the University of Michigan, where he taught architectural history from 1950 to 1988. He has also taught at Wayne State University, Michigan State University, and the University of Victoria (British Columbia). In 1985 he was Margan Professor at the University of Louisville. He took his B.A. with highest honors at Williams College in 1943, and after war service with the 10th Mountain Division, received an M.A. and Ph.D. from Harvard University. His publications include: *Landscape Artist in America: the Life and Work of Jens Jensen* (1964), *Two Chicago Architects and their Clients* (1969), *American Architecture Comes of Age* (1972) and *Gateway Cities and Other Essays* (1989). He is best known for his work on Frank Lloyd Wright, he has presented two papers at Nexus conferences, "Fractal Geometry in the Late Work of Frank Lloyd Wright: the Palmer House" (Nexus 98) and "Mathematics and Music in the Art Glass Windows of Frank Lloyd Wright" (Nexus 2000). Prof. Eaton is completing a longer work on Hardy Cross.