Infinitesimal Rigidity of Convex Polytopes *

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Abstract. Aleksandrov [1] proved that a simple convex d -dimensional polytope, $d \geq 3$, is infinitesimally rigid if the volumes of its facets satisfy a certain assumption of stationarity. We extend this result by proving that this assumption can be replaced by a stationarity assumption on the k -dimensional volumes of the polytope's k -dimensional faces, where $k \in \{1, \dots, d-1\}$.

1. Introduction and Main Result

Let $P \subset \mathbb{R}^d$ be a polytope with interior points in d -dimensional Euclidean space $\mathbb{R}^d$, $d \geq 3$. In particular, P belongs to the set $\mathcal{K}^d$ of convex bodies (nonempty, compact, convex sets) in $\mathbb{R}^d$. We equip $\mathbb{R}^d$ with the usual scalar product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. We denote the unit sphere in $\mathbb{R}^d$ by S^{d-1} .

Let $F_1, \dots, F_n$ be the facets, that is, the $(d-1)$ -dimensional faces of P , and let $u_1, \dots, u_n \in S^{d-1}$ be the corresponding outer unit normal vectors. If $h_i := h(P, u_i)$ denotes the value of the support function of P at u_i , then P can be represented as

$$P = \bigcap_{i=1}^n H^-(u_i, h_i),$$

where $H^-(u_i, h_i) := \{x \in \mathbb{R}^d \mid \langle x, u_i \rangle \leq h_i\}$.

We now consider polytopes of the form

$$P(x) := \bigcap_{i=1}^n H^-(u_i, x_i),$$

* This work is part of the author's thesis [2] written at the Albert-Ludwigs-Universität, Freiburg im Breisgau, Germany.

with $x = (x_1, \dots, x_n) \in \mathbb{R}^d$. If x lies in a sufficiently small neighborhood of $h := (h_1, \dots, h_n)$, then $P(x)$ has the same number of facets as P . Furthermore, for each $i = 1, \dots, n$, the facet $F_i(x)$ of $P(x)$ is parallel to F_i .

In addition, we further assume that P is a *simple* polytope, that is, each vertex of P is contained in exactly d facets of P . Then there is a neighborhood $U(h)$ of h such that for all $x \in U(h)$ the polytopes $P(x)$ and P are analogous (see Lemma 2.4.12 of [8]). If the origin o is an interior point of P , then $U(h)$ can be chosen such that

$$U(h) = \prod_{i=1}^n]h_i - \varepsilon, h_i + \varepsilon[\quad \text{with} \quad \varepsilon \in]0, \min\{h_1, \dots, h_n\}[. \quad (1)$$

Hence, o is also an interior point of $P(x)$ for all $x \in U(h)$.

The term *analogous* means that for an arbitrary $u \in S^{d-1}$ the intersection of P with its supporting hyperplane in direction u has the same dimension as the intersection of $P(x)$ with its supporting hyperplane in the same direction. Furthermore, these faces are parallel. If F is a face of P we denote by $F(x)$ the corresponding face of $P(x)$. We write $\mathcal{F}_i(P)$ for the set of i -dimensional faces of P , where $i \in \{0, \dots, d-1\}$.

If $c = (c_1, \dots, c_d) \in \mathbb{R}^d$ is fixed, then the above shows that for all $t \in \mathbb{R}$ satisfying $\max\{|tc_1|, \dots, |tc_n|\} < \varepsilon$ the polytope $P(h + tc)$ is analogous to P and hence has facets $F_i(c + th)$, $i = 1, \dots, n$. If the $(d-1)$ -dimensional volume V_{d-1} of each facet (considered as a function of t) satisfies the equation

$$\left. \frac{d}{dt} \right|_{t=0} V_{d-1}(F_i(h + tc)) = 0, \quad (2)$$

then $c = (\langle a, u_1 \rangle, \dots, \langle a, u_n \rangle)$ with $a \in \mathbb{R}^d$. This was proved by Aleksandrov [1, pp. 407–411]. We then say that c is the *support vector of a point*. Aleksandrov's result shows that

$$P(h + tc) = \bigcap_{i=1}^n H^-(u_i, h_i + t\langle a, u_i \rangle) = P + ta.$$

Consequently, $P(h + tc)$ and P are equal up to translation.

For t with $\max\{|tc_1|, \dots, |tc_n|\} < \varepsilon$ the polytope $P(h + tc)$ may be viewed as a deformation of the polytope P . Thus, Aleksandrov has proved that a simple polytope satisfying (2) is *infinitesimally rigid*. This means that the deformation is a translation if the derivatives of the facets' volumes are vanishing at the beginning of the deformation (that is, at time $t = 0$).

In this work we generalize Aleksandrov's result. It was asked by Rolf Schneider (personal communication) whether in (2) one may replace the facets and $(d-1)$ -dimensional volume, respectively, by the k -faces and k -dimensional volume V_k for some fixed $k \in \{1, \dots, d-1\}$. That this may be done is proved by the following theorem.

Theorem 1. *Let $P \in \mathcal{K}^d$, $d \geq 3$, be a simple d -dimensional polytope. Let $u_1, \dots, u_n \in S^{d-1}$ be the pairwise disjoint outer normal vectors of P . Define the vector $h \in \mathbb{R}^n$ by $h := (h(P, u_1), \dots, h(P, u_n))$. For $k \in \{1, \dots, d-1\}$ and $c = (c_1, \dots, c_n) \in \mathbb{R}^n$*

assume that

$$\left. \frac{d}{dt} \right|_{t=0} V_k(F(h + tc)) = 0 \quad (3)$$

for all $F \in \mathcal{F}_k(P)$. Then $c = (\langle a, u_1 \rangle, \dots, \langle a, u_n \rangle)$ with $a \in \mathbb{R}^d$.

The proof of this result requires several preliminaries.

In Section 2 we investigate the volume of $P(x)$ as a function of $x \in U(h)$. Here, we study the relationship of the k -fold partial derivatives of the volume and the k -dimensional volumes of the k -faces of $P(x)$.

In Section 3 we relate the volume of $P(x)$ to the notion of k -stress which was introduced by Lee [5].

Section 4 is devoted to the proof of Theorem 1.

2. Simple Polytopes and Volume

In what follows, we, without loss of generality, always assume that o is an interior point of P and $U(h)$ is as in (1). For $x \in U(h)$ let $V(x) := V_d(P(x))$ be the d -dimensional volume of $P(x)$. It is proved in [8, Lemma 5.1.2], that V is a polynomial in x which is homogeneous of degree d . Consequently, for $k \in \{1, \dots, d\}$ and $i_1, \dots, i_k \in \{1, \dots, n\}$, the partial derivatives

$$\frac{\partial^k V}{\partial x_{i_k} \cdots \partial x_{i_1}}(x)$$

exist for all $x \in U(h)$.

The fact that the polytope P is simple implies that for each $(d - k)$ -dimensional face $F \in \mathcal{F}_{d-k}(P)$ there are exactly k facets $F_{i_1}, \dots, F_{i_k}$ of P such that $F = F_{i_1} \cap \dots \cap F_{i_k}$ (see Theorem 12.14 of [3]). The outer normal vectors $u_{i_1}, \dots, u_{i_k}$ belonging to these facets are linearly independent. Furthermore, for all $x \in U(h)$, the face $F(x)$ has the representation $F(x) = F_{i_1}(x) \cap \dots \cap F_{i_k}(x)$.

Our first objective is to clarify the connection between the $(d - k)$ -dimensional volume $V_{d-k}(F(x))$ of $F(x)$ and the partial derivative $(\partial^k V / \partial x_{i_k} \cdots \partial x_{i_1})(x)$. This is done in Theorem 4. In order to prove this theorem we need some preliminaries. In what follows, we set $H(u, \tau) := \{x \in \mathbb{R}^d \mid \langle x, u \rangle = \tau\}$.

Lemma 2. *Let $K \in \mathcal{K}^d$ be a convex body of dimension ≥ 1 . Let $u \in S^{d-1}$. We assume that u is not orthogonal to the linear subspace which is parallel to $\text{aff } K$, the affine hull of K . Let $(\tau_i)_{i \in \mathbb{N}}$ be a sequence of real numbers converging to $\tau \in \mathbb{R}$ and satisfying $K \cap H(u, \tau_i) \neq \emptyset$ for all $i \in \mathbb{N}$. Then $\lim K \cap H(u, \tau_i) = K \cap H(u, \tau)$ in the Hausdorff metric.*

Proof. For the proof we use Theorem 1.8.7 of [8].

Clearly, each limit of a sequence $(x_{i_j})_{j \in \mathbb{N}}$ in $\mathbb{R}^d$ with $x_{i_j} \in K \cap H(u, \tau_{i_j})$ for all $j \in \mathbb{N}$ belongs to $K \cap H(u, \tau)$.

Now let $x \in K \cap H(u, \tau)$. We want to show that there is a sequence $(x_i)_{i \in \mathbb{N}}$ satisfying $x_i \in K \cap H(u, \tau_i)$ for all $i \in \mathbb{N}$ and $\lim x_i = x$.

We first assume that the hyperplane $H(u, \tau)$ supports K . Let $z \in \text{relint } K$, where relint denotes the relative interior. Then, by elementary convexity,

$$(\text{conv}\{z, x\}) \setminus \{x\} \subset \text{relint } K. \quad (4)$$

For $i \in \mathbb{N}$ define

$$\alpha_i := \frac{\tau_i - \tau}{\langle z, u \rangle - \tau}.$$

We have $\alpha_i \geq 0$ since $H(u, \tau)$ supports K . Because $(\alpha_i)_{i \in \mathbb{N}}$ converges to 0, the relation $\alpha_i < 1$ holds if i is sufficiently large. For $i \in \mathbb{N}$ with this property we define $x_i := (1 - \alpha_i)x + \alpha_i z$. Then $x_i \in K$ because of (4). Furthermore, $\langle x_i, u \rangle = \tau_i$ and thus $x_i \in K \cap H(u, \tau_i)$. From $\lim \alpha_i = 0$ we deduce $\lim x_i = x$.

If $H(u, \tau)$ is not a supporting hyperplane of K , then $(\text{relint } K) \cap H(u, \tau) \neq \emptyset$. We then define $K^\pm := K \cap H^\pm(u, \tau)$. The convex bodies K^+ and K^- have the same dimension as K and $x \in K^\pm \cap H(u, \tau)$. Furthermore, the hyperplane $H(u, \tau)$ supports K^+ as well as K^- . For $\tau_i \neq \tau$ the hyperplane $H(u, \tau_i)$ intersects either K^+ or K^- but not both. As before, one can construct a sequence $(x_i)_{i \in \mathbb{N}}$ such that $\lim x_i = x$ and $x_i \in K \cap H(u, \tau_i)$ for all $i \in \mathbb{N}$. $\square$

Lemma 3. *Let $k \in \{1, \dots, d-1\}$ and let $u_1, \dots, u_k, u \in S^{d-1}$, be linearly independent. Let $K \in \mathcal{K}^d$ be a convex body with interior points. If $d_k(u)$ denotes the distance from u to $\text{lin}\{u_1, \dots, u_k\}$, then*

$$\begin{aligned} V_{d-k}(K \cap H(u_1, \tau_1) \cap \dots \cap H(u_k, \tau_k)) \\ = \frac{1}{d_k(u)} \int_{-h(K, -u)}^{h(K, u)} V_{d-k-1}(K \cap H(u_1, \tau_1) \cap \dots \cap H(u_k, \tau_k) \cap H(u, \tau)) d\tau \end{aligned}$$

for all $\tau_1, \dots, \tau_k \in \mathbb{R}$.

Proof. Let $K' := K \cap H(u_1, \tau_1) \cap \dots \cap H(u_k, \tau_k)$ be of dimension $d-k$ (otherwise, there is nothing to prove). Denoting by $\perp$ the orthogonal complement, $\text{aff } K'$ has the form $\text{aff } K' = \text{lin}\{u_1, \dots, u_k\}^\perp + x_0$ with $x_0 \in \text{lin}\{u_1, \dots, u_k\}$. Let u' be the orthogonal projection of u onto $\text{lin}\{u_1, \dots, u_k\}^\perp$.

We apply Fubini's theorem in the linear subspace $\text{lin}\{u_1, \dots, u_k\}^\perp$ which is parallel to $\text{aff } K'$. Furthermore, we notice that $h(K' - x_0, u') = h(K' - x_0, u)$. We obtain

$$\begin{aligned} V_{d-k}(K') &= V_{d-k}(K' - x_0) \\ &= \int_{-h(K' - x_0, -u'/\|u'\|)}^{h(K' - x_0, u'/\|u'\|)} V_{d-k-1}((K' - x_0) \cap H(u'/\|u'\|, \tau)) d\tau \\ &= \frac{1}{\|u'\|} \int_{-h(K' - x_0, -u')}^{h(K' - x_0, u')} V_{d-k-1}((K' - x_0) \cap H(u', \tau)) d\tau \\ &= \frac{1}{\|u'\|} \int_{-h(K', -u) - \langle x_0, u \rangle}^{h(K', u) - \langle x_0, u \rangle} V_{d-k-1}(K' \cap H(u, \tau + \langle x_0, u \rangle)) d\tau \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\|u'\|} \int_{-h(K', -u)}^{h(K', u)} V_{d-k-1}(K' \cap H(u, \tau)) d\tau \\
 &= \frac{1}{\|u'\|} \int_{-h(K, -u)}^{h(K, u)} V_{d-k-1}(K' \cap H(u, \tau)) d\tau.
 \end{aligned}$$

Here, the last step follows because $K' \subset K$. Since $\|u'\| = d_k(u)$, the assertion follows. $\square$

The connection between the $(d-k)$ -dimensional volume of $F_{i_1}(x) \cap \dots \cap F_{i_k}(x)$ and the partial derivative $(\partial^k V / \partial x_{i_k} \dots \partial x_{i_1})(x)$ is given by the following theorem.

Theorem 4. *Let $k \in \{1, \dots, d\}$ and let numbers $i_1, \dots, i_k \in \{1, \dots, n\}$ be chosen such that $F_{i_1} \cap \dots \cap F_{i_k} \in \mathcal{F}_{d-k}(P)$. Then*

$$\frac{\partial^k V}{\partial x_{i_k} \dots \partial x_{i_1}}(x) = [u_{i_1}, \dots, u_{i_k}]^{-1} V_{d-k}(F_{i_1}(x) \cap \dots \cap F_{i_k}(x)) \quad (5)$$

for all $x \in U(h)$. Here, $[u_{i_1}, \dots, u_{i_k}]$ denotes the k -dimensional volume of the parallelepiped spanned by $u_{i_1}, \dots, u_{i_k}$.

Proof. The assertion is proved by induction on k . Let $i \in \{1, \dots, n\}$ and $a \in U(h)$. For $x \in]h_i - \varepsilon, a_i]$ let

$$P_x := P(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n).$$

From $o \in \text{int } P_x$ and $x > 0$, we deduce that $h(P_x, -u_i) = h(P(a), -u_i)$. Hence,

$$\begin{aligned}
 &V(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n) \\
 &= \int_{-h(P_x, -u_i)}^{h(P_x, u_i)} V_{d-1}(P_x \cap H(u_i, \tau)) d\tau \\
 &= \int_{-h(P(a), -u_i)}^x V_{d-1}(P(a) \cap H^-(u_i, x) \cap H(u_i, \tau)) d\tau \\
 &= \int_{-h(P(a), -u_i)}^x V_{d-1}(P(a) \cap H(u_i, \tau)) d\tau.
 \end{aligned}$$

From Lemma 2 we obtain that the function $\tau \mapsto V_{d-1}(P(a) \cap H(u_i, \tau))$ is continuous on $]-h(P(a), -u_i), a_i]$. Thus,

$$\begin{aligned}
 \frac{\partial V}{\partial x_i}(a) &= V_{d-1}(P(a) \cap H(u_i, a_i)) \\
 &= V_{d-1}(F_i(a)).
 \end{aligned}$$

Let $k \in \{2, \dots, d\}$ and assume the assertion to be true for $k-1$. Let $i_1, \dots, i_k \in \{1, \dots, n\}$ and $F_{i_1} \cap \dots \cap F_{i_k} \in \mathcal{F}_{d-k}(P)$. After a suitable renumbering we may assume

that $\{i_1, \dots, i_k\} = \{1, \dots, k\}$. The assumption of induction implies

$$\frac{\partial^{k-1} V}{\partial x_{k-1} \cdots \partial x_1}(x) = [u_1, \dots, u_{k-1}]^{-1} V_{d-k+1}(F_1(x) \cap \cdots \cap F_{k-1}(x)) \quad (6)$$

for all $x \in U(h)$.

Let $a = (a_1, \dots, a_n) \in U(h)$ and $x \in]h_k - \varepsilon, a_k]$. Define

$$a_x := (a_1, \dots, a_{k-1}, x, a_{k+1}, \dots, a_n)$$

and $P_x := P(a_x)$. If $d_{k-1}(u_k)$ denotes the distance from u_k to $\text{lin}\{u_1, \dots, u_{k-1}\}$, then Lemma 3 implies

$$\begin{aligned} d_{k-1}(u_k) V_{d-k+1}(F_1(a_x) \cap \cdots \cap F_{k-1}(a_x)) \\ &= d_{k-1}(u_k) V_{d-k+1}(P_x \cap H(u_1, a_1) \cap \cdots \cap H(u_{k-1}, a_{k-1})) \\ &= \int_{-h(P_x, -u_k)}^{h(P_x, u_k)} V_{d-k}(P_x \cap H(u_1, a_1) \cap \cdots \cap H(u_{k-1}, a_{k-1}) \cap H(u_k, \tau)) d\tau \\ &= \int_{-h(P(a), -u_k)}^x V_{d-k}(P_x \cap H(u_1, a_1) \cap \cdots \cap H(u_{k-1}, a_{k-1}) \cap H(u_k, \tau)) d\tau. \end{aligned}$$

Hence by (6),

$$\begin{aligned} [u_1, \dots, u_k] \frac{\partial^{k-1} V}{\partial x_{k-1} \cdots \partial x_1}(a_x) \\ &= d_{k-1}(u_k) [u_1, \dots, u_{k-1}] \frac{\partial^{k-1} V}{\partial x_{k-1} \cdots \partial x_1}(a_x) \\ &= d_{k-1}(u_k) V_{d-k+1}(F_1(a_x) \cap \cdots \cap F_{k-1}(a_x)) \\ &= \int_{-h(P(a), -u_k)}^x V_{d-k}(P_x \cap H(u_1, a_1) \cap \cdots \cap H(u_{k-1}, a_{k-1}) \cap H(u_k, \tau)) d\tau \\ &= \int_{-h(P(a), -u_k)}^x V_{d-k}(P(a) \cap H(u_1, a_1) \cap \cdots \cap H(u_{k-1}, a_{k-1}) \cap H(u_k, \tau)) d\tau \\ &= \int_{-h(P(a), -u_k)}^{h_k - \varepsilon} V_{d-k}(P(a) \cap H(u_1, a_1) \cap \cdots \cap H(u_{k-1}, a_{k-1}) \cap H(u_k, \tau)) d\tau \\ &\quad + \int_{h_k - \varepsilon}^x V_{d-k}(P(a) \cap H(u_1, a_1) \cap \cdots \cap H(u_{k-1}, a_{k-1}) \cap H(u_k, \tau)) d\tau. \end{aligned}$$

By Lemma 2, the function

$$\tau \mapsto V_{d-k}(P(a) \cap H(u_1, a_1) \cap \cdots \cap H(u_{k-1}, a_{k-1}) \cap H(u_k, \tau))$$

is continuous on $]h_k - \varepsilon, a_k]$. This follows, since for $\tau \in]h_k - \varepsilon, a_k]$ the intersection

$$P(a) \cap H(u_1, a_1) \cap \cdots \cap H(u_{k-1}, a_{k-1}) \cap H(u_k, \tau)$$

is the face of $P(a_1, \dots, a_{k-1}, \tau, a_{k+1}, \dots, a_n)$ corresponding to the face $F_1 \cap \dots \cap F_k$ of P . Consequently,

$$\begin{aligned} [u_1, \dots, u_k] \frac{\partial^k V}{\partial x_k \dots \partial x_1}(a) &= V_{d-k}(P(a) \cap H(u_1, a_1) \cap \dots \cap H(u_{k-1}, a_{k-1}) \cap H(u_k, a_k)) \\ &= V_{d-k}(F_1(a) \cap \dots \cap F_k(a)). \end{aligned}$$

This proves the assertion. $\square$

Theorem 4 yields:

Proposition 5. *Let $k \in \{1, \dots, d\}$. Let $i_1, \dots, i_k \in \{1, \dots, n\}$ be such that $F_{i_1} \cap \dots \cap F_{i_k} = \emptyset$. Then*

$$\frac{\partial^k V}{\partial x_{i_k} \dots \partial x_{i_1}}(x) = 0$$

for all $x \in U(h)$.

Proof. It suffices to prove the assertion for pairwise disjoint numbers $i_1, \dots, i_k \in \{1, \dots, n\}$. Without loss of generality we assume that $\{i_1, \dots, i_k\} = \{1, \dots, k\}$. There exists $s \in \{1, \dots, k-1\}$ such that $F_1 \cap \dots \cap F_s \in \mathcal{F}_{d-s}(P)$ after a suitable renumbering. We further assume that s is the greatest number with this property. This means that each intersection of $s+1$ facets of P with outer normal vectors among $\{u_1, \dots, u_k\}$ is empty. Theorem 4 yields

$$\frac{\partial^s V}{\partial x_s \dots \partial x_1}(x) = [u_1, \dots, u_s]^{-1} V_{d-s}(F_1(x) \cap \dots \cap F_s(x)) \quad (7)$$

for all $x \in U(h)$. Let $a \in U(h)$ and define $a_x := (a_1, \dots, a_s, x, a_{s+2}, \dots, a_n)$, where $x \in]h_{s+1} - \varepsilon, a_{s+1}]$. The choice of s implies that $F_1(a) \cap \dots \cap F_{s+1}(a) = \emptyset$. Thus,

$$\begin{aligned} F_1(a_x) \cap \dots \cap F_s(a_x) &= P(a_x) \cap H(u_1, a_1) \cap \dots \cap H(u_s, a_s) \\ &= P(a) \cap H(u_1, a_1) \cap \dots \cap H(u_s, a_s) \cap H^-(u_{s+1}, x) \\ &= F_1(a) \cap \dots \cap F_s(a) \cap H^-(u_{s+1}, x) \\ &= F_1(a) \cap \dots \cap F_s(a), \end{aligned}$$

if x lies in a sufficiently small leftside neighborhood of a_{s+1} . From (7) we deduce

$$\frac{\partial^s V}{\partial x_s \dots \partial x_1}(a_x) = \frac{\partial^s V}{\partial x_s \dots \partial x_1}(a).$$

Hence,

$$\frac{\partial^{s+1} V}{\partial x_{s+1} \dots \partial x_1}(a) = 0.$$

The assertion follows, since $s \leq k-1$ and $a \in U(h)$ was chosen arbitrarily. $\square$

3. Simple Polytopes and Stress

The main tool for proving the extension of Aleksandrov's result is the notion of k -stress which was introduced by Lee [5] (see also [6] and [7]). The definition of this notion requires some more nomenclature.

As before, P is a simple d -dimensional polytope which contains o in its interior. The polar body of P defines an abstract simplicial complex Δ on $\{1, \dots, n\}$. For $k \in \{1, \dots, d-1\}$ set

$$\Delta_k := \{\{i_1, \dots, i_k\} \subset \{1, \dots, n\} \mid F_{i_1} \cap \dots \cap F_{i_k} \in \mathcal{F}_{d-k}(P)\}.$$

Let

$$\mathbb{Z}_+^n := \{(r_1, \dots, r_n) \in \mathbb{Z}^n \mid r_i \geq 0 \text{ for all } i = 1, \dots, n\}.$$

For $r = (r_1, \dots, r_n) \in \mathbb{Z}_+^n$ define

$$x^r := x_1^{r_1} \cdots x_n^{r_n} \text{ and } \text{supp } x^r := \{i \in \{1, \dots, n\} \mid r_i \neq 0\}.$$

Furthermore, we set

$$r! := r_1! \cdots r_n! \quad \text{and} \quad |r| := r_1 + \dots + r_n.$$

A *linear k -stress* (with respect to $u_1, \dots, u_n$), where $k \in \mathbb{N} \cup \{0\}$, is a polynomial b defined on $U(h)$ of the form

$$b(x) = \sum_{\substack{r \in \mathbb{Z}_+^n \\ |r|=k}} b_r \frac{x^r}{r!},$$

satisfying

$$b_r = 0, \quad \text{if } \text{supp } x^r \notin \Delta, \quad (8)$$

and

$$\sum_{i=1}^n b_{s+e_i} u_i = 0 \quad \text{for all } s \in \mathbb{Z}_+^n \text{ with } |s| = k-1. \quad (9)$$

Here, $e_1, \dots, e_n$ denote the canonical unit vectors of $\mathbb{R}^n$. The set of all linear k -stresses forms a vector space denoted by S_k^l .

For $k \in \{2, \dots, d\}$ let $L_k(\mathbb{R}^n \times \dots \times \mathbb{R}^n; \mathbb{R})$ be the vector space of multilinear real-valued functions defined on the k -fold cartesian product of $\mathbb{R}^n$. We define

$$T_k: S_k^l \rightarrow L_k(\mathbb{R}^n \times \dots \times \mathbb{R}^n; \mathbb{R})$$

by $T_k(b)(e_{i_1}, \dots, e_{i_k}) := b_{e_{i_1} + \dots + e_{i_k}}$, where

$$b(x) := \sum_{\substack{r \in \mathbb{Z}_+^n \\ |r|=k}} b_r \frac{x^r}{r!},$$

and extend $T_k(b)$ multilinearly. Hence, $T_k(b)$ is multilinear and symmetric.

On p. 255 of [6], for $c \in \mathbb{R}^n$ and $k \geq 1$, the mapping $\sigma_c : S_k^l \rightarrow S_{k-1}^l$ was defined by

$$\sigma_c(b)(x) := \sum_{\substack{s \in \mathbb{Z}_+^n \\ |s|=k-1}} \left(\sum_{i=1}^n b_{s+e_i} c_i \right) \frac{x^s}{s!} \quad \text{for all } x \in U(h), \quad (10)$$

where, as before, $b(x) = \sum_{r \in \mathbb{Z}_+^n, |r|=k} b_r (x^r / r!)$. We define

$$\tilde{\omega} := \sigma_{(h_1, \dots, h_n)}.$$

Proposition 5 and the proof of Theorem 20 of [6], yield that the volume V is a linear d -stress. Hence, $\tilde{\omega}^{d-k}(V) \in S_k^l$ for $k = 0, \dots, d$ (see Theorem 8 of [6]).

The aim of this section is to prove the following theorem.

Theorem 6. *Let $k \in \{2, \dots, d\}$, let $c \in \mathbb{R}^n$. The equation*

$$T_k(\tilde{\omega}^{d-k}(V))(e_{i_1}, \dots, e_{i_{k-1}}, c) = 0 \quad (11)$$

holds for all $\{i_1, \dots, i_{k-1}\} \in \Delta_{k-1}$ if and only if c is the support vector of a point.

The proof of Theorem 6 requires some additional results which we derive first. For given $b \in S_k^l$, we define by

$$\Phi_k^b(x) := \sigma_x(b)$$

a linear mapping $\Phi_k^b : \mathbb{R}^n \rightarrow S_{k-1}^l$. The kernel $\ker \Phi_k^b$ of this mapping can be described as follows.

Proposition 7. *Let $k \in \{2, \dots, d\}$ and $c \in \mathbb{R}^n$. Then $c \in \ker \Phi_k^b$ if and only if*

$$T_k(b)(e_{i_1}, \dots, e_{i_{k-1}}, c) = 0 \quad (12)$$

for all $i_1, \dots, i_{k-1} \in \{1, \dots, n\}$.

Proof. Let $c = (c_1, \dots, c_n) \in \ker \Phi_k^b$, where $b(x) = \sum_{r \in \mathbb{Z}_+^n, |r|=k} b_r (x^r / r!)$. Because of (10), this is equivalent to

$$\sum_{i=1}^n b_{s+e_i} c_i = 0 \quad \text{for all } s \in \mathbb{Z}_+^n \text{ with } |s| = k-1. \quad (13)$$

Let $i_1, \dots, i_{k-1} \in \{1, \dots, n\}$ and define $s := e_{i_1} + \dots + e_{i_{k-1}}$. Then $s \in \mathbb{Z}_+^n$ and $|s| = k-1$. Conversely, if $s \in \mathbb{Z}_+^n$ and $|s| = k-1$, then $s = e_{i_1} + \dots + e_{i_{k-1}}$ with suitable $i_1, \dots, i_{k-1} \in \{1, \dots, n\}$.

In both cases, we have

$$\begin{aligned}
 T_k(b)(e_{i_1}, \dots, e_{i_{k-1}}, c) &= \sum_{i=1}^n T_k(b)(e_{i_1}, \dots, e_{i_{k-1}}, e_i) c_i \\
 &= \sum_{i=1}^n b_{e_{i_1} + \dots + e_{i_{k-1}} + e_i} c_i \\
 &= \sum_{i=1}^n b_{s+e_i} c_i.
 \end{aligned}$$

The proposition now follows from (13). $\square$

By Lemma 5.1.2 of [8], the volume function V can be represented as

$$V(x) = \sum_{j_1=1}^n \cdots \sum_{j_d=1}^n a_{j_1 \dots j_d} x_{j_1} \cdots x_{j_d},$$

where the coefficients $a_{j_1 \dots j_d} \in \mathbb{R}$ are symmetric in $j_1, \dots, j_d$. For given $r = (r_1, \dots, r_n) \in \mathbb{Z}_+^n$ with $|r| = d$, there are pairwise disjoint numbers $i_1, \dots, i_m \in \{1, \dots, n\}$ such that $r_{i_s} > 0$ for all $s = 1, \dots, m$ and $r = r_{i_1} e_{i_1} + \dots + r_{i_m} e_{i_m}$. Hence, $\{i_1, \dots, i_m\} = \text{supp } x^r$. We then define

$$a_r := a_{\underbrace{i_1 \dots i_1}_{r_{i_1}} \dots \underbrace{i_m \dots i_m}_{r_{i_m}}}.$$

This is well defined because the coefficients are symmetric. For $r = (r_1, \dots, r_n) \in \mathbb{Z}_+^n$ with $|r| = k$ we then obtain

$$\frac{\partial^k V}{\partial x^r}(x) = d! \sum_{\substack{s \in \mathbb{Z}_+^n \\ |s|=d-k}} a_{r+s} \frac{x^s}{s!},$$

where we defined

$$\frac{\partial^k V}{\partial x^r}(x) := \frac{\partial^k V}{\partial x_n^{r_n} \cdots \partial x_1^{r_1}}(x)$$

for $r = (r_1, \dots, r_n) \in \mathbb{Z}_+^n$ with $|r| = k$.

Next, we define

$$W_k(x) := \sum_{\substack{r \in \mathbb{Z}_+^n \\ |r|=k}} \frac{\partial^k V}{\partial x^r}(h) \frac{x^r}{r!} \quad \text{for } x \in U(h), \quad (14)$$

$k \in \{0, \dots, d\}$. In particular,

$$W_d(x) = d! \sum_{\substack{r \in \mathbb{Z}_+^n \\ |r|=d}} a_r \frac{x^r}{r!} = V(x).$$

The following lemma shows that $\tilde{\omega}^{d-k}(V)$ and W_k are equal up to the factor $(d-k)!$ (see Theorem 21 of [6]).

Lemma 8. *Let $k \in \{0, \dots, d\}$. Then $\tilde{\omega}^{d-k}(V) = (d-k)! W_k$.*

Proof. We have just seen that $\tilde{\omega}^0(V) = V = W_d$. Let $k \in \{0, \dots, d-1\}$ and assume that $\tilde{\omega}^k(V) = k! W_{d-k}$. Then

$$\begin{aligned}
 \tilde{\omega}^{k+1}(V)(x) &= \tilde{\omega}(\tilde{\omega}^k(V))(x) \\
 &= \tilde{\omega}(k! W_{d-k})(x) \\
 &= k! \sum_{\substack{r \in \mathbb{Z}_+^n \\ |r|=d-(k+1)}} \left(\sum_{i=1}^n \frac{\partial^{d-k} V}{\partial x^{r+e_i}}(h) h_i \right) \frac{x^r}{r!} \\
 &= k! \sum_{\substack{r \in \mathbb{Z}_+^n \\ |r|=d-(k+1)}} \frac{d}{dt} \Big|_{t=0} \frac{\partial^{d-(k+1)} V}{\partial x^r}(h + th) \frac{x^r}{r!} \\
 &= k! \sum_{\substack{r \in \mathbb{Z}_+^n \\ |r|=d-(k+1)}} \frac{d}{dt} \Big|_{t=0} (1+t)^{k+1} \frac{\partial^{d-(k+1)} V}{\partial x^r}(h) \frac{x^r}{r!} \\
 &= (k+1)! \sum_{\substack{r \in \mathbb{Z}_+^n \\ |r|=d-(k+1)}} \frac{\partial^{d-(k+1)} V}{\partial x^r}(h) \frac{x^r}{r!} \\
 &= (k+1)! W_{d-(k+1)}(x)
 \end{aligned}$$

for all $x \in U(h)$. This proves the assertion. $\square$

The proof of Theorem 10 requires the following lemma. In this lemma we denote by

$$L(u_1, \dots, u_n) := \left\{ (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n \mid \sum_{i=1}^n \alpha_i u_i = o \right\}$$

the set of *linear dependences* of $u_1, \dots, u_n$. This is a linear subspace of dimension $n-d$.

Lemma 9. *Let $k \in \{2, \dots, d\}$ and*

$$\tilde{\omega}^{d-k}(V)(x) = \sum_{\substack{r \in \mathbb{Z}_+^n \\ |r|=k}} b_r \frac{x^r}{r!}.$$

Then

$$\text{lin}\{(b_{s+e_1}, \dots, b_{s+e_n}) \mid s \in \mathbb{Z}_+^n, |s| = k-1\} = L(u_1, \dots, u_n). \quad (15)$$

Proof. The definition of a k -stress includes that

$$(b_{s+e_1}, \dots, b_{s+e_n}) \in L(u_1, \dots, u_n)$$

for all $s \in \mathbb{Z}_+^n$ with $|s| = k - 1$ (see (9)). Hence,

$$\text{lin}\{(b_{s+e_1}, \dots, b_{s+e_n}) \mid s \in \mathbb{Z}_+^n, |s| = k - 1\} \subset L(u_1, \dots, u_n). \quad (16)$$

The proof of (15) is completed by induction on k . Inserting $k = 2$ into (14) we obtain

$$W_2(x) = \sum_{\substack{r \in \mathbb{Z}_+^n \\ |r|=2}} \frac{\partial^2 V}{\partial x^r} (h) \frac{x^r}{r!}.$$

Using Lemma 8 we see that

$$b_{e_i+e_j} = (d-2)! \frac{\partial^2 V}{\partial x_j \partial x_i} (h).$$

For $\{i, j\} \in \Delta_2$, Theorem 4 then yields

$$\begin{aligned} b_{e_i+e_j} &= (d-2)! [u_i, u_j]^{-1} V_{d-2}(F_i \cap F_j) \\ &= (d-2)! \text{cosec } \theta_{ij} V_{d-2}(F_i \cap F_j). \end{aligned} \quad (17)$$

In this formula, θ_{ij} denotes the angle between u_i and u_j . If $\{i, j\} \notin \Delta_2$ and $i \neq j$, then Proposition 5 implies that

$$b_{e_i+e_j} = (d-2)! \frac{\partial^2 V}{\partial x_j \partial x_i} (h) = 0. \quad (18)$$

For $i \in \{1, \dots, n\}$, we have $\sum_{j=1}^n b_{e_i+e_j} u_j = o$ using (9). Hence,

$$\begin{aligned} b_{2e_i} &= - \sum_{j \neq i} b_{e_i+e_j} \langle u_i, u_j \rangle \\ &= -(d-2)! \sum_{\{i,j\} \in \Delta_2} \cot \theta_{ij} V_{d-2}(F_i \cap F_j). \end{aligned} \quad (19)$$

Summarizing (17), (18), and (19) we obtain

$$b_{e_i+e_j} = \begin{cases} (d-2)! \text{cosec } \theta_{ij} V_{d-2}(F_i \cap F_j) & \text{for } \{i, j\} \in \Delta_2, \\ 0 & \text{for } \{i, j\} \notin \Delta_2, \quad i \neq j, \\ -(d-2)! \sum_{\{i,m\} \in \Delta_2} \cot \theta_{im} V_{d-2}(F_i \cap F_m) & \text{for } i = j. \end{cases}$$

We now consider the $n \times n$ matrix $(b_{e_i+e_j})_{i,j=1,\dots,n}$. Using the above representation of $b_{e_i+e_j}$ we see that

$$\begin{aligned} (W_1, \dots, W_n)^t &\in \ker(b_{e_i+e_j})_{i,j=1,\dots,n} \\ &\Leftrightarrow \sum_{j=1}^n b_{e_i+e_j} W_j = 0 \quad \text{for all } i = 1, \dots, n \\ &\Leftrightarrow \sum_{\{i,j\} \in \Delta_2} b_{e_i+e_j} W_j + b_{2e_i} W_i = 0 \quad \text{for all } i = 1, \dots, n \\ &\Leftrightarrow \sum_{\{i,j\} \in \Delta_2} W_{ij} V_{d-2}(F_i \cap F_j) = 0 \quad \text{for all } i = 1, \dots, n, \end{aligned} \quad (20)$$

where W_{ij} is defined by

$$W_{ij} := W_j \operatorname{cosec} \theta_{ij} - W_i \cot \theta_{ij}.$$

Theorem 4.1.1 in [2] shows that (20) implies

$$\operatorname{rank}(b_{e_i+e_j})_{i,j=1,\dots,n} = n - d$$

(see also p. 11 of [4]). Thus, by (16) and because of

$$\dim L(u_1, \dots, u_n) = n - d,$$

the assertion is true if $k = 2$.

Let $k \in \{2, \dots, d-1\}$ and assume that the assertion is proved for $\tilde{\omega}^{d-k}(V)$. Let

$$\tilde{\omega}^{d-k}(V)(x) = \sum_{\substack{r \in \mathbb{Z}_+^n \\ |r|=k}} b_r \frac{x^r}{r!}$$

and

$$\tilde{\omega}^{d-(k+1)}(V)(x) = \sum_{\substack{r \in \mathbb{Z}_+^n \\ |r|=k+1}} c_r \frac{x^r}{r!}.$$

Because of $\tilde{\omega}^{d-k}(V) = \tilde{\omega}(\tilde{\omega}^{d-(k+1)}(V))$ we have

$$b_r = \sum_{i=1}^n c_{r+e_i} h_i \quad \text{for all } r \in \mathbb{Z}_+^n \text{ with } |r| = k. \quad (21)$$

The assumption of induction yields

$$\operatorname{lin}\{(b_{s+e_1}, \dots, b_{s+e_n}) \mid s \in \mathbb{Z}_+^n, |s| = k-1\} = L(u_1, \dots, u_n). \quad (22)$$

Furthermore, since $o \in \operatorname{int} P$, we have $h_1, \dots, h_n > 0$ and thus

$$\begin{aligned} & \dim \operatorname{lin}\{(c_{s+e_i+e_1} h_1, \dots, c_{s+e_i+e_n} h_n) \mid s \in \mathbb{Z}_+^n, |s| = k-1, i = 1, \dots, n\} \\ &= \dim \operatorname{lin}\{(c_{s+e_i+e_1}, \dots, c_{s+e_i+e_n}) \mid s \in \mathbb{Z}_+^n, |s| = k-1, i = 1, \dots, n\}. \end{aligned} \quad (23)$$

We also know that for each $r \in \mathbb{Z}_+^n$ with $|r| = k$ there are $s \in \mathbb{Z}_+^n, |s| = k-1$, and $i \in \{1, \dots, n\}$ such that $r = s + e_i$. Combining these observations we deduce that

$$\begin{aligned} & \dim L(u_1, \dots, u_n) \\ & \stackrel{(22)}{=} \dim \operatorname{lin}\{(b_{s+e_1}, \dots, b_{s+e_n}) \mid s \in \mathbb{Z}_+^n, |s| = k-1\} \\ & \stackrel{(21)}{=} \dim \operatorname{lin} \left\{ \sum_{i=1}^n (c_{s+e_i+e_1} h_1, \dots, c_{s+e_i+e_n} h_n) \mid s \in \mathbb{Z}_+^n, |s| = k-1 \right\} \\ & \leq \dim \operatorname{lin}\{(c_{s+e_i+e_1} h_1, \dots, c_{s+e_i+e_n} h_n) \mid s \in \mathbb{Z}_+^n, |s| = k-1, i = 1, \dots, n\} \\ & \stackrel{(23)}{=} \dim \operatorname{lin}\{(c_{s+e_i+e_1}, \dots, c_{s+e_i+e_n}) \mid s \in \mathbb{Z}_+^n, |s| = k-1, i = 1, \dots, n\} \\ & = \dim \operatorname{lin}\{(c_{r+e_1}, \dots, c_{r+e_n}) \mid r \in \mathbb{Z}_+^n, |r| = k\} \\ & \stackrel{(16)}{\leq} \dim L(u_1, \dots, u_n), \end{aligned}$$

that is, the linear subspace

$$\text{lin}\{(c_{r+e_1}, \dots, c_{r+e_n}) \mid r \in \mathbb{Z}_+^n, |r| = k\}$$

has the same dimension as $L(u_1, \dots, u_n)$. By (16), this is only possible if

$$\text{lin}\{(c_{r+e_1}, \dots, c_{r+e_n}) \mid r \in \mathbb{Z}_+^n, |r| = k\} = L(u_1, \dots, u_n).$$

Hence, (15) holds true. $\square$

We now define $\Phi_k := \Phi_k^{\tilde{\omega}^{d-k}(V)}$. Using Lemma 9 we are able to determine the kernel of Φ_k .

Theorem 10. *Let $k \in \{2, \dots, d\}$. Then*

$$\ker \Phi_k = \{(\langle a, u_1 \rangle, \dots, \langle a, u_n \rangle) \mid a \in \mathbb{R}^d\}. \quad (24)$$

Proof. Let

$$\tilde{\omega}^{d-k}(V)(x) = \sum_{\substack{r \in \mathbb{Z}_+^n \\ |r|=k}} b_r \frac{x^r}{r!}.$$

Then $c = (c_1, \dots, c_n) \in \ker \Phi_k$ holds if and only if $\sum_{i=1}^n b_{s+e_i} c_i = 0$ for all $s \in \mathbb{Z}_+^n$ with $|s| = k-1$. Lemma 9 shows that this is equivalent to $c \in L(u_1, \dots, u_n)^\perp$. We have $L(u_1, \dots, u_n) = \ker A$, where A is the $d \times n$ matrix whose i th column is the vector u_i , that is, $A := (u_1, \dots, u_n)$. Hence,

$$\begin{aligned} \ker \Phi_k &= L(u_1, \dots, u_n)^\perp \\ &= (\ker A)^\perp \\ &= \text{im } A^t \\ &= \{(\langle a, u_1 \rangle, \dots, \langle a, u_n \rangle) \mid a \in \mathbb{R}^d\}. \end{aligned}$$

Here, $\text{im } A^t$ denotes the image of the transpose A^t of A . $\square$

We are now in a position to prove Theorem 6.

Proof of Theorem 6. It is an immediate consequence of Proposition 7 and Theorem 10 that (11) holds for all $i_1, \dots, i_{k-1} \in \{1, \dots, n\}$ if and only if $c \in \mathbb{R}^n$ is the support vector of a point. Hence, we only have to prove that (11) for all $\{i_1, \dots, i_{k-1}\} \in \Delta_{k-1}$ implies that this equation holds for all $i_1, \dots, i_{k-1} \in \{1, \dots, n\}$.

Assume that (11) holds for all $\{i_1, \dots, i_{k-1}\} \in \Delta_{k-1}$. First let $i_1, \dots, i_{k-1} \in \{1, \dots, n\}$ be pairwise disjoint. If $\{i_1, \dots, i_{k-1}\} \notin \Delta_{k-1}$, then $\{i_1, \dots, i_{k-1}, i\} \notin \Delta$ for all $i \in \{1, \dots, n\}$. Hence,

$$T_k(\tilde{\omega}^{d-k}(V))(e_{i_1}, \dots, e_{i_{k-1}}, e_i) = 0$$

for all $i = 1, \dots, n$ because of $\tilde{\omega}^{d-k}(V) \in S_k^l$ (see (8)). We obtain

$$T_k(\tilde{\omega}^{d-k}(V))(e_{i_1}, \dots, e_{i_{k-1}}, c) = 0. \quad (25)$$

We now drop the assumption of pairwise disjointness and choose arbitrary numbers $i_1, \dots, i_{k-1} \in \{1, \dots, n\}$. We define

$$r := e_{i_1} + \dots + e_{i_{k-1}}. \quad (26)$$

Let $s \in \mathbb{Z}_+^n$, $|s| = k - 1$. Then s has a unique representation $s = e_{j_1} + \dots + e_{j_{k-1}}$ with $j_1, \dots, j_{k-1} \in \{1, \dots, n\}$. For $c = (c_1, \dots, c_n) \in \mathbb{R}^n$ define

$$T_k(s, c) := T_k(\tilde{\omega}^{d-k}(V))(e_{j_1}, \dots, e_{j_{k-1}}, c).$$

We then have

$$\sum_{\substack{s \in \mathbb{Z}_+^n \\ |s|=k-1}} T_k(s, c) \frac{x^s}{s!} = \sigma_c(\tilde{\omega}^{d-k}(V))(x) \quad \text{for all } x \in U(h).$$

This shows that the mapping

$$x \mapsto \sum_{\substack{s \in \mathbb{Z}_+^n \\ |s|=k-1}} T_k(s, c) \frac{x^s}{s!} \quad (27)$$

is a linear $(k - 1)$ -stress.

Since P is a simple polytope, for $m \in \{1, \dots, d\}$ and $\{i_1, \dots, i_m\} \in \Delta_m$, the vectors $u_{i_1}, \dots, u_{i_m}$ are linearly independent. Thus, $u_1, \dots, u_n$ are in general relative position with respect to Δ in the sense of Lee [6, p. 258]. Together with the fact that (27) is a linear $(k - 1)$ -stress, the general relative position of the outer normal vectors as well as Theorem 12 of [6], and (25), imply that

$$T_k(r, c) = \sum_{\{j_1, \dots, j_{k-1}\} \in \Delta_{k-1}} \alpha_{j_1, \dots, j_{k-1}}^{(r, c)} T_k(\tilde{\omega}^{d-k}(V))(e_{j_1}, \dots, e_{j_{k-1}}, c),$$

where $\alpha_{j_1, \dots, j_{k-1}}^{(r, c)} \in \mathbb{R}$ and r is defined by (26).

The assumption that (11) holds for all $\{j_1, \dots, j_{k-1}\} \in \Delta_{k-1}$ now implies

$$T_k(\tilde{\omega}^{d-k}(V))(e_{i_1}, \dots, e_{i_{k-1}}, c) = T_k(r, c) = 0.$$

Hence, (11) holds for all $i_1, \dots, i_{k-1} \in \{1, \dots, n\}$. □

4. Proof of Theorem 1

After these preparations it is not difficult to prove the extension of Aleksandrov's result on the infinitesimal rigidity of convex d -dimensional polytopes with stationary facet volumes (see Theorem 1 on p. 407 of [1]).

Proof of Theorem 1. Without loss of generality we may assume that $o \in \text{int } P$. Let $F \in \mathcal{F}_k(P)$. Since P is a simple polytope, the face F is the intersection of exactly $d - k$ facets of P . Let $u_{i_1}, \dots, u_{i_{d-k}}$ be the outer normal vectors of these facets.

We have

$$\begin{aligned}
 \left. \frac{d}{dt} \right|_{t=0} V_k(F(h+tc)) &= [u_{i_1}, \dots, u_{i_{d-k}}] \left. \frac{d}{dt} \right|_{t=0} \frac{\partial^{d-k} V}{\partial x_{i_{d-k}} \cdots \partial x_{i_1}}(h+tc) \\
 &= [u_{i_1}, \dots, u_{i_{d-k}}] \sum_{i=1}^n c_i \frac{\partial^{d-k+1} V}{\partial x_i \partial x_{i_{d-k}} \cdots \partial x_{i_1}}(h) \\
 &= \frac{[u_{i_1}, \dots, u_{i_{d-k}}]}{(k-1)!} T_{d-k+1}(\tilde{\omega}^{k-1}(V))(e_{i_1}, \dots, e_{i_{d-k}}, c),
 \end{aligned}$$

where the first step follows because of Theorem 4 and where we have applied Lemma 8 in the final step. Thus, (3) is equivalent to

$$T_{d-k+1}(\tilde{\omega}^{k-1}(V))(e_{i_1}, \dots, e_{i_{d-k}}, c) = 0$$

for all $\{i_1, \dots, i_{d-k}\} \in \Delta_{d-k}$. We have $d-k+1 \in \{2, \dots, d\}$, because $k \in \{1, \dots, d-1\}$. Theorem 6 implies that c is the support vector of a point.

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