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The Squaring of the Circle in two Early Norwegian Cathedrals?

The squaring of the circle is impossible, but it can be represented geometrically. One such construction is similar to a diagram used to analyze two early Norwegian cathedrals. Further investigation revealed that it was not quite that simple. However, this pursuit revealed some alternative, but related possibilities, including a way of combining $\sqrt{2}$ and $\sqrt{5}$ – albeit approximately.

Introduction

To construct a square exactly equal in area to a given circle by compass and ruler was one of the three famous geometrical problems of antiquity. A host of circle-squarers have tried their hands at this age-old problem, until it was proven futile when Lindemann in 1882 demonstrated that π is a transcendental number [Struk 1987: 180]. The problem can, however, be represented geometrically, as demonstrated by Helmut Sander [2001]. I immediately recognized his diagram as being very close to a diagram that I have found by analyzing two early Norwegian basilican-plan cathedrals, and which, at first glance, I believed might have been used in determining the ratios between some important dimensions. This spurred me to make further investigations, revealing that it was not quite that simple. However, this pursuit revealed some alternative, but related possibilities, including a way of combining $\sqrt{2}$ and $\sqrt{5}$ – albeit approximately, but close enough to fool a non-mathematician working with small-scale geometry into make a false assumption similar to Le Corbusier's when he was developing the Modulor [Le Corbusier 1954: 229ff].

The Abbey of St. Alban, Selja, and the Cathedral of Stavanger

On the little island of Selja, immediately south of the Stad promontory, are the ruins of a Benedictine monastery. The abbey church, dedicated to St. Alban, contains remnants of the earliest known stone-built basilica in Norway. This church was built around 1100, maybe a little earlier, and served as the cathedral for the first diocese of western Norway until the see was transferred to Bergen about half a century later. A considerably larger cathedral was then built in Bergen, but the remains of this are too scant to make any assumptions about its design, other than that it must have had a central nave with aisles.

In the early twelfth century, a new diocese was established south of Bergen. The cathedral of Stavanger is still standing, with its Anglo-Norman nave almost intact [Hohler 1964; Hohler 1967] (Figures 1 and 2). Despite the apparent inaccuracies of its plan layout, some of the main dimensions indicate the use of simple numerical ratios when setting out the plan. However, other dimensions, as for example the external width, are systematically too far from fitting into this numerical system that they can be ascribed to inaccurate layout.

I therefore looked for ratios generated by geometry, and arrived at a diagram with a square DEFG whose sides are divided into 4 equal parts, and a circle cutting the sides of this square through their outer quarter divisions (Figure 3). The circle diameter, BC, is thus $\sqrt{5}$ of AB, which

Figure 1. Stavanger Cathedral, nave interior towards west. Photo by the author

Figure 2. Stavanger Cathedral, cross section of nave at the third bay from east, compiled by the author from the survey by G. Fischer, 1939-41

is half the square side. If the circle radius is 1, then square DEFG side will be $\sqrt{3.2}$, which is a rough approximation to $\sqrt{\pi}$, and the square area is then somewhat larger than the circle area. Moreover, a square KLMN with side equal to the diameter of the circle generated by this diagram will have approximately the same perimeter as a circle circumscribing the initial square DEFG [Watts and Watts 1987: 269].

$$CB = KL = LM = MN = NK = 2$$

$$OP = \sqrt{5}$$

$$\frac{AB}{CB} = \frac{CB}{OP} \rightarrow \frac{AC}{2} = \frac{2}{\sqrt{5}}$$

$$AC = \frac{4}{\sqrt{5}} = \sqrt{\frac{16}{5}} = \sqrt{3.2} \approx 1.78885\dots$$

$$AB = \sqrt{4 - AC^2} = \sqrt{4 - 3.2} = \sqrt{0.8} \approx 0.894427\dots = \frac{AC}{2}$$

Figure 3. Circle with a diameter $\sqrt{5}/2$ of the square side, or square with side $\sqrt{3.2}$ of the circle radius

In Stavanger, such a figure seems to have determined the external width of the nave (Figure 4), as the width equals the diameter of a circle generated in this way from a square with side twice the distance between the axes through the arcade pier centers at their eastern outset. This distance seems to have been found by dividing the sides of a larger square, $1/\sqrt{2}$ of the external length of the nave at its central axis, by half the diagonal from the square corners (Figure 5 and Figure 6). This subdivision of a square, called “the sacred cut” by Tons Brunés [1967], seems to have been used in building design and town planning in Roman antiquity [Watts and Watts 1987; Watts and Watts 1992].

Figure 4. Possible diagram for determining the exact nave width in relation to the axes of the arcade piers

The geometry of the plan also seems to relate to the elevation (Figure 7), as the height from floor level to the roof ridge is very close to $2/\sqrt{5}$ of the external width of the nave, or the side of the initial square in the diagram for the approximate squaring of the circle mentioned above, which is twice the distance between the axes of the arcade piers. The height of the arcade walls are $\sqrt{2}/4$ of the external nave length, or half the side of the square from which the pier axes seem to have been determined. The original height of the arcade walls and roof ridge can, however, not be ascertained, as the wall heads and roof were completely rebuilt during the restoration of 1867-69. In addition, the settling of the walls, which at least in some parts must have occurred during the first building campaign, makes it difficult to determine the original base level accurately. The wall head level and roof pitch is in accordance with that indicated by the Gothic choir gable, but may have been changed after the fire of 1272. Like most of the early bishops of Norway, the first Stavanger bishop was of the order of Benedictines, and the cathedral shows some similarities with the first St. Alban's church at Selja.

Figure 5. Stavanger Cathedral plan, after Fischer, 1940. Possible geometry governing the dimensions of the nave and original choir and west tower as proposed by the author

Figure 6. Possible geometric relationship of external length of nave to position of arcade piers

Figure 7. Stavanger Cathedral, possible scheme for nave elevation with supposed revision of pier height after settling of the south aisle wall. Numbers refer to a “foot” length of 32.5 cm. Main roof drawn with alternative inclinations. To the right as present, to the left as indicated by a horizontal masonry joint 32 “feet” above floor level

Figure 8. St. Alban's, Selja. Plan after Cato Enger, 1930. Analysis by author, before visit to the site

The ruins there were excavated and meticulously measured in the late 1930s (several of the measured drawings from the 1939-41 survey of Stavanger Cathedral are published in [Fischer 1964], which contains a short English summary of the text), and for comparison, I made a rough analysis of the plan. On the drawing to the scale of 1:100, it seemed that the ratio of internal nave length along its central axis to internal width was $\sqrt{2}$, and the pier axes determined by the “sacred cut” from a square with side the internal length (Figure 8). Even more intriguing was that the external length seemed to be $\sqrt{5}/2$ of the internal length, or corresponding to the approximate “squaring of the circle” discussed above.

Last summer I had the opportunity to visit Selja and make my own measurements. With the present state of the ruins, the length of the nave cannot be determined exactly, but the bases and lower parts of the west front and side walls are fairly well preserved, as are the traces in the west wall of the original arcade walls (Figure 9).

Figure 9. St. Alban's, Selja. Part of the interior west wall. Photo by the author

Although the walls are clad with ashlar of exquisite craftsmanship, there are pronounced inaccuracies in the plan layout. The nave is 14 cm wider at the west end than in the east, and the north wall is some 30 cm longer than the south wall (internal measures). The wall thickness varies between circa 85 cm and 94 cm. The ratios most close to $\sqrt{2}$ are found between the inner lengths of the south and east walls (c. 1260 cm : 887 cm, approximately 1.421), and central long axis and west wall (c. 1275 cm : 901 cm, approximately 1.415). Using the “sacred cut” of a square with side 1275 cm gives a distance between the pier axes of $1275(\sqrt{2}-1)$, approximately 528 cm. The actual measure at the west end is $438 \text{ cm} + (88+89)\text{cm} : 2 = 526.5 \text{ cm}$. However, with a wall thickness of 85 cm at the northwest corner, the $\sqrt{5}$ geometry does not fit the ratio of external to internal length.

A closer approximation to the squaring of the circle can be made by constructing a square with side $(5/4)\sqrt{2} = \sqrt{3.125}$ of the circle radius. This accords with $3 \frac{1}{8}$ as the value of π , which is known as an approximation used by the Babylonians (Figure 10) [Struk 1987: 29].¹ The north wall of the nave of St. Alban's at Selja is c. 1290 cm inside, which by this diagram gives a circle with a diameter 1459.5 cm. Half of the difference is slightly less than 85 cm, which is the thickness of the west wall at the northwest corner. However, the length of the north wall is the dimension furthest from the $\sqrt{2}$ geometry as related to the internal width of the nave, and the thickness of the west wall increases from north to south. Thus, it is impossible to maintain that what appears to be an instance of squaring the circle is more than a mere coincidence.

Figure 10a. The diagram from Figure 3 and a Babylonian approximation for squaring the circle superimposed

Figure 10b. A way of constructing the square from a given circle in accordance with the Egyptian Rhind Papyrus version

Figure 11. Stavanger Cathedral. Alternative diagram for determining the external width of the nave by pure root-two geometry

Back to Stavanger, there is the problem that my assumed geometry relates to the outside faces of the walls, and not the interior dimensions as at Selja. In Stavanger, there is also another possibility for the geometrical determination of the nave width. Instead of the $\sqrt{5}$ geometry suggested, it could be defined by the sides of a regular octagon inscribed in the circle inscribed in the square $1/\sqrt{2}$ of the external length, thus conforming to a “pure” $\sqrt{2}$ geometry (Figure 11). With the dimensions of the nave, the actual difference as generated by the two methods is some 5 cm. On a small scale diagram to, say, 1:100, this is hardly discernible, and even in full scale layout using a cord or a lath on a terrain not completely level, it could be perceived as coinciding

if not checked by algebra (Figure 12). Of course one should not speculate upon the ideas of a long dead master builder, but it is tempting to wonder if he might have believed that there was a connection between $\sqrt{2}$ and $\sqrt{5}$ here, which could be thought to have some esoteric significance.²

Now, there is further the problem of inaccurate plan layout, as the north wall is 90 cm longer than the south wall, with a mean length (central axis) of 31.15 m, which is the measure I have used in the calculations. The width varies between 20.35 m at the east end, 20.40 m at the side doorways, and 19.90 m at the west end (Figure 14). What is more, the arcade piers follow rather uneven lines, wavering in and out westwards to meet the west wall almost 60 cm closer than at their outset in the east. These irregularities could of course be held as an argument for abandoning any attempt at all to deduce mathematically-based design methods for the building. However, the erection of a building of this size and quality would not be started without a preconceived plan, but even small inaccuracies made when setting out the measures on the site would easily lead to deviations of this order.

The irregularities of the plan, and the especially pronounced slithering of the south arcade pier line should not be expected from a building team capable of erecting the solid and long-lasting edifice with the rather unwieldy stone material they had at hand. It has been suggested that an earlier church may have occupied part of the site, and that the bishop did not permit the demolition of this until the eastern part of the new cathedral could be taken into service [Hohler 1967].

If this were the case, the master builder would have to have set out the plan by measuring his way around what was standing in the way of free sight lines. His scheme may have been conceived by geometrical methods, but he would have to employ a measuring rod with all the possibilities for inaccuracies in using this instrument on an uneven terrain. Maybe he was not equally well versed in numerical calculations, and maybe he took shortcuts by taking measures from the wall lines of the old church, which most probably was less than rectangular in plan.

I have used the east wall as my point of departure, as the church must have been built in the normal way from the east westwards, as evidenced by the sculptural detail. The width of the central nave at the eastern wall corresponds very well to the average wall thickness and length along the central axis in ratios of neat numbers. If the wall thickness is (1), the width of the central nave is (8) outside and (6) inside, the length (24) outside and (22) inside. The numbers here set in parentheses represent relational values; the actual unit (1) measure being slightly less than 130 cm.

By using the “sacred cut” on the square with side $1/\sqrt{2}$ of the external length (24), the distance between the pier axes would become 7.029... units, which may have been approximated by (7). A numerical approximation of the geometrical ratios might be (17) as the side of the square inscribed in the circle with diameter the external length of the nave (24). The square side division by the “sacred cut” could then be approximated as $(5)+(7)+(5)=(17)$. By coincidence (?), $\sqrt{5}$ of this approximation gives an external width of 2035 cm, which at the accuracy level of centimetres is identical to the measure generated by the pure $\sqrt{2}$ geometry of an octagon inscribed in the circle with diameter $1/\sqrt{2}$ of the external nave length.

Figure 12. The diagrams of figures 6 and 11 superimposed. AB is the length of the nave along the central axis. AG/HK are divisions of the sides of square CDEF by the Sacred Cut, giving the “ideal” pier axes; MNOP is the square with sides twice the length of GL/HK. RSTU is the ideal external perimeter of nave

Figure 13. Stavanger Cathedral, nave. Diagrams showing the relationship between aisle extension at west end and tower/narthex, respectively at the east end and the chancel walls. The ratios indicate that the aisle extensions were determined geometrically from existing or already-begun parts of the building at either end, using the root-five geometry of Figures 3 and 6

Figure 14. Stavanger Cathedral, overall plan. Numbers refer to approximations by the unit of 32.5 cm, or a quarter of the average wall thickness

If the unit of ratios cited above is divided by four, it gives a “foot” length of close to 32.5 cm. Could this be the unit deduced from some twelfth and thirteenth century churches in Ile de France of 32.48 cm, which some scholars believe to be the *pied royale* as decided by Charlemagne? The chancel arch of Stavanger Cathedral was, according to measured drawings made before the widening in 1867-69, approximately 3.25 m, or 10 such “feet”. With an internal width of the central nave of $(8)-(2)=(6)$, or 24 “feet”, this would divide the internal east wall by the “sacred cut” approximated as $7+10+7$ (Figure 15).

Figure 15. South side junction of nave and chancel, section of internal elevation. Numbers refer to the unit of 32.5 cm.

In the Gothic chancel built after the 1272 fire, a series of measures divisible by 32.5 cm or 26 cm ($4/5$ of 32.5 cm) into whole numbers can be found (Figure 16). The interesting relationship between these two “foot” lengths is that they form the hypotenuse and the longer leg of a

Pythagorean triangle with sides 3:4:5. Such a triangle could have been used as a scaling device, and by drawing lines parallel to the sides, all the sides can be divided into 3 or 5 parts, thus yielding a wider range of numerical approximations to geometrically generated measures than the simple method of halving a length and its partitions successively. In the former Bishop's Palace immediately to the south of the cathedral is a chapel contemporary with the choir. No reliable measured drawings have been made of this building, but some measures I made there last year suggest that it is proportioned with the use of a measure unit of c. 23 cm, or 32,5 cm divided by the square root of two (Figure 16).

Figure 16. Interior of Bishop's Chapel. The width is 24 units of slightly less than 23 cm., or 32.5 cm : $\sqrt{2}$. Important heights in the room are whole number multiples of this unit. Photo by the author, September 1999

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All original measured drawings used belong to the archives of the Norwegian Directorate for Cultural heritage, Oslo.

Notes

1. The Egyptian Rhind papyrus of c. 1500 BC offers an even better approximation to this problem, with the ratio of 8/9 between square side and circle diameter, giving π = approximately 3.1605; see [Struk 1987: 23 and Fig. 12b].
2. Four preserved twelfth-century church roofs in the Norwegian region of Trøndelag have a ratio between width and height above the wallheads of $\sqrt{2}$ (see figure below) [Nilsen 1999: 39].

The ratio can be determined in a simple way by dividing the width (w) into four parts, and using the outer quarter division points as centers for arcs of $\frac{3}{4}w$ starting at the outer ends and meeting at the ridge. On further elaboration of the diagram, several of the roof truss joints seem to correspond to significant points in the geometry. The arithmetical “beauty” of this triangle is that, if w is 2, it combines $\sqrt{2}$ (height), $\sqrt{3}$ (roof slope) and $\sqrt{4}$ ($=2$, the width). This may also have been the pitch of the nave roof of Stavanger Cathedral as originally planned.

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