On Grünbaum's Conjecture about Inner Illumination of Convex Bodies*

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Abstract. In 1964 Grünbaum conjectured that any primitive set illuminating from within a convex body in E^d , $d \geq 3$, has at most 2^d points. This was confirmed by V. Soltan in 1995 for the case $d = 3$. Here we give a negative answer to Grünbaum's conjecture for all $d \geq 4$, by constructing a convex body $K \subset E^d$ with primitive illuminating sets of an arbitrarily large cardinality.

1. Introduction and Main Results

In what follows, K means a convex body (i.e., a compact convex set with nonempty interior) in the d -dimensional linear space E^d . By $\text{bd } K$ and $\text{int } K$ we denote the boundary and the interior of K , respectively; $\text{rint } F$ means the relative interior of a convex set $F \subset E^d$ and $\text{conv } G$ denotes the convex hull of a set $G \subset E^d$; $[x, z]$ and $]x, z[$ stand for the closed and open line segments with the endpoints x and z , and $\langle x, z \rangle$ denotes the scalar product of x and z .

The concept of illumination of a convex body from within (inner illumination, in other terminology) was introduced by P. Soltan [8]. A set $F \subset \text{bd } K$ *illuminates* a convex body K *from within* provided for every point $x \in \text{bd } K$ there is a point $y \in F$ distinct from x such that the open line segment $]x, y[$ is contained in $\text{int } K$. Soltan [8] proved that any

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convex body K in E^d , $d \geq 1$, can be illuminated from within by at most $d + 1$ points, with exactly $d + 1$ points if and only if K is a simplex. (See the monograph [1] for a wider discussion on inner and exterior illumination of convex bodies.)

Following Grünbaum [3], a set F illuminating a convex body K from within is called *primitive* if no proper subset of F illuminates K from within. For short, we will say that F is a *primitive illuminating set* of K or, equivalently, that F *primitively illuminates* K . Grünbaum [3] (see also p. 423 of [4]) suggested the question about the maximum number of points in a primitive illuminating set of a convex body $K \subset E^d$. This maximum is easily shown to be 4 for $d = 2$ (see p. 423 of [4]). Although even a proof for the existence of the maximum for $d \geq 3$ was lacking, Grünbaum [3] conjectured that any primitive illuminating set of a convex body in E^d , $d \geq 3$, has at most 2^d points.

V. Soltan [9] confirmed this conjecture for the case $d = 3$. More exactly, he proved that any primitive illuminating set of a convex body in E^3 has at most eight points, and only convex polytopes combinatorially equivalent to the 3-cube have primitive illuminating sets of eight points (placed at their vertices).

In this paper we give a negative answer to Grünbaum's conjecture for all $d \geq 4$, by proving the following assertion.

Theorem 1. *For any positive integers $m \geq 2$ and $d \geq 4$, there exists a convex body $K_m \subset E^d$ with a primitive illuminating set of at least m points.*

The respective bodies are obtained by a slight modification of cyclic polytopes in E^d . Namely, given any even number of points $z_1, \dots, z_{2r}$ on the moment curve in E^d , we extend the line segments $[z_1, z_2], \dots, [z_{2r-1}, z_{2r}]$ to very thin spindle-shaped bodies $V_{1,2}, \dots, V_{2r-1,2r}$ such that the convex hull of $V_{1,2} \cup \dots \cup V_{2r-1,2r}$ is a convex body primitively illuminated by $z_1, \dots, z_{2r}$ and the points z_{2i-1}, z_{2i} illuminate each other for all $i = 1, \dots, r$ (for the technical details, see the proof of Theorem 1).

A more elaborated technique of this modification enables us to prove even the following theorem.

Theorem 2. *For each $d \geq 4$, there is a convex body $K \subset E^d$ such that for any positive integer m it has a primitive illuminating set of at least m points.*

In connection with this result, one could ask whether there exist convex bodies having infinite primitive illuminating sets. The following simple proposition shows that this is impossible.

Proposition 1. *Any primitive illuminating set of a convex body $K \subset E^d$ is finite.*

A more restrictive notion of inner illumination was introduced by Hadwiger [5]: a convex d -polytope $P \subset E^d$ is said to be *illuminated* by its *vertices* provided for any vertex x of P there is another vertex z of P such that $[x, z] \subset \text{int } P$. Hadwiger [5] asked whether a convex d -polytope in E^d illuminated by its vertices has at least $2d$ vertices. Mani [6] proved that for $d \leq 7$ the answer to Hadwiger's question is affirmative, while for higher dimensions he showed the existence of a convex d -polytope illuminated by its vertices and having about $d + 2\sqrt{d}$ vertices.

Proposition 2. *A convex d -polytope $P \subset E^d$ primitively illuminated by its vertices has at least $2d$ vertices.*

At the end of this section we pose the following problem.

Problem. Does there exist a least positive integer $k(d)$, $d \geq 1$, such that any convex d -polytope in E^d primitively illuminated by its vertices has at most $k(d)$ vertices?

Clearly, $k(1) = 2$, $k(2) = 4$, and from [9] it follows that $k(3) = 8$.

2. Proofs

To prove Theorems 1 and 2 we need some auxiliary lemmas. Recall that a subset Q of a convex body K is called *extreme* provided for any points $x, z \in K$ the property $]x, z[\cap Q \neq \emptyset$ implies $x, z \in Q$. Clearly, both the extreme subset Q and its complement $K \setminus Q$ are convex sets. The following lemma is obvious.

Lemma 1. *For any boundary point x of a convex body $K \subset E^d$, the set*

$$Q_x := \bigcup \{[v, w] : x \in]v, w[\subset K\}$$

is the minimal (under inclusion) extreme subset of K containing x in its relative interior: $x \in \text{rint } Q_x$.

Lemma 2. *Let $K \subset E^d$ be a convex body and let G be a subset of $\text{bd } K$ such that $K = \text{conv } G$. A set $F \subset \text{bd } K$ illuminates K from within if and only if F illuminates G , i.e., if for any point $g \in G$ there is a point $f \in F$ satisfying $]f, g[\subset \text{int } K$.*

Proof. Since the “only if” part of the assertion is trivial, it remains to prove the “if” part. Let G be a subset of $\text{bd } K$ such that $K = \text{conv } G$, and let a set $F \subset \text{bd } K$ illuminate G .

Choose an arbitrary point $x \in \text{bd } K$. If $x \in G$, then x is illuminated by a point from F . Assume that $x \notin G$, and denote by Q_x the least under inclusion extreme subset of K containing x in its relative interior (see Lemma 1). The set Q_x contains a point $g \in G$, since otherwise $\text{conv } G \subset K \setminus Q_x$, contrary to $K = \text{conv } G$. Let $f \in F$ be a point illuminating g . Due to the construction of Q_x , there is a point $h \in Q_x$ such that $x \in]g, h[$. From here and the inclusion $]f, g[\subset \text{int } K$ it easily follows that $]f, x[\subset \text{int } K$. Hence F illuminates K from within. $\square$

Two more lemmas deal with some properties of the moment curve $M \subset E^d$, defined parametrically by $x(t) = (t, t^2, \dots, t^d)$, $t \geq 0$. It is well known (see, e.g., [2] and p. 61 of [4]) that for a given positive integer k , with $2k \leq d$, any k points in M determine a $(k-1)$ -dimensional face of the convex set $\text{conv } M$. We need a particular case of this assertion, namely $k = 1$.

Lemma 3. Let $z' = x(\eta)$ and $z'' = x(\theta)$, $0 < \eta \leq \theta$, be two points on the moment curve $M \subset E^d$, $d \geq 4$. Then the half-space $H(z', z'')$ defined by

$$\left. \begin{aligned} H(z', z'') &= \{v \in E^d : \langle v, b \rangle \leq \beta\}, & b &= (b_1, \dots, b_d), \\ b_1 &= -2\eta\theta(\eta + \theta), & b_2 &= \eta^2 + 4\eta\theta + \theta^2, & b_3 &= -2(\eta + \theta), \\ b_4 &= 1, & b_5 &= \dots = b_d = 0, & \beta &= -(\eta\theta)^2, \end{aligned} \right\} \quad (1)$$

supports M such that $H(z', z'') \cap M = \{z', z''\}$.

Proof. Indeed, for any point $z = (t, t^2, \dots, t^d) \in M$ one has

$$\langle z, b \rangle - \beta = b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 - \beta = (t - \eta)^2 (t - \theta)^2.$$

Since the last expression is nonnegative and equals 0 if and only if $t = \eta$ or $t = \theta$, we conclude that M lies in the closed half-space $\{v \in E^d : \langle v, b \rangle \geq \beta\}$ complementary to $H(z', z'')$ such that $H(z', z'') \cap M = \{z', z''\}$. $\square$

Lemma 4. Let $z_1 = x(t_1)$, $z_2 = x(t_2)$, $z_0 = x(t_0)$, $0 < t_1 < t_2 < t_0$, be three points on the moment curve $M \subset E^d$, $d \geq 4$. There is a real number $\varepsilon > 0$ such that the ball with radius ε and center z_1 is disjoint to any half-space $H(z', z'')$ of the form (1), with $t_2 \leq \eta \leq \theta \leq t_0$.

Proof. Clearly, the distance from z_1 to $H(z', z'')$ equals $(\langle z_1, b \rangle - \beta) / \|b\|$. One has

$$\langle z_1, b \rangle - \beta = (t_1 - \eta)^2 (t_1 - \theta)^2 \geq (t_1 - t_2)^4$$

and

$$\begin{aligned} \|b\| &= \langle b, b \rangle^{1/2} = (4\eta^2\theta^2(\eta + \theta)^2 + (\eta^2 + 4\eta\theta + \theta^2)^2 + 4(\eta + \theta)^2 + 1)^{1/2} \\ &\leq (16t_0^6 + 36t_0^4 + 16t_0^2 + 1)^{1/2}. \end{aligned}$$

Hence

$$\mu := \frac{(t_1 - t_2)^4}{(16t_0^6 + 36t_0^4 + 16t_0^2 + 1)^{1/2}} \leq \frac{\langle z_1, b \rangle - \beta}{\|b\|},$$

and any real number $\varepsilon \in]0, \mu[$ satisfies the condition of Lemma 2. $\square$

In the proof of Theorems 1 and 2 we use the following notation. For any points z_i, z_j on the moment curve M , $w_{i,j}$ denotes the middle point of the line segment $[z_i, z_j]$. If $\varepsilon_{i,j}$ is a real positive number, then $U_{i,j}$ means the closed ball with center $w_{i,j}$ and radius $\varepsilon_{i,j}$, and $C_{i,j}$ denotes the open convex cone with apex z_i generated by $\text{int } U_{i,j}$:

$$C_{i,j} = \{(1 - \lambda)z_i + \lambda v : v \in \text{int } U_{i,j}, \lambda > 0\}. \quad (2)$$

Put

$$V_{i,j} = \text{conv}(U_{i,j} \cup [z_i, z_j]). \quad (3)$$

In what follows, $H(z', z'')$ stands for a half-space of the form (1).

Proof of Theorem 1. Let m be any positive integer and let $z_1 = x(t_1), \dots, z_{2r} = x(t_{2r})$, $0 < t_1 < \dots < t_{2r}$, be any $2r$ points on the moment curve $M \subset E^d$, with $2r \geq m$. Due to Lemma 3, the middle point $w_{p,q}$ of a line segment $[z_p, z_q]$, $1 \leq p < q \leq 2r$, lies outside any half-space $H(z_i, z_j)$, $1 \leq i < j \leq 2r$, $\{i, j\} \neq \{p, q\}$. Hence there is a real number $\varepsilon_{p,q}$ such that the closed ball $U_{p,q}$ with center $w_{p,q}$ and radius $\varepsilon_{p,q}$ is disjoint to $H(z_i, z_j)$ for all i, j satisfying $1 \leq i < j \leq 2r$, $\{i, j\} \neq \{p, q\}$.

We claim that the set $Z := \{z_1, \dots, z_{2r}\}$ primitively illuminates the convex body

$$K := \text{conv}(V_{1,2} \cup V_{3,4} \cup \dots \cup V_{2r-1,2r}).$$

First we observe that any line segment $[z_i, z_j]$, $1 \leq i < j \leq 2r$, with $\{i, j\}$ not in

$$\Omega := \{\{1, 2\}, \{3, 4\}, \dots, \{2r-1, 2r\}\},$$

is a one-dimensional face of K (due to the choice of the numbers $\varepsilon_{p,q}$). This implies that two points $z_m, z_n \in Z$ illuminate each other if and only if $\{m, n\} \in \Omega$. Hence no proper subset of Z illuminates Z and, subsequently, no proper subset of Z illuminates K from within.

It remains to show that Z illuminates the whole body K . Indeed, it is easily seen that any convex body $V_{i,i+1}$, $\{i, i+1\} \in \Omega$, is illuminated from within by the set $\{z_i, z_{i+1}\}$. As a result, the set

$$G := \bigcup \{\text{bd } K \cap V_{i,i+1} : \{i, i+1\} \in \Omega\}$$

is illuminated by Z . Since $K = \text{conv } G$, we get from Lemma 2 that Z illuminates the whole body K . $\square$

Proof of Theorem 2. The required convex body $K \subset E^d$ will be represented as the union of an increasing sequence $K_1 \subset \dots \subset K_r \subset \dots$ of convex bodies like those in the proof of Theorem 1. The bodies $K_1, \dots, K_r, \dots$ are constructed recursively, by induction on r .

Choose two distinct points $z_1 = x(t_1)$ and $z_0 = x(t_0)$, $0 < t_1 < t_0$, on the moment curve $M \subset E^d$, and let $M_0 \supset M_1 \supset \dots$ be a decreasing sequence of arcs of M such that $M_0 = \{x(t) : t_1 \leq t \leq t_0\}$ and $\bigcap \{M_i : i \geq 0\} = \{z_0\}$.

Initial step. The first body K_1 is determined by four points

$$z_1 = x(t_1), \quad z_2 = x(t_2), \quad z_3 = x(t_3), \quad z_0 = x(t_0), \quad 0 < t_1 < t_2 < t_3 < t_0,$$

on the moment curve M . Namely, put

$$K_1 := \text{conv}(V_{1,0} \cup V_{2,3}),$$

where $V_{1,0}$ and $V_{2,3}$ are as in (3) and real numbers $\varepsilon_{1,0} > 0$ and $\varepsilon_{2,3} > 0$ are defined as follows:

(a₁) $\varepsilon_{1,0}$ is the radius of a ball $U_{1,0}$ disjoint to any of the half-spaces

$$H(z_1, z_2), \quad H(z_1, z_3), \quad \text{and} \quad H(x(\eta), x(\theta)), \quad t_2 \leq \eta \leq \theta \leq t_0,$$

and it is so small that the intersection $M \cap C_{1,0}$ is a subset of M_1 and contains none of z_2, z_3 ;

(a₂) $\varepsilon_{2,3}$ is the radius of a ball $U_{2,3}$ disjoint to any of the half-spaces

$$H(z_i, z_j), \quad i, j = 0, 1, 2, 3, \quad \{i, j\} \neq \{2, 3\},$$

$$\text{and } H(x(\eta), x(\theta)), \quad t_3 \leq \eta \leq \theta \leq t_0.$$

The existence of $\varepsilon_{1,0}$ and $\varepsilon_{2,3}$ follows from Lemmas 3 and 4. Indeed, let $\varepsilon > 0$ be a real number such that the ball with center z_1 and radius ε is disjoint to any half-space $H(x(\eta), x(\theta))$, $t_2 \leq \eta \leq \theta \leq t_0$. Since both z_1 and z_0 lie in all closed half-spaces complementary to these half-spaces, the middle point $w_{1,0}$ of the line segment $[z_1, z_0]$ is at a distance at least $\varepsilon/2$ from any $H(x(\eta), x(\theta))$, $t_2 \leq \eta \leq \theta \leq t_0$. Now, by Lemma 3, one can choose $\varepsilon_{1,0} \in]0, \varepsilon/2[$ such that the ball $U_{1,0}$ with center $w_{1,0}$ and radius $\varepsilon_{1,0}$ is disjoint to both half-spaces $H(z_1, z_2)$ and $H(z_1, z_3)$. The existence of $\varepsilon_{2,3}$ can be shown similarly.

It is easily seen (compare with the proof of Theorem 1) that the points z_0, z_1, z_2, z_3 primitively illuminate K_1 , and z_p illuminates z_q , $p, q = 0, 1, 2, 3$, if and only if $\{p, q\} = \{0, 1\}$ or $\{p, q\} = \{2, 3\}$.

Inductive step. Assume that for a given positive integer $r (\geq 2)$ some $3r - 2$ points

$$z_1 = x(t_1), \quad \dots, \quad z_{3r-3} = x(t_{3r-3}), \quad z_0 = x(t_0) \in M, \quad t_1 < \dots < t_{3r-3} < t_0,$$

and some real positive numbers $\varepsilon_{1,0}, \varepsilon_{2,3}, \varepsilon_{4,0}, \varepsilon_{5,6}, \dots, \varepsilon_{3r-5,0}, \varepsilon_{3r-4,3r-3}$ are chosen such that:

- (b₁) $M \cap C_{3i-2,0}$ is a subset of M_{3i-2} , $i = 1, \dots, r - 1$,
- (b₂) the points $z_1, \dots, z_{3i}$ lie in the set $M_0 \setminus C_{3i-2,0}$, $i = 1, \dots, r - 1$,
- (b₃) the points $z_{3i+1}, \dots, z_{3r-3}$ lie in the set $M \cap C_{3i-2,0}$, $i = 1, \dots, r - 2$,
- (b₄) $\varepsilon_{3i-2,0}$ is the radius of a ball $U_{3i-2,0}$, $i = 1, \dots, r - 1$, disjoint to any of the half-spaces

$$H(z_p, z_q), \quad p, q = 0, 1, \dots, 3i, \quad \{p, q\} \neq \{0, 3i - 2\},$$

as well as to any of the half-spaces $H(x(\eta), x(\theta))$, $t_{3i-1} \leq \eta \leq \theta \leq t_0$,

- (b₅) $\varepsilon_{3i-1,3i}$ is the radius of a ball $U_{3i-1,3i}$, $i = 1, \dots, r - 1$, disjoint to any of the half-spaces

$$H(z_p, z_q), \quad p, q = 0, 1, \dots, 3i, \quad \{p, q\} \neq \{3i - 1, 3i\},$$

as well as to any of the half-spaces $H(x(\eta), x(\theta))$, $t_{3i} \leq \eta \leq \theta \leq t_0$.

Now we choose three new points

$$z_{3r-2} = x(t_{3r-2}), \quad z_{3r-1} = x(t_{3r-1}), \quad z_{3r} = x(t_{3r}), \quad t_{3r-3} < t_{3r-2} < t_{3r-1} < t_{3r} < t_0,$$

and real positive numbers $\varepsilon_{3r-2,0}, \varepsilon_{3r-1,3r}$ such that:

- (c₁) $M \cap C_{3r-2,0}$ is a subset of M_{3r-2} ,
- (c₂) the points $z_1, \dots, z_{3r}$ lie in the set $M_0 \setminus C_{3r-2,0}$,
- (c₃) the points $z_{3r-2}, z_{3r-1}, z_{3r}$ lie in the set $M \cap C_{3r-2,0}$,

(c₄) $\varepsilon_{3r-2,0}$ is the radius of a ball $U_{3r-2,0}$ disjoint to any of the half-spaces

$$H(z_p, z_q), \quad p, q = 0, 1, \dots, 3r, \quad \{p, q\} \neq \{0, 3r-2\},$$

as well as to any of the half-spaces $H(x(\eta), x(\theta)), t_{3r-1} \leq \eta \leq \theta \leq t_0$,

(c₅) $\varepsilon_{3r-1,3r}$ is the radius of a ball $U_{3r-1,3r}$ disjoint to any of the half-spaces

$$H(z_p, z_q), \quad p, q = 0, 1, \dots, 3r, \quad \{p, q\} \neq \{3r-1, 3r\},$$

as well as to any of the half-spaces $H(x(\eta), x(\theta)), t_{3r} \leq \eta \leq \theta \leq t_0$.

As a result (compare with the proof of Theorem 1), the convex body

$$K_r := \text{conv}(V_{1,0} \cup V_{2,3} \cup V_{4,0} \cup V_{5,6} \cup \dots \cup V_{3r-2,0} \cup V_{3r-1,3r})$$

is primitively illuminated by any of the sets

$$Z_i := \{z_0, z_{3i-2}\} \cup \{z_2, z_3; z_5, z_6; \dots; z_{3i-1}, z_{3i}\}, \quad i = 1, \dots, r,$$

such that two points $z_p, z_q \in Z_i$ illuminate each other if and only if $\{p, q\}$ is one of the pairs $\{0, 3i-2\}, \{2, 3\}, \{5, 6\}, \dots, \{3i-1, 3i\}$.

Now put $K = K_1 \cup K_2 \cup \dots$. From the construction above it follows that K is a convex body illuminated from within by the set $Z := \{z_0, z_1, \dots\}$ such that:

- (d₁) any point z_{3m-2} , $m \geq 1$, illuminates z_0 and all points z_n , $n \geq 3m+1$,
- (d₂) for any $m \geq 1$ the points z_{3m-1} and z_{3m} illuminate each other,
- (d₃) for any pair of points $z_p, z_q \in Z$ not mentioned in (d₁) and (d₂), i.e., such that

$$\{p, q\} \notin \{\{0, 3m-2\}, \{3m-2, n\}, m \geq 1, n \geq 3m+1\} \cup \{\{3m-1, 3m\}, m \geq 1\},$$

the line segment $[z_p, z_q]$ is in the boundary of K .

Hence for any positive integer $m \geq 1$ the set

$$\{z_0, z_{3m-2}\} \cup \{z_2, z_3; z_5, z_6; \dots; z_{3m-1}, z_{3m}\}$$

of cardinality $2m+2$ primitively illuminates K . □

Proof of Proposition 1. It is sufficient to show that any set F illuminating a convex body K from within has a finite subset also illuminating K from within. To do this, consider for any point $x \in F$ the set $I(x)$ of all points in $\text{bd } K$ illuminated by x :

$$I(x) := \{z \in \text{bd } K :]x, z[\subset \text{int } K\}.$$

We claim that $I(x)$ is a relatively open set in the boundary of K . Indeed, let v be any point in $I(x)$, and denote by w the middle point of the open line segment $]x, v[$. Since $w \in \text{int } K$, there is an open ball $U \subset K$ with center w . Clearly, the set $N(v) = \{z \in \text{bd } K :]x, z[\cap U\}$ is a relatively open neighborhood of v in $\text{bd } K$.

The fact that F illuminates K from within means that the family of sets $\{I(x) : x \in F\}$ covers $\text{bd } K$. By the compactness argument, one can choose a finite subset F' of F such that $\text{bd } K = \bigcup \{I(x) : x \in F'\}$. In other words, the set F' illuminates K from within. □

Proof of Proposition 2. It is shown in [6] (see also [7]) that if a convex d -polytope $P \subset E^d$ is illuminated by its vertex set $V(P)$ in such a way that any vertex of P illuminates exactly one other vertex, then P has at least $2d$ vertices. Hence it is sufficient to prove that if P is primitively illuminated by $V(P)$, then any vertex of P illuminates exactly one other vertex.

Indeed, assume for contradiction that there are three vertices $v_1, v_2, v_3 \in V(P)$ such that v_2 illuminates both v_1 and v_3 . Since the set $V(P)$ primitively illuminates P and $\{v_1, v_2\}$ illuminates $\{v_1, v_2, v_3\}$, there is a new vertex $v_4 \in V(P) \setminus \{v_1, v_2, v_3\}$ illuminated by v_3 and not illuminated by any of v_1, v_2 . By a similar argument, for any positive integer $r \geq 3$ there is a vertex $v_{r+1} \in V(P) \setminus \{v_1, \dots, v_r\}$ illuminated by v_r and not illuminated by any of $v_1, \dots, v_{r-1}$. The last is in contradiction to the finiteness of $V(P)$. $\square$

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