The Missing Boundaries of the Santaló Diagrams for the Cases (d, ω, R) and $(\omega, R, r)^*$

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Abstract. In this paper we solve two open problems posed by Santaló: to obtain complete systems of inequalities for some triples of measures of a planar convex set.

1. Introduction

There is abundant literature on geometric inequalities for planar figures. These inequalities connect several geometric quantities and in many cases determine the extremal sets which satisfy the equality conditions.

Each new inequality obtained is interesting on its own, but it is also possible to ask if a collection of inequalities concerning several geometric magnitudes is large enough to determine the existence of the figure. Such a collection is called a *complete system of inequalities*: a system of inequalities relating all the geometric characteristics such that, for any set of numbers satisfying those conditions, a planar figure with these values of the characteristics exists in the given class.

Santaló [7] considered the area, the perimeter, the diameter, the minimal width, the circumradius, and the inradius of a planar convex K (A , p , d , ω , R and r , respectively). He tried to find complete systems of inequalities concerning either two or three of these measures.

He found that, for pairs of measures, the known classic inequalities between them already form a complete system. However, the situation was different for triples of measures. He gave a list of all the known inequalities involving triples of measures,

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studying in which cases there were complete systems of inequalities: (A, p, ω) , (A, p, r) , (A, p, R) , (A, d, ω) , (p, d, ω) , and (d, r, R) . He left the remaining cases as open problems.

The main tool that Santaló used was to map the set of convex bodies into a subset (diagram) of the unit square $[0, 1] \times [0, 1] \subset E^2$, following an idea of Blaschke. The extremal sets for the inequalities considered were mapped into the boundary points of the diagram.

Among all the 14 open triples, he studied specially the cases (d, ω, R) and (ω, R, r) , and he remarked that there was at least one new inequality missing for each one. He also represented the diagram for these two cases, but part of the boundaries was not determined.

In this paper we solve both problems, closing the boundaries of the two diagrams.

Also the case (d, ω, r) has been recently solved in [5]. So there are still 11 open cases: (A, p, d) , (A, d, r) , (A, d, R) , (A, ω, r) , (A, ω, R) , (A, r, R) , (p, d, r) , (p, d, R) , (p, ω, r) , (p, ω, R) , and (p, r, R) .

The first historical case of this type of problem was raised by Blaschke [2]. He tried to find a complete system of inequalities for the volume, the integral mean curvature, and the surface area of a convex body in the three-dimensional Euclidean space. He realized that the well-known geometric inequalities of Minkowski and the isoperimetric inequality were not enough to determine a complete system. The problem is still open. Several contributions to this problem were made by many mathematicians (Hadwiger, Bieri, Groemer, Sangwine-Yager).

More information about systems of inequalities can be found in [4] and [6]–[8].

2. Results

We state here a number of inequalities. The results of Theorems 1 and 2 are new, and are established in Section 3. There follows a list of inequalities which are well known in the classical literature of convexity. These are referred to later in the paper. We then state Theorem 3, which gives a new contribution to the Santaló problem.

Theorem 1. *Let K be a compact convex set in the Euclidean plane E^2 . Then*

$$(4R^2 - d^2)d^4 \leq 4\omega^2 R^4, \quad (1)$$

with equality when and only when K is an isosceles triangle.

Theorem 2. *Let K be a compact convex set in the Euclidean plane E^2 . Then*

$$(4r - \omega)(\omega - 2r) \leq \frac{2r^3}{R}, \quad (2)$$

with equality when and only when K is an isosceles triangle.

Fig. 1. The Reuleaux triangle and the Yamanouti set.

For the diameter, the width, and the circumradius of a compact convex set K , the relationships between pairs of these geometric measures are

$$d \leq 2R, \quad \text{equality for centrally symmetric sets,} \quad (3)$$

$$R \leq d/\sqrt{3}, \quad \text{equality for the equilateral triangle and Yamanouti sets,} \quad (4)$$

$$\omega \leq d, \quad \text{equality for sets of constant width,} \quad (5)$$

$$\omega \leq 2R, \quad \text{equality for the circle.} \quad (6)$$

For the width, the circumradius, and the inradius of a compact convex set K , the relationships between pairs of these geometric measures are (6) and

$$\omega \leq 3r, \quad \text{equality for the equilateral triangle,} \quad (7)$$

$$2r \leq \omega, \quad \text{equality for centrally symmetric sets,} \quad (8)$$

$$r \leq R, \quad \text{equality for the circle.} \quad (9)$$

Besides, in [7], Santaló shows that the width, the circumradius, and the inradius of a planar convex set satisfy

$$\omega \leq R + r, \quad (10)$$

where equality is attained for all the figures obtained from an equilateral triangle, replacing the edges by three equal circular arcs. We then have a one-parameter family of piecewise-circular equilateral 3-gons which goes from the circle to the equilateral triangle.

We prove the following theorem:

Theorem 3. *Inequalities (1) and (3)–(6) form a complete system of inequalities for the case (d, ω, R) . Inequalities (2), (6), and (8)–(10) form a complete system of inequalities for the case (ω, R, r) .*

3. Proofs of the Inequalities

In this section we prove Theorems 1 and 2. We shall require the following result (see [1] and [9]):

Fig. 2. Construction of the isosceles triangle.

Lemma 1. *Let $T = \triangle XYZ$ be an acute-angled triangle, with $\angle Y \leq \angle X \leq \angle Z$. Then there exists an isosceles triangle $T' = \triangle XYZ'$ satisfying*

$$\begin{aligned} A(T) &= A(T'), & d(T) &= d(T'), & \omega(T) &= \omega(T'), \\ p(T) &\leq p(T'), & R(T) &\leq R(T'), & r(T) &\geq r(T'). \end{aligned}$$

Proof. Since $\angle Z$ is the largest angle, it follows that $d = XY$. Let l be the straight line through Z which is parallel to XY . Now we translate the vertex Z along the line l till we obtain the new isosceles triangle $T' = \triangle XYZ'$ with $YZ' = XY = d$ (see Fig. 2). Hence, $d(T) = d(T')$.

In any triangle the width ω is in the direction perpendicular to the longest side, XY , so it is clear that $\omega(T) = \omega(T')$, and then it follows that $A(T) = A(T')$. Now, the inequality on the perimeter results from a well-known shortest path problem. Since the equality $2A = rp$ holds for each triangle, we can deduce easily that $r(T') \leq r(T)$.

Finally, let P and P' be the circumcenters of T and T' , respectively. We note that P and P' both lie on the perpendicular bisector of the line segment XY . Since $\angle Z' < \angle Z$, we deduce that $PZ' > PZ = R(T)$, and so P' is farther away from XY than point P . Hence $R(T') > R(T)$. $\square$

Proof of Theorem 1. We may suppose that $\text{int}(K) \neq \emptyset$, because otherwise either $K = \emptyset$ or K is a line segment. For if $K = \emptyset$, then inequality (1) holds trivially; if K is a line segment, then $d = 2R$ and, again, the inequality is trivially true.

It is known (see, for example, [3]) that the circumcircle of a set K either contains two points of the boundary of K which are diametrically opposite, or it contains three points of K that form the vertices of an acute-angled triangle.

In the first case we have $d = 2R$, and the inequality is true. Hence we may assume that K contains an acute-angled triangle T with $R(T) = R(K)$, and, since T is contained in K , it is clear that $d(T) \leq d(K)$ and $\omega(T) \leq \omega(K)$.

We consider $f_1(d) = (4R^2 - d^2)d^4$. It is easy to see that $f_1'(d) = d^3(16R^2 - 6d^2)$. Since it always holds that $R \leq d/\sqrt{3}$, we have $f_1'(d) \leq 0$. So, $(4R^2 - d^2)d^4$ is a decreasing function of d for fixed R . Therefore,

$$\frac{d(K)^4}{\omega(K)^2 R(K)^4} (4R(K)^2 - d(K)^2) \leq \frac{d(T)^4}{\omega(T)^2 R(T)^4} (4R(T)^2 - d(T)^2).$$

So, it suffices to prove the inequality for acute-angled triangles. Let $T = \triangle XYZ$ be an acute-angled triangle, for which we may suppose that $\angle Y \leq \angle X \leq \angle Z$. From

such a triangle T and using Lemma 1, we obtain a new isosceles triangle T' verifying $R(T') > R(T)$, $d(T') = d(T)$, and $\omega(T') = \omega(T)$.

Now, we consider the function $f_2(R) = (1/R^4)(4R^2 - d^2)$. We can check that $f_2'(R) = (4/R^5)(d^2 - 2R^2)$. Again, since $R \leq d/\sqrt{3}$, it holds that $f_2'(R) > 0$. So $(1/R^4)(4R^2 - d^2)$ is an increasing function of R for fixed d . Then

$$\frac{d(T)^4}{\omega(T)^2 R(T)^4} (4R(T)^2 - d(T)^2) \leq \frac{d(T')^4}{\omega(T')^2 R(T')^4} (4R(T')^2 - d(T')^2).$$

Let $\alpha = \angle X = \angle Z$. Since in any triangle the width ω is in the direction perpendicular to the longest side, we have $\omega = d \sin(2\alpha)$. Also, $R = d/(2 \sin \alpha)$. So, for any isosceles triangle we have

$$\frac{d(T')^4}{\omega(T')^2 R(T')^4} (4R(T')^2 - d(T')^2) = \frac{2^4 \sin^4 \alpha}{d^2 \sin^2(2\alpha)} \left(\frac{d^2}{\sin^2 \alpha} - d^2 \right) = 4.$$

Hence, we conclude that

$$\frac{1}{\omega^2 R^4} (4R^2 - d^2) d^4 \leq 4,$$

with equality when and only when K is an isosceles triangle. $\square$

Proof of Theorem 2. We may suppose that $\text{int}(K) \neq \emptyset$, because otherwise either $K = \emptyset$ or K is a line segment (and in both cases inequality (2) holds trivially).

It is known (see [3]) that the incircle of a set K meets the boundary of K either in two diametrically opposite points, or in three points forming the vertices of an acute-angled triangle. In the first case we have $\omega = 2r$, and the inequality is true. Hence we may assume that K is contained in a triangle T formed by three lines of support common to K and the circle, so $r(T) = r(K)$. Further, since T contains K , it is clear that $\omega(K) \leq \omega(T)$ and $R(K) \leq R(T)$.

We now consider $f_1(\omega) = (4r - \omega)(\omega - 2r)$. Since $\omega \leq 3r$ by (7), we have $f_1'(\omega) \geq 0$. So $(4r - \omega)(\omega - 2r)$ is an increasing function of ω for fixed r . Therefore,

$$(4r(K) - \omega(K))(\omega(K) - 2r(K)) \frac{R(K)}{r(K)^3} \leq (4r(T) - \omega(T))(\omega(T) - 2r(T)) \frac{R(T)}{r(T)^3}.$$

From such a triangle T and Lemma 1, we can obtain again a new isosceles triangle T' with $\omega(T') = \omega(T)$, $r(T') \leq r(T)$, and $R(T') \geq R(T)$.

Now, we consider the function $f_2(r) = (4r - \omega)(\omega - 2r)/r^3$. Since $\omega/3 \leq r \leq \omega/2$, $f_2(r)$ is a decreasing function of r for fixed ω . Then

$$(4r(T) - \omega(T))(\omega(T) - 2r(T)) \frac{R(T)}{r(T)^3} \leq (4r(T') - \omega(T'))(\omega(T') - 2r(T')) \frac{R(T')}{r(T')^3}.$$

Let $\alpha = \angle X = \angle Z$ and let L be the length of the longest sides of T' . It is easy to see that $\omega = L \sin(2\alpha)$, $R = L/(2 \sin \alpha)$, and $r = L \cos \alpha \tan(\alpha/2)$. We then obtain

$$(4r(T') - \omega(T'))(\omega(T') - 2r(T')) \frac{R(T')}{r(T')^3} = 2.$$

Hence, we conclude that

$$(4r - \omega)(\omega - 2r)\frac{R}{r^3} \leq 2,$$

with equality when and only when K is an isosceles triangle. $\square$

4. Complete Systems of Inequalities for (d, ω, R) and (ω, R, r)

Let (a_1, a_2, a_3) be any triple of the measures A, p, d, ω, R , and r for compact convex sets in the Euclidean plane E^2 . Using Blaschke's approach, Santaló observed that the problem of finding a complete system of inequalities for (a_1, a_2, a_3) can be expressed by mapping each compact convex set K to a point $(x, y) \in [0, 1] \times [0, 1]$, where $x = f_1(a_1, a_2, a_3)$ and $y = f_2(a_1, a_2, a_3)$. By Blaschke's convergence theorem, the range of this map $D(K)$ is a closed subset of the square $[0, 1] \times [0, 1]$. Each of the optimal inequalities relating a_1, a_2, a_3 determines part of the boundary of $D(K)$. When the set of inequalities determines the whole boundary of $D(K)$, then these inequalities form a complete system; if some inequality is missing, some part of the boundary of $D(K)$ remains unknown.

Proof of Theorem 3.

The case (d, ω, R) . Let

$$x = \frac{\omega}{2R} \quad \text{and} \quad y = \frac{d}{2R}.$$

Clearly, $0 \leq x \leq 1$ and $0 \leq y \leq 1$.

The inequality $\omega \leq d$ assures us that $x \leq y$, and this inequality is optimal only for the sets with $\omega = d$, that is, for the sets of constant width. However, for sets with $\omega = d$ we have $R \leq \omega/\sqrt{3}$, with equality for the Reuleaux triangle (see [3]) and so $x \geq \sqrt{3}/2$.

Therefore, we have the part of the boundary of $D(K)$ from the point $P = (\sqrt{3}/2, \sqrt{3}/2)$ (a Reuleaux triangle) to $C = (1, 1)$ (a circle), determined by the line $y = x$; the sets of constant width are mapped to the points of the line segment $\overline{PC}$ (see Fig. 3).

The centrally symmetric sets satisfy $d = 2R$, that is, $y = 1$ (with $0 \leq x \leq 1$); so, the centrally symmetric convex sets are mapped to the points of the line segment $\overline{SC}$,

Fig. 3. The Santaló diagram for the case (d, ω, R) .

with $S = (0, 1)$ (a segment). The second part of the boundary is determined by the line $y = 1$.

Now we consider inequality (4). It is equivalent to $y \geq \sqrt{3}/2$. So, the third part of the boundary is determined by the line $y = \sqrt{3}/2$, from the point $T = (\frac{3}{4}, \sqrt{3}/2)$ (the equilateral triangle) to the point P (the Reuleaux triangle); the Yamanouti sets are mapped to the points of the line segment $\overline{TP}$.

It is easy to see that inequality (1) can be translated to

$$4y^4(1 - y^2) \leq x^2.$$

The corresponding equation completes the boundary of $D(K)$: the curve $x^2 = 4y^4(1 - y^2)$ connects point S (corresponding to the segments) with point T (corresponding to the equilateral triangle), and the isosceles triangles are mapped to the points of this curve.

Thus, we have determined the boundary of the domain corresponding to the planar convex sets with given diameter, width, and circumradius. Now it is easy to find examples of convex sets which are mapped into any of the interior points in the Santaló diagram. This completes the proof of the case (d, ω, R) .

The case (ω, R, r) . Let

$$x = \frac{\omega}{2R} \quad \text{and} \quad y = \frac{r}{R}.$$

Clearly, $0 \leq x \leq 1$ and $0 \leq y \leq 1$.

The inequality $\omega \geq 2r$ gives $x \geq y$, and the points of the line $y = x$ represent all the sets with $\omega = 2r$, that is, the centrally symmetric convex sets (see Fig. 4). Therefore, we have the upper part of the boundary of $D(K)$, the diagonal OC from the origin (a segment) to $C = (1, 1)$ (a circle).

Now we consider inequality (10). It is equivalent to $y \geq 2x - 1$. So, the second part of the boundary is determined by the line $y = 2x - 1$, from point T (the equilateral triangle) to point C (the circle); the piecewise-circular equilateral 3-gons are mapped to the points of the line segment $\overline{TC}$.

If we use inequality (7), we obtain $y \geq 2/3x$. However, the equality sign holds just for the equilateral triangle, that is, for the point $T = (\frac{3}{4}, \frac{1}{2})$, and therefore this bound is

Fig. 4. The Santaló diagram for the case (ω, R, r) .

redundant because there are no figures for which the equality holds. Hence, (7) can be removed.

It is easy to see that inequality (2) can be translated to

$$2(2y - x)(x - y) \leq y^3,$$

and then the corresponding equation closes the boundary of $D(K)$; the curve $2(2y - x)(x - y) = y^3$ joins the origin (corresponding to the segment) to point T (the equilateral triangle), and the isosceles triangles are mapped to the points of this curve.

Thus, we have determined the boundary of the domain corresponding to the plane convex sets with given width, circumradius, and inradius. This completes the proof of the Theorem 3. $\square$

References

1. P. W. Awyong. An Inequality Relating the Circumradius and Diameter of Two-Dimensional Lattice-Point-Free Convex Bodies. *Amer. Math. Monthly* **106**(3) (1999), 252–255.
2. W. Blaschke. Eine Frage über konvexe Körper. *Jahresber. Deutsch. Math.-Verein.* **25** (1916), 121–125.
3. T. Bonnesen and W. Fenchel. *Theorie der Konvexer Körper*. Springer-Verlag, Berlin (1934, 1974); Chelsea, New York (1948).
4. J. D. Burago and V. A. Zalgaller. *Geometric Inequalities*. Springer-Verlag, Berlin (1988) (Russian original: 1980).
5. M. A. Hernández Cifre. Which Is the Minimal Family of Inequalities to Guarantee the Existence of Planar Convex Sets from Their Width, Diameter and Inradius? Preprint.
6. J. R. Sangwine-Yager. The Missing Boundary of the Blaschke Diagram. *Amer. Math. Monthly* **96** (3) (1989), 233–237.
7. L. Santaló. Sobre los sistemas completos de desigualdades entre tres elementos de una figura convexa plana. *Math. Notae* **17** (1961), 82–104.
8. R. Schneider. *Convex Bodies: The Brunn–Minkowski Theory*. Cambridge University Press, Cambridge (1993).
9. P. R. Scott. A Family of Inequalities for Convex Sets. *Bull. Austral. Math. Soc.* **20** (1979), 237–245.

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