Decomposing the Secondary Cayley Polytope*

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Abstract. The vertices of the secondary polytope of a point configuration correspond to its regular triangulations. The Cayley trick links triangulations of one point configuration, called the Cayley polytope, to the fine mixed subdivisions of a tuple of point configurations. In this paper we investigate the secondary polytope of this Cayley polytope. Its vertices correspond to all regular mixed subdivisions of a tuple of point configurations. We demonstrate that it equals the Minkowski sum of polytopes, which we call mixed secondary polytopes, whose vertices correspond to regular-cell configurations.

1. Introduction

Regular triangulations and mixed subdivisions play an important role in algebraic geometry (see [S1], [S2], [IR], [LW], and [V]) and are used in homotopy continuation methods for solving polynomial systems (see [HS] and [VVC]). The vertices of the secondary polytope (see [GKZ]) are in a one-to-one correspondence to regular triangulations of a point configuration. We show that for mixed subdivisions, this secondary polytope has a degenerate structure: it can be Minkowski decomposed in nontrivial polytopes that we call *mixed secondary polytopes*. If the underlying point configurations are in general position with respect to each other, then no two mixed secondary polytopes have parallel edges. This is not the case with the Minkowski summands obtained by the decompositions of fiber polytopes presented in Theorem 1.5 and Proposition 3.1 of [BS].

In Sections 2 and 3 a brief introduction is given to regular triangulations and regular mixed subdivisions stressing the properties that are important to us. We define the *I*-mixed secondary polytopes in Section 4 and prove in Section 5 that the sum of these

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I -mixed secondary polytopes is the secondary polytope. The connection between bistellar flips and the edges of the I -mixed secondary polytopes is explained in Section 6. We conclude with an example in Section 7.

2. Preliminaries on Triangulations

A *triangulation* $\mathcal{T}$ of a finite point configuration $A \subset \mathbb{R}^d$ is a collection of cells $I \subset A$ with cardinality $d + 1$ and $\dim(\text{conv}(I)) = d$ such that $\bigcup_{I \in \mathcal{T}} \text{conv}(I) = \text{conv}(A)$, and $\forall I, J \in \mathcal{T}$, $\text{conv}(I \cap J) = \text{conv}(I) \cap \text{conv}(J)$. A triangulation $\mathcal{T}$ is called *regular* (or *coherent*) if there exists a lifting vector $\omega \in \mathbb{R}^A$ such that the lifted simplices $\{(\mathbf{a}, \omega(\mathbf{a})) \mid \mathbf{a} \in I\} \subset \mathbb{R}^{d+1}$ with $I \in \mathcal{T}$ are the upper convex hull facets of the lifted point configuration $\{(\mathbf{a}, \omega(\mathbf{a})) \mid \mathbf{a} \in A\} \subset \mathbb{R}^{d+1}$. (see [L] for more on regular triangulations.)

A *circuit* Z is a minimal affinely dependent point configuration. For every circuit Z , there is, up to a real factor, a unique affine relation between its points: $\sum_{\mathbf{z} \in Z} \gamma_{\mathbf{z}} \mathbf{z} = 0$ and $\sum_{\mathbf{z} \in Z} \gamma_{\mathbf{z}} = 0$. (see [Z] for more on circuits.) We fix $|\gamma_{\mathbf{x}}| = \text{vol}(\text{conv}(Z \setminus \{\mathbf{x}\}))$, where vol denotes the $(\#Z - 2)$ -dimensional volume scaled to be one for the unit simplices. A circuit has a unique Radon partition ($Z_+ := \{\mathbf{z} \in Z \mid \gamma_{\mathbf{z}} > 0\}$, $Z_- := \{\mathbf{z} \in Z \mid \gamma_{\mathbf{z}} < 0\}$) and two $(\#Z - 2)$ -dimensional triangulations $\mathcal{T}_+ = \{Z \setminus \{\mathbf{z}\} \mid \mathbf{z} \in Z_+\}$ and $\mathcal{T}_- = \{Z \setminus \{\mathbf{z}\} \mid \mathbf{z} \in Z_-\}$.

Let J be a face of a cell in a triangulation $\mathcal{T}$. The *link* of the face J in this triangulation $\mathcal{T}$ is the family $F^{(1)}, F^{(2)}, \dots, F^{(s)}$ of subsets of A for which the simplices $(F^{(j)} \cup J)$ for all j are precisely the cells of $\mathcal{T}$ which have J as a face. A triangulation is called *flippable* over a circuit Z if cells of a triangulation $\mathcal{T}_+$ of Z are faces of cells of $\mathcal{T}$ and the links of all $J \in \mathcal{T}_+$ are equal. We call $F^{(1)}, F^{(2)}, \dots, F^{(s)}$ the link of Z .

Definition 2.1. Given a circuit Z and a triangulation $\mathcal{T}$ flippable over Z and $F^{(1)}, F^{(2)}, \dots, F^{(s)}$ the link of Z , then the *bistellar flip* of $\mathcal{T}$ over Z , denoted $\text{flip}_Z(\mathcal{T})$, is the triangulation obtained by removing all cells of $\mathcal{T}$ of the form $I = I_+ \cup F^{(j)}$ where $I_+ \in \mathcal{T}_+$ and $j \in \{1, \dots, s\}$ and replacing them with all cells of the form $I_- \cup F^{(j)}$ where $I_- \in \mathcal{T}_-$ and $j \in \{1, \dots, s\}$.

A cell I of $\mathcal{T}$ is *involved* in a bistellar flip flip_Z if $I \notin \text{flip}_Z(\mathcal{T})$, i.e., if $\#(Z \setminus I) = 1$.

Theorem 2.2. Let $\mathcal{T}$ be a triangulation of a point configuration A that is flippable over a circuit $Z := \{\mathbf{z}^{(1)}, \mathbf{z}^{(2)}, \dots, \mathbf{z}^{(r)}\} \subset A$ and let $F^{(j)} \subset A$ be a part of the link of Z , then, for any $I^{(i)} := Z \setminus \{\mathbf{z}^{(i)}\}$ and $I^{(i')} := Z \setminus \{\mathbf{z}^{(i')}\}$ in $\mathcal{T}_+ \cup \mathcal{T}_-$,

$$\frac{\text{vol}(\text{conv}(F \cup I^{(i)}))}{\text{vol}(\text{conv}(F \cup I^{(i')}))} = \left| \frac{\gamma_{\mathbf{z}^{(i)}}}{\gamma_{\mathbf{z}^{(i')}}} \right|.$$

Proof. By uniqueness of the affine dependence on a circuit, the result will be true if we prove that

$$\sum_{I^{(i)} \in \mathcal{T}_+} \text{vol}(\text{conv}(F^{(j)} \cup I^{(i)})) \mathbf{z}^{(i)} = \sum_{I^{(i)} \in \mathcal{T}_-} \text{vol}(\text{conv}(F^{(j)} \cup I^{(i)})) \mathbf{z}^{(i)}.$$

For proving this, we write

$$\begin{aligned} \sum_{I^{(i)} \in \mathcal{T}_+} \text{vol}(\text{conv}(F^{(j)} \cup I^{(i)})) \mathbf{z}^{(i)} &= \left(\sum_{I^{(i)} \in \mathcal{T}_+} \text{vol}(\text{conv}(F^{(j)} \cup I^{(i)})) \right) \left(\sum_{\mathbf{z}^{(i)} \in Z} \mathbf{z}^{(i)} \right) \\ &\quad - \sum_{I^{(i)} \in \mathcal{T}_+} \sum_{\mathbf{z} \in I^{(i)}} \text{vol}(\text{conv}(F^{(j)} \cup I)) \mathbf{z}. \end{aligned} \quad (1)$$

The equality holds because $I^{(i)} = Z \setminus \{\mathbf{z}^{(i)}\}$. We can rewrite (1) by replacing the $\mathcal{T}_+$'s by $\mathcal{T}_-$. We will prove that the right-hand side of this newly obtained equation equals the right-hand side of the original (1). The first part is equal because it is

$$\text{vol}(\text{conv}(F^{(j)} \cup Z)) \sum_{\mathbf{z}^{(i)} \in Z} \mathbf{z}^{(i)}.$$

The second part is also equal: they both are

$$(d+1) \text{vol}(\text{conv}(F^{(j)} \cup Z)) \times \text{centroid}(\text{conv}(F^{(j)} \cup Z))$$

since the centroid of a polytope can be computed as the average sum of centroids of simplices in any triangulation of this polytope and since the centroid of a simplex is the average of its vertices. $\square$

Secondary polytopes were introduced by Gel'fand et al. Chapter 7 of [GKZ] gives a good overview of their work on secondary polytopes. The vertices of a secondary polytope correspond to the regular triangulations of a polytope. This property was used in [dL], [TI], and [MII] to enumerate all regular triangulations of point configurations. Secondary polytopes are generalized by *fiber polytopes* (see [BS] and Lecture 9 of [Z]) and can be seen as the projections of *universal polytopes* (see [dLHSS]).

Definition 2.3. The *characteristic function* of a triangulation $\mathcal{T}$ of a point configuration $A \subset \mathbb{R}^d$ is¹

$$\varphi_{\mathcal{T}}: A \rightarrow \mathbb{R}: \mathbf{a} \mapsto \sum_{I \in \mathcal{T} | \mathbf{a} \in I} \text{vol}(\text{conv}(I)),$$

where vol is a translation-invariant volume scaled to be 1 for the unit simplex.

Note that characteristic functions can be regarded as $\#A$ -dimensional vectors.

Definition 2.4. The *secondary polytope* of a point configuration $A \subset \mathbb{R}^d$ is

$$\Sigma(A) := \text{conv}(\{\varphi_{\mathcal{T}} \mid \mathcal{T} \text{ a triangulation of } A\}).$$

¹ With $\sum_{X|Y}$ we mean the summation over all X for which Y holds.

Theorem 2.5 (Theorem 1.7, page 221, and Theorem 2.11, page 233, of [GKZ]). *Let $A \subset \mathbb{R}^d$ be a point configuration, then:*

1. *The vertices of $\Sigma(A)$ correspond to regular triangulations of A . Moreover, the normal cones of $\Sigma(A)$ are exactly the cones of lifting vectors inducing the regular triangulations.*
2. *The edges of $\Sigma(A)$ correspond to the bistellar flips between regular triangulations.*

We conclude this section with a property of secondary polytopes that can be proven similar to Proposition 3.1 in [BS].

Theorem 2.6. *Let A be a point configuration in $\mathbb{R}^d$ and let $F \subset A$ be a face of A , then $\Sigma(F)$, where F is considered in a suitable lower-dimensional space, is a Minkowski-summand of $\Sigma(A)$, i.e., there exists a factor $\lambda \in \mathbb{R}^+$ and a polytope $P \subset \mathbb{R}^A$ such that $\Sigma(A) = P + \lambda \cdot \Sigma(F)$.*

3. Preliminaries on Mixed Subdivisions

Let $(A_1, A_2, \dots, A_n)$ be an n -tuple of point configurations with $A_i \subset \mathbb{R}^d$ for all i . Let $\mathbf{e}^{(0)}, \mathbf{e}^{(1)}, \dots, \mathbf{e}^{(n-1)}$ be the canonical affine basis for $\mathbb{R}^{n-1}$. We consider the *Cayley embedding* of $(A_1, A_2, \dots, A_n)$ to be the point configuration $\mathcal{C}(A_1, A_2, \dots, A_n) := \bigcup_{i=1}^n A_i \times \{\mathbf{e}^{(i-1)}\}$ in $\mathbb{R}^d \times \mathbb{R}^{n-1}$. A (regular) *fine mixed subdivision* $\mathcal{S}$ of $(A_1, A_2, \dots, A_n)$ with $A_i \subset \mathbb{R}^d$ is a collection of cells $C = (C_1, C_2, \dots, C_n)$ with $C_i \subset A_i$ such that the Cayley embedding $\mathcal{C}(C_1, C_2, \dots, C_n) := \bigcup_i C_i \times \{\mathbf{e}^{(i-1)}\}$ of these cells make a (regular) triangulation of $\mathcal{C}(A_1, A_2, \dots, A_n)$. Since we only consider *fine* mixed subdivisions, we omit the word *fine*. Our definition is consistent with those in [S2] and [GKZ] due to the Cayley trick [S1, Lemma 5.2], [GKZ, Proposition 1.7, page 274], [VGC, Proposition 3.9]. All properties of (regular) triangulations can be formulated for (regular) mixed subdivisions.

A *subtuple* of a tuple $(A_1, A_2, \dots, A_n)$ is a tuple $(A_i \mid i \in I) := (A_{l_1}, A_{l_2}, \dots, A_{l_m})$ where $I := \{l_1, l_2, \dots, l_m\} \subset \{1, 2, \dots, n\}$. A mixed subdivision $\mathcal{S}$ of a tuple $(A_1, A_2, \dots, A_n)$ induces a mixed subdivision of any subtuple $(A_i \mid i \in I)$:

$$\mathcal{S}(A_i \mid i \in I) := \left\{ (C_i \mid i \in I) \mid C \in \mathcal{S} \text{ and } \dim\left(\sum_{i \in I} C_i\right) = d \right\}.$$

This is explained by the fact that a triangulation of a polytope induces a triangulation of a lower dimension of every face. In our case, $\mathcal{S}(A_i \mid i \in I)$ corresponds to a triangulation of $\mathcal{C}(A_i \mid i \in I)$ which is a face of $\mathcal{C}(A_1, A_2, \dots, A_n)$.

A tuple $Z = (Z_1, Z_2, \dots, Z_n)$ is a *mixed circuit* if $\mathcal{C}(Z_1, Z_2, \dots, Z_n)$ is a circuit. We denote the affine coefficients of this (mixed) circuit by $\gamma_{i,z}$ for $\mathbf{z} \in Z_i$. Note that since $\sum_{\mathbf{z} \in Z_i} \gamma_{i,z} = 0$ we have $\#Z_i \neq 1$ for all i . A mixed subdivision $\mathcal{S}$ is flippable over Z if its corresponding triangulation is flippable over $\mathcal{C}(Z_1, Z_2, \dots, Z_n)$. Two mixed subdivisions are connected by a bistellar flip flip_Z if their corresponding triangulations are connected by a bistellar flip over $\mathcal{C}(Z_1, Z_2, \dots, Z_n)$.

A cell $C \in \mathcal{S}$ is *involved* in a bistellar flip flip_Z if $C \notin \text{flip}_Z(\mathcal{S})$, i.e., if $\sum_{i=1}^n \#(Z_i \setminus C_i) = 1$.

Definition 3.1. For a mixed subdivision $\mathcal{S}$ the *characteristic function* is

$$\varphi_{\mathcal{S}} = (\varphi_{\mathcal{S},1}, \varphi_{\mathcal{S},2}, \dots, \varphi_{\mathcal{S},n})$$

with

$$\varphi_{\mathcal{S},i}: A_i \rightarrow \mathbb{R}: \mathbf{a} \mapsto \sum_{C \in \mathcal{S} | \mathbf{a} \in C_i} \text{vol}(\text{conv}(\mathcal{C}(C_1, C_2, \dots, C_n))).$$

For notational convenience from now on we denote the volume of the Cayley embedding of a cell C briefly with $\text{vol}(C)$. Note that this volume only depends on those C_i with $\#C_i > 1$. The characteristic function of a mixed subdivision is a tuple of functions each mapping one point configuration to $\mathbb{R}$. It can be regarded as a $(\sum_{i=1}^n \#A_i)$ -dimensional vector. Clearly, $\varphi_{\mathcal{S}} = \varphi_{\mathcal{T}}$ for a triangulation $\mathcal{T}$ corresponding to a mixed subdivision $\mathcal{S}$.

Definition 3.2. The *secondary polytope* of a tuple of point configurations $(A_1, A_2, \dots, A_n)$ with $A_i \subset \mathbb{R}^d$ is

$$\Sigma(A_1, A_2, \dots, A_n) := \text{conv}(\{\varphi_{\mathcal{S}} \mid \mathcal{S} \text{ a mixed subdivision of } (A_1, A_2, \dots, A_n)\}).$$

Similar to Theorem 2.5 we have:

Theorem 3.3. For a tuple of point configurations $(A_1, A_2, \dots, A_n)$:

1. The vertices of $\Sigma(A_1, A_2, \dots, A_n)$ correspond to regular mixed subdivisions.
2. The edges of $\Sigma(A_1, A_2, \dots, A_n)$ correspond to bistellar flips between regular mixed subdivisions.

The following theorem is a consequence of Theorem 2.6 but can also be regarded as a corollary of Main Theorem 4.6.

Theorem 3.4. For a tuple of point configurations $(A_1, A_2, \dots, A_n)$ and a subtuple $(A_i \mid i \in I)$ with $I \subset \{1, 2, \dots, n\}$, $\Sigma(A_i \mid i \in I)$, naturally embedded in a $(\sum_{i=1}^n \#A_i)$ -dimensional space, is a Minkowski summand of $\Sigma(A_1, A_2, \dots, A_n)$.

4. Decomposing the Secondary Cayley Polytope

In this section we define mixed secondary polytopes and present our main theorem, stating that the secondary polytope of a tuple of point configurations can be Minkowski-decomposed into the mixed secondary polytopes of its subtuples.

Definition 4.1. The *type* of a tuple of point configurations (cell or mixed circuit) $X = (X_1, X_2, \dots, X_n)$, denoted by $\tau(X)$, is the set of indices $\{i \in \{1, 2, \dots, n\} \mid \#X_i > 1\}$.

Definition 4.2. For $I \subset \{1, 2, \dots, n\}$ the I -mixed characteristic function of a mixed subdivision $\mathcal{S}$ of $(A_1, A_2, \dots, A_n)$ is the tuple $\varphi_S^I = (\varphi_{S,1}^I, \varphi_{S,2}^I, \dots, \varphi_{S,n}^I)$ where

$$\varphi_{S,i}^I: A_i \rightarrow \mathbb{R}: \mathbf{x} \mapsto \sum_{C \in \mathcal{S} | \tau(C) \cup \{i\} = I, \mathbf{x} \in C_i} \text{vol}(C).$$

Theorem 4.3. $\varphi_{S,i} = \sum_{I \subset \{1,2,\dots,n\}} \varphi_{S,i}^I$.

Proof.

$$\begin{aligned} \varphi_{S,i}(\mathbf{x}) &= \sum_{C \in \mathcal{S} | \mathbf{x} \in C_i} \text{vol}(C) \\ &= \sum_{I \subset \{1,2,\dots,n\} | i \in I} \sum_{C \in \mathcal{S} | \mathbf{x} \in C_i \wedge \tau(C) \cup \{i\} = I} \text{vol}(C) \\ &= \sum_{I \subset \{1,2,\dots,n\} | i \in I} \varphi_{S,i}^I(\mathbf{x}). \end{aligned}$$

From Definition 4.2 we know that $\varphi_{S,i}^I(\mathbf{x}) = 0$ if $i \notin I$. Thus $\varphi_{S,i}(\mathbf{x}) = \sum_{I \subset \{1,2,\dots,n\}} \varphi_{S,i}^I(\mathbf{x})$. $\square$

The I -mixed characteristic function does not depend on the point configurations A_i for which $i \notin I$. Moreover, $\varphi_{S,i}^I = \mathbf{0}$ if $i \in I$ and $\varphi_S^I = \varphi_{S(A_i | i \in I)}^I$.

Definition 4.4. The *mixed secondary polytope* of a tuple $(A_1, A_2, \dots, A_n)$ of point configurations with $A_i \subset \mathbb{R}^d$ is

$$\begin{aligned} \Xi(A_1, A_2, \dots, A_n) \\ := \text{conv}(\{\varphi_S^{\{1,\dots,n\}} \mid \mathcal{S} \text{ is a regular mixed subdivision of } (A_1, A_2, \dots, A_n)\}). \end{aligned}$$

Theorem 4.5. If $n > d + 1$, then $\Xi(A_1, A_2, \dots, A_n) = \{\mathbf{0}\}$.

Proof. Since a cell C of a mixed subdivision $\mathcal{S}$ corresponds to an $(n+d-1)$ -dimensional simplex, we have $\sum \#C_i = n+d$ and, $\forall i, \#C_i \geq 1$, thus $\#\tau(C) \leq d$. If $n > d+1$, then there is no cell $C \in \mathcal{S}$ such that $\tau(C) \cup \{i\} = \{1, 2, \dots, n\}$. $\square$

Theorem 4.6 (Main Theorem). The secondary polytope of a tuple of point configurations $(A_1, A_2, \dots, A_n)$ can be decomposed in mixed secondary polytopes

$$\Sigma(A_1, A_2, \dots, A_n) = \sum_{I \subset \{1,2,\dots,n\}} \Xi(A_i \mid i \in I), \quad (2)$$

where the summands are embedded in the same $(\sum_{i=1}^n \#A_i)$ -dimensional space.

The right inclusion $\subset$ in (2) follows from Theorem 4.3. We postpone the proof of the left inclusion to the next section.

5. Mixed Secondary Polytopes versus Bistellar Flips

In this section we focus on the difference between the characteristic functions of two neighboring mixed subdivisions $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)} = \text{flip}_Z(\mathcal{S}^{(1)})$. From what we know of a bistellar flip, there exist tuples of point configurations $F^{(1)}, F^{(2)}, \dots, F^{(s)}$ with, $\forall k, j, F_k^{(j)} \cap Z_k = \emptyset$ such that the cells of $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)}$ involved in the bistellar flip flip_Z can be written as

$$C^{(j,k,\mathbf{z})} = (Z_1 \cup F_1^{(j)}, Z_2 \cup F_2^{(j)}, \dots, Z_k \setminus \{\mathbf{z}\} \cup F_k^{(j)}, \dots, Z_n \cup F_n^{(j)})$$

for $j = 1, \dots, s, \quad k = 1, \dots, n, \quad k = 1, \dots, n, \quad \mathbf{z} \in Z_k, \quad (3)$

where $\gamma_{k,\mathbf{z}} > 0$ for cells of $\mathcal{S}^{(1)}$ and $\gamma_{k,\mathbf{z}} < 0$ for $\mathcal{S}^{(2)}$. Using Theorem 2.2 we can write the volume of (3) as

$$\text{vol}(C^{(j,k,\mathbf{z})}) = f^{(j)} \cdot |\gamma_{k,\mathbf{z}}|,$$

where $f^{(j)}$ is a constant independent of $\mathbf{z}$. Using these notations we formulate the following theorem:

Theorem 5.1. *Given a mixed subdivision $\mathcal{S}^{(1)}$ and a neighbor $\mathcal{S}^{(2)} = \text{flip}_Z(\mathcal{S}^{(1)})$, then*

$$\varphi_{\mathcal{S}^{(1)},i}(\mathbf{x}) - \varphi_{\mathcal{S}^{(2)},i}(\mathbf{x}) = -\gamma_{i,\mathbf{x}} \cdot \sum_{j=1}^s f^{(j)}. \quad (4)$$

Proof. We only need the volumes of cells involved in the bistellar flip to express the difference between the characteristic function:

$$\begin{aligned} \varphi_{\mathcal{S}^{(1)},i}(\mathbf{x}) - \varphi_{\mathcal{S}^{(2)},i}(\mathbf{x}) &= \sum_{j=1}^s \sum_{k=1}^n \sum_{\mathbf{z} \in Z_k | \mathbf{x} \in C_i^{(j,k,\mathbf{z})}} \text{sign}(\gamma_{k,\mathbf{z}}) \cdot \text{vol}(C^{(j,k,\mathbf{z})}) \\ &= \sum_{j=1}^s \left(\sum_{k=1 | i \neq k}^n \sum_{\mathbf{z} \in Z_k | \mathbf{x} \in F_i^{(j)} \cup Z_i} f^{(j)} \cdot \gamma_{k,\mathbf{z}} + \sum_{\mathbf{z} \in Z_i | \mathbf{x} \in F_i^{(j)} \cup Z_i \setminus \{\mathbf{z}\}} f^{(j)} \cdot \gamma_{i,\mathbf{z}} \right) \\ &= \sum_{j=1}^s f^{(j)} \cdot \left(\underbrace{\sum_{k=1 | i \neq k}^n \sum_{\mathbf{z} \in Z_k | \mathbf{x} \in F_i^{(j)} \cup Z_i} \gamma_{k,\mathbf{z}}}_{=0} + \sum_{\mathbf{z} \in Z_i | \mathbf{x} \in F_i^{(j)} \cup Z_i \setminus \{\mathbf{z}\}} \gamma_{i,\mathbf{z}} \right) \quad (5) \end{aligned}$$

$$= \sum_{j=1}^s f^{(j)} \cdot \left(\underbrace{\sum_{\mathbf{z} \in Z_i | \mathbf{x} \in F_i^{(j)}} \gamma_{i,\mathbf{z}}}_{=0} + \sum_{\mathbf{z} \in Z_i | \mathbf{x} \in Z_i \setminus \{\mathbf{z}\}} \gamma_{i,\mathbf{z}} \right) \quad (6)$$

$$\begin{aligned}
&= \sum_{j=1}^s f^{(j)} \cdot \sum_{\mathbf{z} \in Z_i | \mathbf{z} \neq \mathbf{x} \in Z_i} \gamma_{i,\mathbf{z}} \\
&= -\gamma_{i,\mathbf{x}} \cdot \sum_{j=1}^s f^{(j)}, \quad \text{where } \gamma_{i,\mathbf{x}} = 0 \quad \text{for } \mathbf{x} \notin Z_i.
\end{aligned}$$

The simplifications of (5) and (6) are based on the observation that $\sum_{\mathbf{z} \in Z_k} \gamma_{k,\mathbf{z}} = 0$. $\square$

Observe that the factor $\sum_{j=1}^s f^{(j)}$ in (4) does not depend on $\mathbf{x}$ and, thus, is a constant, scaling the vector γ .

Theorem 5.2. *Given a mixed subdivision $\mathcal{S}^{(1)}$ and a neighbor $\mathcal{S}^{(2)} = \text{flip}_Z(\mathcal{S}^{(1)})$, then*

$$\varphi_{\mathcal{S}^{(1)},i}^I(\mathbf{x}) - \varphi_{\mathcal{S}^{(2)},i}^I(\mathbf{x}) = \left(\sum_{j | I = \tau(F^{(j)}) \cup \tau(Z)} f^{(j)} \right) \cdot (-\gamma_{i,\mathbf{x}}).$$

Proof. The only cells that influence $\varphi_{\mathcal{S}^{(1)},i}^I(\mathbf{x}) - \varphi_{\mathcal{S}^{(2)},i}^I(\mathbf{x})$ are those that are involved in the flip and obey the restrictions of Definition 4.2, i.e., $\tau(C^{(j,k,\mathbf{z})}) \cup \{i\} = I$. Note that $\tau(C^{(j,k,\mathbf{z})})$ is independent of the choice of $\mathbf{z}$. Hence we can make the same simplifications as done in (5) and (6) of the proof of Theorem 5.1:

$$\varphi_{\mathcal{S}^{(1)},i}^I(\mathbf{x}) - \varphi_{\mathcal{S}^{(2)},i}^I(\mathbf{x}) = \left(\sum_{j \in \{1, \dots, s\} | \tau(C^{(j,i,\mathbf{z})}) \cup \{i\} = I} f^{(j)} \right) \cdot (-\gamma_{i,\mathbf{x}}).$$

Since, $\forall k, \#Z_k \neq 1$, from (3) it follows that $\tau(C^{(j,i,\mathbf{z})}) \cup \{i\} = \tau(Z) \cup \tau(F^{(j)})$, and this completes the proof. $\square$

Corollary 5.3. *Given a mixed subdivision $\mathcal{S}^{(1)}$ and a neighbor $\mathcal{S}^{(2)} = \text{flip}_Z(\mathcal{S}^{(1)})$, then for all $I \subset \{1, 2, \dots, n\}$ we have*

$$\varphi_{\mathcal{S}^{(1)}}^I - \varphi_{\mathcal{S}^{(2)}}^I = \frac{\sum_{j \in \{1, \dots, s\} | \tau(C^{(j,i,\mathbf{z})}) \cup \{i\} = I} f^{(j)}}{\sum_{j \in \{1, \dots, s\}} f^{(j)}} (\varphi_{\mathcal{S}^{(1)}} - \varphi_{\mathcal{S}^{(2)}}).$$

Proof. This follows directly from Theorems 4.3, 5.1, and 5.2. $\square$

We can now prove Theorem 4.6.

Proof. We only need to prove the right inclusion $\supset$. Let $\mathbf{f}$ be a vertex of

$$\sum_{I \subset \{1, 2, \dots, n\}} \Xi(A_i \mid i \in I). \quad (7)$$

The vertex $\mathbf{f}$ can be written as

$$\sum_{I \subset \{1, \dots, n\}} \varphi_{\mathcal{S}^{(I)}}^I.$$

Let $\mathbf{v}$ be any vector for which the inproduct $\langle \cdot, \mathbf{v} \rangle$ with the vertices of (7) attains a maximum in $\mathbf{f}$. Let $\varphi_{\mathcal{S}^*}$ be the vertex maximizing this inproduct on $\Sigma(A_1, A_2, \dots, A_n)$. We need to prove that $\mathbf{f}$ equals $\varphi_{\mathcal{S}^*}$.

For every $I \subset \{1, 2, \dots\}$ we can build a chain of neighboring regular mixed subdivisions

$$\mathcal{S}^{(I)} = \mathcal{S}^{(I,1)}, \mathcal{S}^{(I,2)}, \dots, \mathcal{S}^{(I,s_I)} = \mathcal{S}^*$$

using the simplex algorithm on $\Xi(A_i \mid i \in I)$ such that

$$\langle \varphi_{\mathcal{S}^{(I,j+1)}}, \mathbf{v} \rangle > \langle \varphi_{\mathcal{S}^{(I,j)}}, \mathbf{v} \rangle \quad \text{for } j = 1, 2, \dots, s_I.$$

Applying Corollary 5.3 gives

$$\langle \varphi_{\mathcal{S}^{(I,j+1)}}^I, \mathbf{v} \rangle \geq \langle \varphi_{\mathcal{S}^{(I,j)}}^I, \mathbf{v} \rangle \quad \text{for } j = 1, 2, \dots, s_I,$$

and thus

$$\langle \varphi_{\mathcal{S}^*}^I, \mathbf{v} \rangle \geq \langle \varphi_{\mathcal{S}^{(I)}}^I, \mathbf{v} \rangle. \quad (8)$$

Summing (8) over all I 's gives

$$\sum_{I \subset \{1, \dots, n\}} \langle \varphi_{\mathcal{S}^*}^I, \mathbf{v} \rangle = \left\langle \sum_{I \subset \{1, \dots, n\}} \varphi_{\mathcal{S}^*}^I, \mathbf{v} \right\rangle = \langle \varphi_{\mathcal{S}^*}, \mathbf{v} \rangle \geq \left\langle \sum \varphi_{\mathcal{S}^{(I)}}^I, \mathbf{v} \right\rangle = \langle \mathbf{f}, \mathbf{v} \rangle.$$

However, since the inproduct with $\mathbf{v}$ on (7) is maximized in $\mathbf{f}$, and $\varphi_{\mathcal{S}^*}$ is also contained in (7) we have that $\mathbf{f}$ must be equal to $\varphi_{\mathcal{S}^*}$. $\square$

Corollary 5.4. *If the points in the point configurations are in general position, then no two mixed secondary polytopes have parallel edges.*

Proof. If the points are in general position, then, for every circuit Z , $\dim(\sum_{i \in \tau(Z)} Z_i) = d$ and the link contains only the empty set $F^{(0)} = \emptyset$. If $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)} := \text{flip}_Z(\mathcal{S}^{(1)})$ are two neighboring mixed subdivisions, then $\varphi_{\mathcal{S}^{(1)}}^I - \varphi_{\mathcal{S}^{(2)}}^I = \varphi_{\mathcal{S}^{(1)}} - \varphi_{\mathcal{S}^{(2)}}$ if $I = \tau(Z)$ and $\varphi_{\mathcal{S}^{(1)}}^I - \varphi_{\mathcal{S}^{(2)}}^I = \mathbf{0}$ otherwise. Hence every edge of $\Sigma(A_1, A_2, \dots, A_n)$ belongs to exactly one of the mixed secondary polytopes. $\square$

6. Mixed Secondary Polytopes versus Regular Mixed-Cell Configurations

In this section we see that the vertices and edges of an I -mixed secondary polytope play a similar role as for the secondary polytope.

Definition 6.1. A cell $C = (C_1, C_2, \dots, C_n)$ is called I -mixed if $I = \tau(C)$. A (regular) I -mixed-cell configuration is the set

$$\mathcal{S}^I := \{(C_i \mid i \in I) \mid C \in \mathcal{S} \text{ and } \tau(C) = I\}$$

of a (regular) mixed subdivision $\mathcal{S}$.

This definition generalizes the definition of mixed-cell configurations of [MV]. If $I = \{1, 2, \dots, n\}$ we call an I -mixed cell a *full-mixed cell* and an I -mixed-cell configuration a *full-mixed-cell configuration*.

Theorem 6.2. *Given a circuit Z , two neighboring regular mixed subdivisions $\mathcal{S}^{(1)}, \mathcal{S}^{(2)} = \text{flip}_Z(\mathcal{S}^{(1)})$, and $I \subset \{1, \dots, n\}$ with $\#I \leq d$, then*

$$\mathcal{S}^{(1)I} = \mathcal{S}^{(2)I} \iff \varphi_{\mathcal{S}^{(1)}}^I = \varphi_{\mathcal{S}^{(2)}}^I. \quad (9)$$

Proof. We denote the cells involved in the bistellar flip flip_Z between $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)}$ by $C^{(j,i,z)}$ as in (3).

$$\begin{aligned} \mathcal{S}^{(1)I} \neq \mathcal{S}^{(2)I} &\stackrel{(1)}{\iff} \exists i, j, \mathbf{z} \in Z_i: \gamma_{i,\mathbf{z}} > 0, \tau(C^{(j,i,\mathbf{z})}) = I \quad \text{and} \quad C_i^{(j,i,\mathbf{z})} \notin \mathcal{S}^{(2)I} \\ &\stackrel{(2)}{\iff} \exists i, j, \mathbf{z} \in Z_i: \gamma_{i,\mathbf{z}} > 0, \tau(C^{(j,i,\mathbf{z})}) = I \quad \text{and} \quad i \in I \\ &\stackrel{(3)}{\iff} \exists j: \tau(F^{(j)}) \cup \tau(Z) = I \quad \text{and} \quad \exists i: \#(Z_i \cup F_i^{(j)}) > 2 \\ &\stackrel{(4)}{\iff} \exists j: \tau(F^{(j)}) \cup \tau(Z) = I \\ &\stackrel{(5)}{\iff} \varphi_{\mathcal{S}^{(1)}}^I \neq \varphi_{\mathcal{S}^{(2)}}^I. \end{aligned}$$

1. A difference between $\mathcal{S}^{(1)I}$ and $\mathcal{S}^{(2)I}$ can only be caused by a cell $C^{(j,i,\mathbf{z})}$ involved in the bistellar flip between $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)}$.
2. Since a bistellar flip only affects $C_i^{(j,i,\mathbf{z})}$ we have $C_i^{(j,i,\mathbf{z})} \notin \mathcal{S}^{(2)I} \Leftrightarrow i \in I$.
3. This follows from, $\forall i, \#Z_i \neq 1$.
4. If $\tau(F^{(j)}) \cup \tau(Z) = I$, then there is always an i such that $\#(Z_i \cup F_i^{(j)}) > 2$ because $\sum \#(Z_i \cup F_i^{(j)}) = n + d + 1, \forall i: \#(Z_i \cup F_i^{(j)}) \geq 1$ and the number of i 's for which $\#(Z_i \cup F_i^{(j)}) > 1$ is smaller than $\#I (\leq d)$.
5. This follows from Theorem 5.2. □

Theorem 6.3. *The set of regular mixed subdivisions $\{\mathcal{S}^{(1)}, \mathcal{S}^{(2)}, \dots, \mathcal{S}^{(r)}\}$ whose I -mixed-cell configurations are equal for a given $I \subset \{1, 2, \dots, n\}$ (i.e., $\mathcal{S}^{(1)I} = \mathcal{S}^{(2)I} = \dots = \mathcal{S}^{(r)I}$) is interconnected by bistellar flips.*

Proof. Consider the union of normal cones on the vertices $\varphi_{\mathcal{S}^{(1)}}, \varphi_{\mathcal{S}^{(2)}}, \dots, \varphi_{\mathcal{S}^{(r)}}$. This union is the set of lifting vectors inducing one I -mixed-cell configuration. One can express this set as the solution of a system of homogeneous linear inequalities. Consequently this union of normal cones is convex, and thus all normal cones are interconnected by facets. These facets correspond to the bistellar flips between the regular mixed subdivisions. □

Theorem 6.4. *Given a set of all regular mixed subdivisions $\{\mathcal{S}^{(1)}, \mathcal{S}^{(2)}, \dots, \mathcal{S}^{(r)}\}$ whose I -mixed-characteristic functions are equal for a given $I \subset \{1, 2, \dots, n\}$ (i.e., $\varphi_{\mathcal{S}^{(1)}}^I = \varphi_{\mathcal{S}^{(2)}}^I = \dots = \varphi_{\mathcal{S}^{(r)}}^I$), then the vertices $\varphi_{\mathcal{S}^{(1)}}, \varphi_{\mathcal{S}^{(2)}}, \dots, \varphi_{\mathcal{S}^{(r)}}$ are interconnected by edges.*

Proof. This follows directly from Theorem 4.6 and the basic properties of Minkowski sums. □

Theorems 6.3 and 6.4 allows us to generalize Theorem 6.2 for nonneighboring regular mixed-cell configurations.

Theorem 6.5. *Given two regular mixed subdivisions $\mathcal{S}^{(1)}$, $\mathcal{S}^{(2)}$ and $I \subset \{1, \dots, n\}$ with $\#I \leq d$, then*

$$\mathcal{S}^{(1)I} = \mathcal{S}^{(2)I} \iff \varphi_{\mathcal{S}^{(1)}}^I = \varphi_{\mathcal{S}^{(2)}}^I. \quad (10)$$

Proof. If $\mathcal{S}^{(1)I} = \mathcal{S}^{(2)I}$, then Theorem 6.3 ensures that one can construct a path of regular mixed-cell configurations from $\mathcal{S}^{(1)}$ to $\mathcal{S}^{(2)}$ all sharing the same I -mixed-cell configuration. Using Theorem 6.2 we know that they all have the same characteristic function. On the other hand, if $\varphi_{\mathcal{S}^{(1)}}^I = \varphi_{\mathcal{S}^{(2)}}^I$, then Theorem 6.4 ensures that one can construct a path of regular mixed-cell configurations from $\mathcal{S}^{(1)}$ to $\mathcal{S}^{(2)}$ all sharing the same I -characteristic function. Using Theorem 6.2 we know that they all have the same I -mixed-cell configuration. $\square$

Theorem 6.6. *For a tuple of point configurations $(A_1, A_2, \dots, A_n)$ from $\mathbb{R}^d$ with $n \leq d$:*

1. *The vertices of $\Xi(A_1, A_2, \dots, A_n)$ correspond to the full-mixed-cell configurations of $(A_1, A_2, \dots, A_n)$.*
2. *The edges of $\Xi(A_1, A_2, \dots, A_n)$ correspond to bistellar flips involving full-mixed cells.*

Proof. This follows from Theorem 6.5. $\square$

7. Two Examples

Example 7.1. Consider the following point configurations $A_1 = A_2 = \{(0, 0), (1, 0), (1, 1)\}$ and $A_3 = \{(0, 0), (1, 0)\}$. Figure 1 shows the secondary polytope of (A_1, A_2, A_3) with, for each vertex, a regular mixed subdivision. Figures 2–8 denote the mixed secondary polytopes. Note that $\Xi(A_1)$, $\Xi(A_2)$, and $\Xi(A_3)$ (Figs. 2–4) are singletons corresponding to the only triangulation of A_1 , A_2 , and A_3 . The vertices of $\Xi(A_1, A_2)$, $\Xi(A_1, A_3)$, and $\Xi(A_2, A_3)$ (Figs. 5–7) correspond to mixed-cell configurations $\mathcal{S}^{\{1,2\}}$, $\mathcal{S}^{\{1,3\}}$, and $\mathcal{S}^{\{2,3\}}$. The mixed-cell configurations are depicted by drawing their cells as $\sum_{i \in I} C_i$.

Example 7.2. Let $A_1 = A_2 = \dots = A_n$ all be the same pair $\{0, 1\} \subset \mathbb{R}^1$. Then the Cayley-embedding $\mathcal{C}(A_1, A_2, \dots, A_n)$ is $\Delta_1 \times \Delta_n$, the product of a line-segment and an n -simplex. The secondary polytope is known to be the n -dimensional permutohedron (see Lecture 9 of [Z] and Chapter 7 of [GKZ]), which is the Minkowski sum of $n(n-1)/2$ line-segments (i.e., a zonotope). These line-segments are the mixed secondary polytopes $\Xi(A_i, A_j)$ with $1 \leq i < j \leq n$. The mixed secondary polytopes $\Xi(A_i)$ with $i \in \{1, \dots, n\}$ of only one point configuration are (nonzero) points. According to Theorem 4.5, the mixed secondary polytopes $\Xi(A_i \mid i \in I)$ with $\#I > 2$ are all $\{\mathbf{0}\}$.

Fig. 1. $\Sigma(A_1, A_2, A_3)$.

Fig. 2. $\Xi(A_1)$.

Fig. 3. $\Xi(A_2)$.

Fig. 4. $\Xi(A_3)$.

Fig. 5. $\Xi(A_1, A_2, A_3)$. Fig. 6. $\Xi(A_1, A_3)$. Fig. 7. $\Xi(A_2, A_3)$. Fig. 8. $\Xi(A_1, A_2, A_3)$.

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