Shellings and the Lower Bound Theorem

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Abstract. We use shellings to give an elementary proof of the lower bound theorem for simplicial polytopes including the case of equality.

1. Introduction

A *stacked* d -polytope P_s is a simplicial polytope obtained from a simplex by successive addition of pyramids over facets. Equivalently, P_s is a simplicial polytope having a triangulation such that all interior faces are $(d - 1)$ -dimensional. The number of k -faces ($1 \leq k \leq d - 1$) of P_s is some function $\varphi_k(n, d)$, where n is the vertex number of P_s . In particular, $\varphi_1(n, d) = nd - \binom{d+1}{2}$.

The lower bound theorem (LBT) asserts that for a simplicial d -polytope P with n vertices, the number $f_k(P)$ of k -faces of P satisfies the inequality $f_k(P) \geq \varphi_k(n, d)$ for all k , $1 \leq k \leq d - 1$. Moreover, if $d \geq 4$ and equality for any one value of k occurs, then P is stacked. The LBT was proved by Walkup [11] for $d = 4$ and $d = 5$. Barnette [1], [2] proved it for all d , but the case of equality only for $k = d - 1$; the remaining cases of k were proved by Billera and Lee [4]. Using the rigidity theory of frameworks, Kalai [7] generalized the LBT to triangulated manifolds, and using a weaker (combinatorial) form of rigidity, Tay [10] generalized the LBT to pseudomanifolds; the case of equality refers to Kalai's result. So in contrast to the upper bound theorem proved by McMullen [8] via shellings, no elementary proof of the LBT including the case of equality is known so far.

An easy reduction (see [2] or [6, p. 127]) shows that the LBT is equivalent to its case $k = 1$. As is well known, this case is closely related to shellings: a *shelling* of the boundary complex ∂P of P is an ordering $F_1, F_2, \dots, F_m$ of the facets of P so that

$F_{i+1} \cap \bigcup_{t=1}^i F_t$ is a nonempty union of $(d-2)$ -faces of F_{i+1} for all i , $1 \leq i \leq m-1$. It is well known that ∂P is shellable (see, e.g., p. 173 of [9] or p. 488 of [3]). For $1 \leq i \leq m$, let $\mathcal{K}_i$ be the complex spanned by $F_1, \dots, F_i$, so that $\mathcal{K}_m = \partial P$. Then going from $\mathcal{K}_i$ to $\mathcal{K}_{i+1}$ by adding F_{i+1} is a *shelling operation*, in particular, a *j-operation* if F_{i+1} and $\mathcal{K}_i$ have j common $(d-2)$ -faces, where $1 \leq j \leq d$. Let h_j be the number of j -operations among the $m-1$ shelling operations. Then $n = d + h_1$, and $f_1(P) = \binom{d}{2} + (d-1)h_1 + h_2$, and the LBT is equivalent to

Theorem. *If P is a simplicial d -polytope ($d \geq 3$), then $h_1 \leq h_2$, and, for $d \geq 4$, equality occurs only if P is stacked.*

This is a simple property of shellings, and the aim of this paper is to give an elementary proof of it using only shellings. It is easily seen that $h_1 = h_2$ for $d = 3$, so assume $d \geq 4$.

It follows from the definitions that $\mathcal{K}_i$ is a triangulated $(d-1)$ -ball for $1 \leq i \leq m-1$, so the complex $\partial\mathcal{K}_i$ of its outer faces is a triangulated $(d-2)$ -sphere. Moreover, $\mathcal{K}_m = \partial P$ is a triangulated $(d-1)$ -sphere, so $\partial\mathcal{K}_m = \emptyset$. Comparing $\partial\mathcal{K}_i$ and $\partial\mathcal{K}_{i+1}$, we say that a face of $\partial\mathcal{K}_i$ *disappears* if it is not a face of $\partial\mathcal{K}_{i+1}$. Since the common $(d-2)$ -faces of $\partial\mathcal{K}_i$ and F_{i+1} disappear, a j -operation makes precisely those faces of $\partial\mathcal{K}_i$ containing some $(d-j-1)$ -face disappear. Moreover, a disappearing $(d-4)$ -face of $\partial\mathcal{K}_i$ must lie in precisely three $(d-2)$ -faces of $\partial\mathcal{K}_i$, hence in precisely three $(d-3)$ -faces of $\partial\mathcal{K}_i$.

2. Chains

The proof of the theorem is based on the following structure on $\partial\mathcal{K}_i$.

Definition. For an interior $(d-2)$ -face A of $\mathcal{K}_i$ ($1 \leq i \leq m-1$), let $C_i(A)$ be the complex spanned by the common $(d-3)$ -faces of A and $\partial\mathcal{K}_i$. Then $C_i(A)$ is a *chain* of $\partial\mathcal{K}_i$.

The definition is illustrated in Fig. 1 showing complexes $\partial\mathcal{K}_1, \dots, \partial\mathcal{K}_4$ for some 4-polytope. A j -operation is marked by j . Chains are represented by bold edge-chains; an edge belonging to several chains is drawn several times. The vertex f is needed later.

Fig. 1

Note that a chain may be empty. A $(d-3)$ -face of $\partial\mathcal{K}_i$ may belong to several chains, but there is at most one chain containing two given $(d-3)$ -faces.

3. A Local Property of a Family of Chains

Let f be a $(d-4)$ -face of $\partial\mathcal{K}_i$. Each chain C of $\partial\mathcal{K}_i$ containing f is an f -chain. By the definition of a chain, f then lies in one or in two $(d-3)$ -faces of C . In the latter case, C is also called an f -bichain.

Let $\mathcal{A}_i$ be the family of chains of $\partial\mathcal{K}_i$, let i be fixed throughout this section, and consider a subfamily $\mathcal{I}_i \subset \mathcal{A}_i$. We estimate the number n of $(d-3)$ -faces of $\partial\mathcal{K}_i$ containing f . Clearly, $n \geq 3$, and $n \geq n(\mathcal{I}_i)$, the number of $(d-3)$ -faces containing f that lie in some f -chain of $\mathcal{I}_i$. In Fig. 1, e.g., $n(\mathcal{A}_2) = 2$, $n(\mathcal{A}_3) = 3$, and $n(\mathcal{A}_4) = 4$.

We use f -bichains to improve the estimation. The $(d-3)$ -faces of $\partial\mathcal{K}_i$ containing f can be naturally ordered $g_1, g_2, \dots, g_s = g_1$ so that two subsequent ones span a $(d-2)$ -face of $\partial\mathcal{K}_i$, and g_1 may be chosen arbitrarily. For a given f -bichain C of $\mathcal{I}_i$, choose $g_1 \in C$, and let g_t be the other $(d-3)$ -face of C containing f . By definition of a chain, g_1 and g_t span an interior $(d-2)$ -face of $\mathcal{K}_i$, hence $t \neq 2, s-1$, so that $\{g_2, \dots, g_{t-1}\}$ and $\{g_{t+1}, \dots, g_{s-1}\}$ are nonempty sets called *sides* of C ; thus a side is a nonempty set of $(d-3)$ -faces. If C has a side S such that no $(d-3)$ -face of S lies in a chain of $\mathcal{I}_i$, that is, S is disjoint from each chain of $\mathcal{I}_i$, then C is a *counting* f -bichain, and then C guarantees that an extra $(d-3)$ -face of $\partial\mathcal{K}_i$ containing f exists, in addition to those lying in the f -chains of $\mathcal{I}_i$. Note that if two counting f -bichains exist, then these two extra $(d-3)$ -faces are different. Letting $c(\mathcal{I}_i)$ be the number of counting f -bichains of $\mathcal{I}_i$, we thus have $n \geq \max\{3, n(\mathcal{I}_i)\} + c(\mathcal{I}_i)$, which is at least four if an f -bichain exists, and is three if f lies in precisely three $(d-3)$ -faces, in particular, if f is a disappearing face.

We now define the *value* $V_f(\mathcal{I}_i)$ of $\mathcal{I}_i$ in f by

$$V_f(\mathcal{I}_i) := \max\{3, n(\mathcal{I}_i)\} + c(\mathcal{I}_i) - 3. \quad (3.1)$$

The definition is such that it ensures

Lemma 3.1. *If $\mathcal{I}_i$ contains an f -bichain, then $V_f(\mathcal{I}_i) \geq 1$, and if f is disappearing, then $V_f(\mathcal{I}_i) = 0$.*

In Fig. 1, e.g., $V_f(\mathcal{A}_4) = 1$, $V_f(\mathcal{A}_2) = 1$, and $V_f(\mathcal{A}_3) = 2$.

Lemma 3.2. *We have $V_f(\mathcal{I}_i) \leq V_f(\mathcal{A}_i)$.*

Proof. A proof is needed only if $\mathcal{I}_i$ contains an f -bichain. Then $n(\mathcal{I}_i) \geq 2$. If $n(\mathcal{I}_i) = 2$, then $\mathcal{I}_i$ has a unique f -bichain C , which is counting, so $V_f(\mathcal{I}_i) = 1$. Since C also belongs to $\mathcal{A}_i$, Lemma 3.1 yields $V_f(\mathcal{A}_i) \geq 1$, as claimed.

If $n(\mathcal{I}_i) \geq 3$, we must show that $n(\mathcal{I}_i) + c(\mathcal{I}_i) \leq n(\mathcal{A}_i) + c(\mathcal{A}_i)$. We must consider only those f -bichains $C_1, \dots, C_r$ of $\mathcal{I}_i$ that are counting as chains of $\mathcal{I}_i$, but not counting as chains of $\mathcal{A}_i$. Each such f -bichain C_k has a side S_k disjoint from each chain of $\mathcal{I}_i$, and so the sides S_k are mutually disjoint. On the other hand, each S_k contains a $(d-3)$ -face of some chain of $\mathcal{A}_i$. So $n(\mathcal{A}_i) \geq n(\mathcal{I}_i) + r$, and the assertion follows. $\square$

4. The Value of a Family of Chains

Again, let i be fixed, and consider a subfamily $\mathcal{I}_i \subset \mathcal{A}_i$. We will include the important properties of $\mathcal{I}_i$ in a number called value.

Let $f_1, \dots, f_z$ be some ordering of the $(d-4)$ -faces of $\partial\mathcal{K}_i$. Determine the value of $\mathcal{I}_i$ in f_1 , then remove from $\mathcal{I}_i$ all its f_1 -chains, and determine the value of that subfamily in f_2 , and so on. More precisely, let $\mathcal{I}_i^{(1)} := \mathcal{I}_i$, and let $\mathcal{I}_i^{(k)}$ be that subfamily of $\mathcal{I}_i$ where all the f_1 -chains, f_2 -chains, $\dots$, f_{k-1} -chains are removed. Then the *value* $V(\mathcal{I}_i)$ of $\mathcal{I}_i$ is defined as

$$V(\mathcal{I}_i) := \min \sum_{k=1}^z V_{f_k}(\mathcal{I}_i^{(k)}),$$

where the minimum is taken over all the orderings of the $(d-4)$ -faces of $\partial\mathcal{K}_i$. In Fig. 1, e.g., the values of $\mathcal{A}_1, \dots, \mathcal{A}_4$ are 0, 1, 2, 1, respectively.

Lemma 4.1. *If $\mathcal{I}_i \subset \mathcal{A}_i$, then $V(\mathcal{I}_i) \leq V(\mathcal{A}_i)$. Moreover, $V(\mathcal{I}_i) = V(\mathcal{A}_i)$ if $\mathcal{I}_i$ arises from $\mathcal{A}_i$ by removing all its f -chains for some f with $V_f(\mathcal{A}_i) = 0$.*

Proof. The first part follows immediately from Lemma 3.2. As to the second part, let $f_1, \dots, f_z$ be an ordering of the $(d-4)$ -faces of $\partial\mathcal{K}_i$ with $f = f_1$. By Lemma 3.2, $V_f(\mathcal{I}_i) = V_f(\mathcal{A}_i) = 0$, thus $\sum_{k=1}^z V_{f_k}(\mathcal{I}_i^{(k)}) = \sum_{k=1}^z V_{f_k}(\mathcal{A}_i^{(k)})$. Since $\mathcal{I}_i$ has no f -chain, we may neglect $f = f_1$ in determining $V(\mathcal{I}_i)$, thus $V(\mathcal{I}_i) \geq V(\mathcal{A}_i)$. $\square$

5. Proof of $h_1 \leq h_2$

Comparing the values $V(\mathcal{A}_i)$ and $V(\mathcal{A}_{i+1})$, we show that each 1-operation increases the value, and only 2-operations decrease it, each at most by one.

A shelling operation $\tau: \mathcal{K}_i \rightarrow \mathcal{K}_{i+1}$ transforms a chain $C_i(A)$ into the chain $C_{i+1}(A)$, where $C_i(A) \supset C_{i+1}(A)$. If $C_i(A) = C_{i+1}(A)$, then $C_i(A)$ *remains unchanged*. Moreover, each disappearing $(d-2)$ -face of $\partial\mathcal{K}_i$ generates a *new* chain of $\partial\mathcal{K}_{i+1}$. So $\mathcal{A}_{i+1}$ consists of the transformed chains of $\mathcal{A}_i$ and the new chains.

Lemma 5.1. *Let $\mathcal{I}_i \subset \mathcal{A}_i$ be a family of chains remaining unchanged, so that also $\mathcal{I}_i \subset \mathcal{A}_{i+1}$. Then the value of $\mathcal{I}_i$ is the same in both cases.*

Proof. In determining the value of $\mathcal{I}_i$, it is enough to regard those $(d-4)$ -faces of $\partial\mathcal{K}_i$ (resp. $\partial\mathcal{K}_{i+1}$) which lie in a chain of $\mathcal{I}_i$, and so are common faces of $\partial\mathcal{K}_i$ and $\partial\mathcal{K}_{i+1}$. Moreover, as is easily seen, the property that an f -bichain of $\mathcal{I}_i$ is counting does not depend on whether we regard $\partial\mathcal{K}_i$ or $\partial\mathcal{K}_{i+1}$. $\square$

Lemma 5.2. *If $\tau: \mathcal{K}_i \rightarrow \mathcal{K}_{i+1}$ is a j -operation for some $j \geq 3$, then $V(\mathcal{A}_i) \leq V(\mathcal{A}_{i+1})$.*

Proof. Recall that τ makes the faces containing some $(d-j-1)$ -face e disappear,

where $d - j - 1 \leq d - 4$. Thus a $(d - 3)$ -face of $\partial\mathcal{K}_i$ disappears if and only if it contains e , that is, if and only if it contains a disappearing $(d - 4)$ -face.

Thus, letting $\mathcal{I}_i \subset \mathcal{A}_i$ be the family of chains remaining unchanged, $\mathcal{I}_i$ arises from $\mathcal{A}_i$ by removing all its f -chains for each disappearing $(d - 4)$ -face f . Hence Lemmas 3.1 and 4.1 yield $V(\mathcal{I}_i) = V(\mathcal{A}_i)$.

Now, $\mathcal{I}_i \subset \mathcal{A}_{i+1}$ also, and Lemmas 5.1 and 4.1 yield $V(\mathcal{I}_i) \leq V(\mathcal{A}_{i+1})$. $\square$

Lemma 5.3. *If $\tau: \mathcal{K}_i \rightarrow \mathcal{K}_{i+1}$ is a 1-operation, then $V(\mathcal{A}_i) + 1 \leq V(\mathcal{A}_{i+1})$.*

Proof. Recall that τ makes one $(d - 2)$ -face disappear, so the chains of $\mathcal{A}_i$ remain unchanged, and one new chain $C \in \mathcal{A}_{i+1}$ arises, which is a counting f -bichain for each of its $(d - 4)$ -faces f . Thus $\mathcal{A}_{i+1} = \mathcal{A}_i \cup \{C\}$. Moreover, in determining $V(\mathcal{A}_{i+1})$, we may regard the $(d - 4)$ -faces of $\partial\mathcal{K}_i$, since they belong to $\partial\mathcal{K}_{i+1}$, and the other $(d - 4)$ -faces of $\partial\mathcal{K}_{i+1}$ do not lie in any chain.

So, let $f_1, \dots, f_k, \dots$ be an ordering of the $(d - 4)$ -faces of $\partial\mathcal{K}_i$, remove from $\mathcal{A}_i$ and from $\mathcal{A}_{i+1}$ its f_1 -chains, $\dots$, f_{k-1} -chains in order to get $\mathcal{A}_i^{(k)}$ (resp. $\mathcal{A}_{i+1}^{(k)}$), and compare $V_{f_k}(\mathcal{A}_i^{(k)})$ and $V_{f_k}(\mathcal{A}_{i+1}^{(k)})$. If f_k does not lie in C , then clearly these values are equal. For the first f_k lying in C it is easily seen that $V_{f_k}(\mathcal{A}_i^{(k)}) + 1 \leq V_{f_k}(\mathcal{A}_{i+1}^{(k)})$. For the following f_k , the values are again equal, since then C is removed. This proves the assertion. $\square$

Lemma 5.4. *If $\tau: \mathcal{K}_i \rightarrow \mathcal{K}_{i+1}$ is a 2-operation, then $V(\mathcal{A}_i) - 1 \leq V(\mathcal{A}_{i+1})$.*

Proof. Recall that τ makes one $(d - 3)$ -face g disappear and the two $(d - 2)$ -faces A_1 and A_2 containing g . So, for each $C \in \mathcal{A}_i$, its transform $\tau(C) \in \mathcal{A}_{i+1}$ is spanned by the $(d - 3)$ -faces of C without g ; precisely the chains not containing g remain unchanged. The $(d - 3)$ -faces of A_1 (resp. A_2) without g span two new chains C_1 and C_2 . Moreover, in determining $V(\mathcal{A}_{i+1})$, we may regard the $(d - 4)$ -faces of $\partial\mathcal{K}_i$, since they belong to $\partial\mathcal{K}_{i+1}$, and the other $(d - 4)$ -faces of $\partial\mathcal{K}_{i+1}$ do not lie in any chain.

So, let $f_1, \dots, f_k, \dots, f_z$ be an ordering of the $(d - 4)$ -faces of $\partial\mathcal{K}_i$, and remove from $\mathcal{A}_i$ (resp. from $\mathcal{A}_{i+1}$) its f_1 -chains, $\dots$, f_{k-1} -chains in order to get $\mathcal{A}_i^{(k)}$ (resp. $\mathcal{A}_{i+1}^{(k)}$). It is enough to show

$$\sum_{k=1}^z V_{f_k}(\mathcal{A}_i^{(k)}) - 1 \leq \sum_{k=1}^z V_{f_k}(\mathcal{A}_{i+1}^{(k)}). \quad (5.1)$$

Let f_q be the first f_k lying in g , and consider f_k for $1 \leq k \leq q - 1$. Then $\mathcal{A}_{i+1}^{(k)}$ consists of the transformed chains of $\mathcal{A}_i^{(k)}$ and possibly C_1 or C_2 . Thus $V_{f_k}(\mathcal{A}_i^{(k)}) = V_{f_k}(\mathcal{A}_{i+1}^{(k)})$ if $f_k \notin C_1, C_2$. Note that f_k cannot lie in both C_1 and C_2 because of $1 \leq k \leq q - 1$. If $f_k \in C_1$ occurs, then C_1 is a counting f_k -bichain of $\mathcal{A}_{i+1}^{(k)}$, hence for the first such f_k we have

$$V_{f_k}(\mathcal{A}_i^{(k)}) + 1 \leq V_{f_k}(\mathcal{A}_{i+1}^{(k)}), \quad (5.2)$$

as is easily seen; for each following such f_k we have $V_{f_k}(\mathcal{A}_i^{(k)}) \leq V_{f_k}(\mathcal{A}_{i+1}^{(k)})$, since then C_1 is removed. The same is true if $f_k \in C_2$ occurs.

Consider now f_q . Let the $(d-3)$ -faces of $\partial\mathcal{K}_i$ containing f_q be naturally ordered $g_1, g_2, \dots, g_s = g_1$ with $g = g_1$, and let $g_2 \in C_2, g_{s-1} \in C_1$. Still, $\mathcal{A}_{i+1}^{(q)}$ consists of the transformed chains of $\mathcal{A}_i^{(q)}$ and possibly C_1 or C_2 . We may assume that $\mathcal{A}_i^{(q)}$ has a chain containing g , for otherwise the chains of $\mathcal{A}_i^{(q)}$ remain unchanged, and Lemma 3.2 yields $V_{f_k}(\mathcal{A}_i^{(k)}) \leq V_{f_k}(\mathcal{A}_{i+1}^{(k)})$ for $k \geq q$, which together with the above inequalities yields (5.1).

In order to compare $V_{f_q}(\mathcal{A}_i^{(q)})$ and $V_{f_q}(\mathcal{A}_{i+1}^{(q)})$, we go into the details of (3.1). Except g , each $(d-3)$ -face of any chain of $\mathcal{A}_i^{(q)}$ also belongs to some chain of $\mathcal{A}_{i+1}^{(q)}$. On the other hand, because of C_1 and C_2 , it may happen that g_j ($j = 2, s-1$) does not lie in any chain of $\mathcal{A}_i^{(q)}$ but lies in some chain of $\mathcal{A}_{i+1}^{(q)}$. Put $\varepsilon_j := 1$ in that case, and $\varepsilon_j := 0$ otherwise. Then, in the notation of (3.1),

$$n(\mathcal{A}_{i+1}^{(q)}) = n(\mathcal{A}_i^{(q)}) - 1 + \varepsilon_2 + \varepsilon_{s-1}.$$

Next, let C be a counting f_q -bichain of $\mathcal{A}_i^{(q)}$, with transform $\tau(C)$.

If $g \notin C$, then $\tau(C) = C$ is also an f_q -bichain. If $\varepsilon_j = 1$, then g_j might cause $\tau(C)$ to be not counting as a chain of $\mathcal{A}_{i+1}^{(q)}$. However, g_j lies in C or lies on the same side of C as g , and, by assumption, g belongs to some chain of $\mathcal{A}_i^{(q)}$. Thus the other side of C is disjoint from each chain of $\mathcal{A}_i^{(q)}$ and from g_2 and g_{s-1} , hence $\tau(C)$ is also counting.

If $g \in C$, then $\tau(C)$ is no more an f_q -bichain. Thus, in the notation of (3.1),

$$c(\mathcal{A}_{i+1}^{(q)}) = c(\mathcal{A}_i^{(q)}) - \varepsilon,$$

where ε is the number of counting f_q -bichains C of $\mathcal{A}_i^{(q)}$ with $g \in C$. Since each chain is uniquely determined by two of its $(d-3)$ -faces, at most two such C exist. Clearly, $g_2, g_{s-1} \notin C$, and g_2 and g_{s-1} lie on different sides of C , one of which is disjoint from each chain of $\mathcal{A}_i^{(q)}$, say that side containing g_2 . Then g_2 does not belong to any chain of $\mathcal{A}_i^{(q)}$. Then either $\varepsilon_2 = 1$, or g_2 does not belong to any chain of $\mathcal{A}_{i+1}^{(q)}$, that is, C_2 has already been removed, and (5.2) is valid for some $k \leq q-1$. Thus if $n(\mathcal{A}_i^{(q)}) \geq 3$, where we may overlook the maximum in (3.1), we have

$$\sum_{k=1}^q V_{f_k}(\mathcal{A}_i^{(k)}) - 1 \leq \sum_{k=1}^q V_{f_k}(\mathcal{A}_{i+1}^{(k)}).$$

If $n(\mathcal{A}_i^{(q)}) \leq 2$, then clearly $V_{f_q}(\mathcal{A}_i^{(q)}) \leq 1$, and we also obtain that inequality.

Finally, $\mathcal{A}_i^{(q+1)}$ has no more a chain containing g , thus its chains remain unchanged. Moreover, $\mathcal{A}_i^{(q+1)} \subset \mathcal{A}_{i+1}^{(q+1)}$, hence Lemma 3.2 yields $V_{f_k}(\mathcal{A}_i^{(k)}) \leq V_{f_k}(\mathcal{A}_{i+1}^{(k)})$ for $k > q$, which implies (5.1). $\square$

To complete the proof of $h_1 \leq h_2$, note that $\mathcal{A}_1 = \emptyset$, thus $V(\mathcal{A}_1) = 0$. Moreover, $\mathcal{K}_m$ is the boundary complex of our polytope, so $\partial\mathcal{K}_m = \emptyset$, hence each $(d-4)$ -face of $\partial\mathcal{K}_{m-1}$ disappears, and Lemma 3.1 yields $V(\mathcal{A}_{m-1}) = 0$. Pursue now the value $V(\mathcal{A}_i)$ during a shelling. It starts with $V(\mathcal{A}_1) = 0$, each 1-operation increases it, only 2-operations decrease it, each at most by one, and it ends with $V(\mathcal{A}_{m-1}) = 0$. Thus $h_1 \leq h_2$.

6. The Case of Equality for $d \geq 4$

For $d \geq 4$, let the polytope P satisfy $h_1 = h_2$. In order to show that P is stacked, we use induction on d , that is, we prove it for $d \geq 5$ assuming it to be true for $d - 1$, and we prove it for $d = 4$.

Let v be a vertex of P , let $F_1, \dots, F_r, \dots, F_m$ be a shelling of ∂P with $v \in F_i$ if and only if $1 \leq i \leq r$ (a small modification of the original proof of shellability described on p. 173 of [9] shows that such shellings exist, see [5] or p. 243 of [12], and let $P|v$ be the vertex-figure of P at v . Then $\partial \mathcal{K}_r$ is isomorphic to $\partial(P|v)$. Let $h_j^{(d-1)}$ be the number of j -operations in a shelling of $\partial(P|v)$.

Lemma 6.1. *We have $V(\mathcal{A}_r) = 0$, and $h_1^{(d-1)} = h_2^{(d-1)}$.*

Proof. As we have shown, $h_1^{(d-1)} \leq h_2^{(d-1)}$. The shelling of ∂P induces a shelling of $\partial(P|v)$, where j -operations induce j -operations, so that $h_j^{(d-1)}$ is the number of j -operations of ∂P up to $\mathcal{K}_r$, for $j = 1, 2$. As we have seen at the end of Section 5, the value $V(\mathcal{A}_r)$ estimates the difference between the number of the remaining 2- and 1-operations of ∂P , thus $V(\mathcal{A}_r) \leq (h_2 - h_2^{(d-1)}) - (h_1 - h_1^{(d-1)})$, and the assertion follows from $h_1 = h_2$. $\square$

We show next that P has an edge e containing v and lying in precisely $d - 1$ facets. In fact, for $d \geq 5$, $h_1^{(d-1)} = h_2^{(d-1)}$ implies that $P|v$ is stacked by the induction hypothesis, hence some vertex of $P|v$ lies in precisely $d - 1$ ($d - 2$)-faces, which yields the claimed edge of P . For $d = 4$, however, the chains of $\mathcal{A}_r$ coincide with the edges of $\partial \mathcal{K}_r$. Thus $V(\mathcal{A}_r) = 0$ implies that $\partial \mathcal{K}_r$ has a vertex lying in only three edges, and so lying in only three 2-faces, which yields the claimed edge e of P also in this case.

Let v' be the other vertex of e , and let $F_1, \dots, F_{d-1}$ be the facets of P containing e . The complex spanned by them has, apart from v and v' , precisely $d - 1$ vertices, which span a $(d - 2)$ -simplex A_0 .

We show that A_0 is a face of P . If not, extend the sequence $F_1, \dots, F_{d-1}$ to a shelling $F_1, \dots, F_{d-1}, \dots, F_m$ (which exists as above), and consider $\partial \mathcal{K}_i$ for $i \geq d - 1$. We introduce a *modified definition of chains*. Let the only chain of $\partial \mathcal{K}_{d-1}$ be the complex ∂A_0 , which is spanned by the common $(d - 3)$ -faces of $\partial \mathcal{K}_{d-1}$ and A_0 . Let the chains of $\partial \mathcal{K}_i$, $i > d - 1$, be the complexes spanned by the common $(d - 3)$ -faces of $\partial \mathcal{K}_i$ and A , where $A = A_0$, or A is an interior $(d - 2)$ -face of $\mathcal{K}_i$ but not of $\mathcal{K}_{d-1}$, that is, A is a disappearing $(d - 2)$ -face of some $\partial \mathcal{K}_p$, $d - 1 \leq p < i$. Since A_0 is not a face of any $\partial \mathcal{K}_i$, the results of Section 3–5 are also valid for those chains of $\partial \mathcal{K}_i$, $i \geq d - 1$. Thus, since the only chain of $\partial \mathcal{K}_{d-1}$ is an f -bichain for each of its $(d - 4)$ -faces f , we have $V(\mathcal{A}_{d-1}) = 1$. Moreover, in our shelling, one 1-operation and one 2-operation occur up to $\mathcal{K}_{d-1}$, thus the considerations at the end of Section 5 yield $V(\mathcal{A}_{d-1}) \leq (h_2 - 1) - (h_1 - 1) = 0$, which is a contradiction.¹

Let S be the $(d - 1)$ -simplex spanned by A_0 and v , so that all the $(d - 2)$ -faces of S are also faces of P ; let S' be the same for v' . If both these simplices are facets of P , then

¹ In his equality discussion in [1], Barnette also finds such an edge e , and shows that A_0 is a face of P .

P is a simplex, and we are done. If not, then the hyperplane spanned by one of them, say S , dissects P into two simplicial polytopes P_1 and P_2 with common facet S . Take a shelling of ∂P_1 with last facet S and a shelling of ∂P_2 with first facet S , and combine them to obtain a shelling of ∂P . Then $h_1 = h_1(P_1) + h_1(P_2)$ and $h_2 = h_2(P_1) + h_2(P_2)$. Because of $h_1(P_1) \leq h_2(P_1)$ and $h_1(P_2) \leq h_2(P_2)$, the assumption $h_1 = h_2$ yields $h_1(P_1) = h_2(P_1)$ and $h_1(P_2) = h_2(P_2)$.

Iteration now shows that P has a triangulation such that all interior faces are $(d - 1)$ -dimensional, that is, P is stacked.

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