On the Maximum Number of Equilateral Triangles, I*

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Abstract. The following problem was posed by Erdős and Purdy: “What is the maximum number of equilateral triangles determined by a set of n points in $\mathbb{R}^d$?” New bounds for this problem are obtained for dimensions 2, 4, and 5. In addition it is shown that for $d = 2$ the maximum is attained by subsets of the regular triangle lattice.

1. Introduction

In 1946 Erdős [6] posed the seemingly innocent problem of finding the maximum number of unit segments that can be determined by a set of n points in the plane. Even though the solution has long been out of reach, the quest for solving the problem generated a considerable amount of research in combinatorial geometry (see [12] for a discussion of this problem, and [2] and [15] for the latest developments).

Since then many problems sharing the same spirit have been raised (see [12]); problems concerning the maximum number of geometric objects that can be determined by an n -point set in a euclidean space. In this article we address one of them.

“What is the maximum number of equilateral triangles that can be determined by n points in the plane?”

This problem was proposed several times by Erdős and Purdy [8]–[11], and it is also mentioned in [4].

For an arbitrary finite set P denote by $E(P)$ the number of triplets in P that are the vertices of an equilateral triangle, we define

$$E(n) = \max_{\substack{|P|=n \\ P \subseteq \mathbb{R}^2}} E(P).$$

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It is easy to see that $E(n) \leq \frac{2}{3}\binom{n}{2}$ since every pair of points in the plane are the vertices of at most two equilateral triangles; also by considering a suitable set of the regular triangular grid one can easily obtain $E(n) > \frac{1}{6}n^2$. In fact Elekes and Erdős [5] proved that for every triangle T , the maximum number of triangles similar to T determined by n points in the plane is $\Theta(n^2)$.

The aim of this paper is to refine these bounds. We show in Section 2 that

$$0.1955 \leq \frac{1}{3} - \frac{\sqrt{3}}{4\pi} \leq \liminf_{n \rightarrow \infty} \frac{E(n)}{n^2} \leq \frac{1}{4} = 0.25,$$

next we prove in Section 3 that $E(n)$ is achieved by subsets of the equilateral triangle lattice. Finally, in Section 4, we discuss the same problem on higher dimensions where we find lower bounds for dimensions 4 and 5, and a new upper bound for the five-dimensional space.

Throughout the paper we use the following notation. For any P finite set, $G = (P, \Delta(P))$ denotes the 3-uniform hypergraph with vertex set P , where $\Delta(P)$ is the set of triplets in P determining equilateral triangles. For $x \in P$ we denote by $\deg_{\Delta}(x)$ the number of triangles in $\Delta(P)$ with x as one of its vertices.

2. Bounds on the Number of Triangles

Although the order of magnitude of the function $E(n)$ is known to be quadratic, it is not even known whether $\lim_{n \rightarrow \infty} (E(n)/n^2)$ exists. The aim of this section is to give new bounds for $E(n)$.

For every $x \in P$ define $N_x = P \cap R_x(P) \setminus \{x\}$ where R_x denotes the counterclockwise rotation of a $\pi/3$ angle with center at x . We have that

$$\deg_{\Delta}(x) = |N_x| = |P \cap R_x(P)| - 1.$$

Theorem 1. $E(n) \leq \lfloor ((n-1)/2)^2 \rfloor$.

Proof. It is enough to prove that any n -point set P has a point w with $\deg_{\Delta}(w) \leq \lfloor (n-1)/2 \rfloor$. Consider points $x, y \in P$ whose distance realizes the diameter of P . First assume there is a point $z \in P$ such that xyz is an equilateral triangle. By maximality of the segment xy , P must be contained in the Reuleaux triangle T with vertices x, y , and z . Let $X = R_x(T) \cap T$, $Y = R_y(T) \cap T$, and $Z = R_z(T) \cap T$. Observe that $N_x \subset X$, $N_y \subset Y$, $N_z \subset Z$ and since $x \notin N_x$, $y \notin N_y$, $z \notin N_z$ we have that N_x, N_y , and N_z are pairwise disjoint. So

$$\deg_{\Delta}(x) + \deg_{\Delta}(y) + \deg_{\Delta}(z) = |N_x| + |N_y| + |N_z| \leq n,$$

hence at least one of the points x, y , or z has $\deg_{\Delta} \leq \lfloor n/3 \rfloor \leq \lfloor (n-1)/2 \rfloor$.

Now suppose no point in P forms an equilateral triangle together with x and y . It follows that $y \notin N_x$ and $x \notin N_y$. We shall prove by contradiction that $|N_x \cap N_y| \leq 1$. So we assume there are two points u and v in $N_x \cap N_y$. This means there are points $u_x, v_x, u_y, v_y \in P$ such that all triangles $xu_xu, xv_xv, yu_yu, yv_yv$ are equilateral. Notice that the segments u_yu_x and v_yv_x are obtained from yx by $(\pi/3)$ -counterclockwise

rotation with centers in u and v , respectively. So $v_x v_y$ and $u_x u_y$ are parallel and have the same length, i.e., $u_x v_x v_y u_y$ is a parallelogram. This gives us a contradiction since one of the diagonals of the parallelogram is longer than any of its sides, including those which by assumption have maximal length in P . Hence $|N_x \cap N_y| \leq 1$.

Therefore

$$|N_x| + |N_y| \leq |N_x \cup N_y| + 1 \leq |P \setminus \{x, y\}| + 1 = n - 1,$$

so either $|N_x| \leq \lfloor (n - 1)/2 \rfloor$ or $|N_y| \leq \lfloor (n - 1)/2 \rfloor$, i.e., one of x or y satisfies $\deg_\Delta \leq \lfloor (n - 1)/2 \rfloor$. $\square$

Denote by Λ the lattice generated by 1 and $e^{i\pi/3}$. The following result estimates the number of equilateral triangles determined by large clusters of points in Λ .

Theorem 2. *Let K be a compact set with finite perimeter and area $\sqrt{3}/2$. Define $f_K: \mathbb{C} \rightarrow \mathbb{C}$ as $f_K(z) = \text{Area}(K \cap R_z(K))$. If K_n is a similar copy of K intersecting Λ in exactly n points, then*

$$E(K_n \cap \Lambda) = \left(\frac{4}{9} \int_K f_K(z) dz \right) n^2 + O(n^{3/2}).$$

Proof. Given a compact set L with finite area and perimeter, we have that $|rL \cap \Lambda| = (2/\sqrt{3})r^2 \text{Area}(L) + O(r)$ (where rL denotes the magnification of L by a factor $r \geq 1$), thus

$$E(K_n \cap \Lambda) = \frac{1}{3} \sum_{x \in K_n \cap \Lambda} |N_x| = \frac{2\sqrt{3}}{9} \sum_{x \in K_n \cap \Lambda} [\text{Area}(K_n \cap R_x(K_n)) + O(\sqrt{n})].$$

It can be seen that all error terms are uniformly bounded, so

$$E(K_n \cap \Lambda) = \frac{4n^2}{9} \sum_{x \in K_n \cap \Lambda} \left[\frac{\sqrt{3}}{2n} \text{Area} \left(\frac{1}{\sqrt{n}} K_n \cap R_{x/\sqrt{n}} \left(\frac{1}{\sqrt{n}} K_n \right) \right) \right] + O(n^{3/2}).$$

Note that the sum above is a Riemann approximation for the function $f_{(1/\sqrt{n})K_n}$ on the region $(1/\sqrt{n})K_n$, thus

$$E(K_n \cap \Lambda) = \frac{4n^2}{9} \left(\int_{(1/\sqrt{n})K_n} f_{(1/\sqrt{n})K_n}(z) dz + O \left(\frac{1}{\sqrt{n}} \right) \right) + O(n^{3/2}),$$

now since $\text{Area}((1/\sqrt{n})K_n) = \sqrt{3}/2 + O(1/\sqrt{n})$ then

$$\int_{(1/\sqrt{n})K_n} f_{(1/\sqrt{n})K_n}(z) dz = \int_K f_K(z) dz + O \left(\frac{1}{\sqrt{n}} \right)$$

and the result follows. $\square$

The integral $\int_K f_K(z) dz$ can be precisely determined when K is an equilateral triangle, a regular hexagon, or a circular disk. For these sets we obtained $\frac{1}{6}$, $\frac{7}{36}$, and $\frac{1}{3} - \sqrt{3}/4\pi \approx 0.1955$ as the corresponding quadratic coefficients of $E(K_n \cap \Lambda)$. Hence, together with Theorem 1, we have the following bounds for $E(n)$:

$$\left(\frac{1}{3} - \frac{\sqrt{3}}{4\pi}\right)n^2 + O(n^{3/2}) \leq E(n) \leq \frac{1}{4}n^2.$$

Concerning the lower bound, it seems that the circular disk maximizes $\int_K f_K(z) dz$ over all compact sets K with finite perimeter and area $\sqrt{3}/2$; furthermore we conjecture that

Conjecture 1. $\lim_{n \rightarrow \infty} (E(n)/n^2)$ exists, and its value is $\frac{1}{3} - \sqrt{3}/4\pi$.

We also believe that the following stronger conjecture is true.

Conjecture 2. For any n -point set P , there is $x \in P$ with $\deg_\Delta(x) \leq (\frac{2}{3} - \sqrt{3}/2\pi)(n + \sqrt{n})$.

3. Extremal Sets

The best evidence supporting Conjecture 1 is the next theorem, where we prove that to determine $E(n)$, it is sufficient to restrict our attention to subsets P of the equilateral triangle lattice Λ .

Theorem 3. $E(n)$ is attained by subsets of the regular triangular lattice.

Proof. Consider a system of coordinates with axis forming a $\pi/3$ angle (see Fig. 1). Take

Fig. 1. Λ and the shaded regions.

a set $P = \{p_1, p_2, \dots, p_n\}$ of n points in the plane ($p_j = (x_j, y_j)$ for all $j = 1, 2, \dots, n$). Assume without loss of generality that the minimum distance between points of P is $\sqrt{3}/2$.

By the Dirichlet Simultaneous Approximation Theorem, there is an integer $1 \leq t \leq 4^n$ such that, for every $1 \leq j \leq n$,

$$0 < \{tx_j + \frac{1}{4}\} < \frac{1}{2} \quad \text{and} \quad 0 < \{ty_j + \frac{1}{4}\} < \frac{1}{2},$$

where $\{x\}$ denotes the fractional part of x .

Let $p'_j = tp_j + (\frac{1}{4}, \frac{1}{4})$ for $1 \leq j \leq n$, and set $P' = \{p'_1, p'_2, \dots, p'_n\}$. By the choice of t all points of P' are in the shaded regions indicated in Fig. 1. Moreover since any two points in P' are at a distance at least $\sqrt{3}/2$ (by assumption on the minimum distance in P), then each connected region contains at most one point of P' . In particular, all points of P' lie in the interior of a minimal triangle of Λ pointing upward. Now consider the set $Q = \{q_1, q_2, \dots, q_n\}$ of n points in the plane where $q_j = (\lfloor tx_j + \frac{1}{4} \rfloor, \lfloor ty_j + \frac{1}{4} \rfloor)$. Notice that the function $p'_j \mapsto q_j$ is one to one, and that all points of Q lie on Λ .

To conclude the proof of the theorem we verify that $E(Q) \geq E(P)$. Clearly $E(P) = E(P')$. Now suppose that the points p'_j, p'_k, p'_l are the vertices of an equilateral triangle. Let T_j and T_k be the minimal triangles of Λ to which p'_j and p'_k belong (see Fig. 2). Then the locus of all points of the plane forming an equilateral triangle with one interior point of T_j and one interior point of T_k is the union of the interiors of the triangles T and T' in Fig. 2, where a and b are the unique two points forming equilateral triangles with q_j and q_k . Since p'_l must be in the interior of a minimal equilateral triangle of Λ pointing upward, then p'_l must lie in the interior of the shaded region indicated in Fig. 2. Therefore either $q_l = a$ or $q_l = b$, which means that the points q_j, q_k, q_l are the vertices of an equilateral triangle, and hence $E(Q) \geq E(P')$. $\square$

Fig. 2. Illustration for the proof of Theorem 3.

4. Higher Dimensions

For $d \geq 2$ let $E_d(n)$ denote the maximum number of equilateral triangles determined by n points in $\mathbb{R}^d$. Similarly define $F_d(n)$ where the maximum is restricted to the number of unit equilateral triangles.

A generalized Lenz's construction [8] shows that n^3 is actually the correct order of magnitude of both functions $E_d(n)$ and $F_d(n)$ when $d \geq 6$. Regarding the unitary case for $d = 2$ the best results are $n^{1+c/\log \log n} \leq F_2(n) \leq O(n^{4/3})$ (both bounds follow from the unitary distance problem [6], [14]). For $d = 3, 4$, the upper bounds $E_3(n) = O(n^{2.2})$, $F_3(n) = O(n^{1.8+\epsilon})$, and $F_4(n) = O(n^{65/23})$ were proven by Akutsu et al. [1]. Purdy [13] proved as a special case of a more general result that $E_4(n) = O(n^{2.96})$.

Now we give an upper bound for $E_5(n)$ which in turn improves the bound above for $E_4(n)$.

Theorem 4. $E_5(n) = O(n^{3-1/9})$.

Proof. Suppose that for a sufficiently large n there is an n -point set P determining at least $n^{3-1/9}$ equilateral triangles. Consider the 3-uniform hypergraph $G = (P, \Delta(P))$. By Erdős's theorem [7], G contains a subhypergraph H isomorphic to $K_3^{(3)}$ (the 3-uniform hypergraph consisting of nine vertices partitioned into three equal size classes, and whose set of 3-edges is formed by all triangles with exactly one vertex in each class).

Assume $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3$ are the vertices of H and all triplets of the form $\{x_j, y_k, z_l\}$ are the corresponding 3-edges. Assume without loss of generality that all sides of triangles in H have unit length.

Then for all $j \in \{1, 2, 3\}$ the set $\{y_1, y_2, y_3, z_1, z_2, z_3\}$ is contained in the four-dimensional unit sphere S_{x_j} with center in x_j . This implies that the three-dimensional spheres $S_{1,2} := S_{x_1} \cap S_{x_2}$ and $S_{1,3} := S_{x_1} \cap S_{x_3}$ also contain the set $\{y_1, y_2, y_3, z_1, z_2, z_3\}$. Note that the spheres $S_{1,2}$ and $S_{1,3}$ are distinct, since their centers are the middle points of the segments x_1x_2 and x_1x_3 , respectively. Hence $S := S_{1,2} \cap S_{1,3}$ is a two-dimensional sphere containing $\{y_1, y_2, y_3, z_1, z_2, z_3\}$. Let $V_{1,2}$ be the set of points in S equidistant to y_1 and y_2 , and similarly define $V_{1,3}$. Then $V_{1,2}$ and $V_{1,3}$ are distinct circles contained in S . To get a contradiction observe that $\{z_1, z_2, z_3\} \subseteq V_{1,2} \cap V_{1,3}$ but $|V_{1,2} \cap V_{1,3}| \leq 2$. $\square$

To complete this section we present examples in dimensions 4 and 5 with a large number of unit equilateral triangles.

Theorem 5. $F_4(n) = \Omega(n^2)$ and $F_5(n) = \Omega(n^{7/3})$.

Proof. Let $A \subseteq \mathbb{R}^2$ be the set of vertices of $\lfloor n/8 \rfloor$ unit squares centered at the origin. Define $P \subseteq \mathbb{R}^2 \times \mathbb{R}^2$ by $P = (A \times \{0\}) \cup (\{0\} \times A)$. Notice that if a_1a_2 is a unit segment in A , then the triangle $(a, 0), (0, a_1), (0, a_2)$ is equilateral for any $a \in A$. Therefore

$$F_4(n) \geq F(P) \geq 32 \left\lfloor \frac{n}{8} \right\rfloor^2 = \Omega(n^2).$$

For the second part consider the sphere $S \subseteq \mathbb{R}^3$ of radius $1/\sqrt{2}$ centered at the origin. Clarkson et al. [3] proved the existence of an $\lfloor n/2 \rfloor$ -point set $B \subseteq S$ which determines $\Omega(n^{4/3})$ unit distances. Define the set $Q = (A \times \{0\}) \cup (\{0\} \times B) \subseteq \mathbb{R}^2 \times \mathbb{R}^3$. Again if $b_1 b_2$ is a unit segment in B , then the triangle $(a, 0), (0, b_1), (0, b_2)$ is equilateral for any $a \in A$. Therefore

$$F_5(n) \geq F(Q) \geq n \cdot \Omega(n^{4/3}) = \Omega(n^{7/3}). \quad \square$$

References

1. T. Akutsu, H. Tamaki, and T. Tokuyama. Distribution of Distances and Triangles in a Point Set and Algorithms for Computing the Largest Common Point Sets. *Discrete & Computational Geometry* **20** (1998), 307–331.
2. P. Brass. On Point Sets with Many Unit Distances in Few Directions. *Discrete & Computational Geometry* **19** (1998), 355–366.
3. K. L. Clarkson, H. Edelsbrunner, L. J. Guibas, M. Sharir, and E. Welzl. Combinatorial Complexity Bounds for Arrangements of Curves and Spheres. *Discrete & Computational Geometry* **5** (1990), 99–160.
4. H. T. Croft, K. J. Falconer, and R. K. Guy. *Unsolved Problems in Geometry*. Springer-Verlag, New York, 1991.
5. G. Elekes and P. Erdős. Similar Configurations and Pseudogrids. In *Intuitive Geometry*. Colloquia Mathematica Societatis János Bolyai. North-Holland, Amsterdam, 1994, pp. 85–104.
6. P. Erdős. On Sets of Distances of n Points. *American Mathematical Monthly* **53** (1946), 248–250.
7. P. Erdős. On Some Problems of Graphs and Generalized Graphs. *Israel Journal of Mathematics* **2** (1964), 183–190.
8. P. Erdős. On Some Problems of Elementary and Combinatorial Geometry. *Annali di Matematica Pura Applicata Serie Quarta* **103** (1975), 99–108.
9. P. Erdős and G. Purdy. Some Extremal Problems in Geometry. *Journal of Combinatorial Theory* **10** (1971), 246–252.
10. P. Erdős and G. Purdy. Some Extremal Problems in Geometry, III. *Proc. 6th Southeastern Conference on Combinatorics, Graph Theory and Computation*, 1975, pp. 291–308.
11. P. Erdős and G. Purdy. Some Extremal Problems in Geometry, IV. *Proc. 7th Southeastern Conference on Combinatorics, Graph Theory and Computation*, 1976, pp. 307–322.
12. J. Pach and P. K. Agarwal. *Combinatorial Geometry*. Wiley, New York, 1995.
13. G. Purdy. Repeated Angles in E^4 . *Discrete and Computational Geometry* **3** (1988), 73–75.
14. J. Spencer, E. Szemerédi, and W. Trotter, Jr. Unit Distances in the Euclidean Plane. In *Graph Theory and Combinatorics* (B. Bollobás, ed.). Academic Press, New York, 1984, pp. 293–303.
15. L. A. Székely. Crossing Numbers and Hard Erdős Problems in Discrete Geometry. *Combinatorics, Probability & Computing* **7** (1997), 353–358.

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