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Parallelotopes of Maximum Volume in a Simplex

M. Lassak

Instytut Matematyki i Fizyki ATR,
Kaliskiego 7, 85-796 Bydgoszcz, Poland
lassak@atr.bydgoszcz.pl

Abstract. It is proved that the maximum possible volume of a parallelotope contained in a d-dimensional simplex S is equal to $(d!/d^d)\,\mathrm{vol}(S)$. A description of all the parallelotopes of maximum volume contained in S is given.

Introduction

Let S be a nondegenerate simplex in Euclidean d-dimensional space E^d. Denote by a a vertex of S and by $v_1, \ldots, v_d$ the vectors determining the edges of S at a. The parallelotope P_a with vertex a and the edges at a determined by vectors $(1/d)v_1, \ldots, (1/d)v_d$ is a subset of S and has volume $d!/d^d$ times the volume of S. Our intuition says that there is no parallelotope contained in S of greater volume. This conjecture is confirmed in Section 1. In Section 2 we describe all parallelotopes of maximum volume contained in S. We also show that all those parallelotopes are inscribed in S. Section 3 discusses the motivation of the present research and comments on some analogous results and problems, in particular, the problem of finding large simplices in a cube.

We denote by $\mathrm{aff}(K)$ and by $\mathrm{conv}(K)$ the affine and the convex hull of a set $K \subset E^d$. The symbol $\mathrm{vol}(C)$ denotes the volume of a convex body $C \subset E^d$. By kC we mean the homothetic copy of C of ratio k and homothety center at the center of gravity of C.

Two sets in E^d are said to be *affinely equivalent* if one of them is an image of the other under an affine nondegenerate transformation.

1. The Maximum Volume of a Parallelotope Contained in a Simplex

By a *cylinder parallel to a direction* δ we mean the union of a family of segments parallel to δ whose endpoints are in two different parallel hyperplanes. The segments are called

δ-*segments* of the cylinder. If the above two hyperplanes are perpendicular to δ, then the cylinder is called *rectangular*. The smallest rectangular cylinder parallel to δ and containing a given compact set X is denoted by $\mathrm{cyl}(X, \delta)$.

Lemma 1. *Let $S \subset E^d$ be a nondegenerate simplex and let $Y \subset E^d$ be a cylinder parallel to a direction δ. If $-(1/d)S \subset Y \subset S$, then the simplex $-(1/d)S$ contains exactly one δ-segment from among the δ-segments of Y.*

Proof. Since for every cylinder there exists an affinely equivalent rectangular cylinder, we limit our considerations to the case when Y is a rectangular cylinder.

From the definition of $\mathrm{cyl}(X, \delta)$ we see that, in order to show the existence of a δ-segment of Y contained in $-(1/d)S$, it is sufficient to show this in the special case of $\mathrm{cyl}(-(1/d)S, \delta)$ in place of Y. Thus let $Y = \mathrm{cyl}(-(1/d)S, \delta)$ from now on. Denote by G and H the two hyperplanes supporting Y which are perpendicular to δ. Observe that each of the hyperplanes G and H contains at least one vertex of the simplex $-(1/d)S$.

We consider three cases in order to prove the existence of a δ-segment of Y in $-(1/d)S$.

Case 1: *exactly one vertex of $-(1/d)S$ is in G and exactly d vertices of $-(1/d)S$ are in H* (*or vice-versa*). Denote by g this vertex of $-(1/d)S$ which is in G, and by F the opposite face of $-(1/d)S$. Since d vertices of F are in H, we see that $F \subset H$. Let F' be the perpendicular projection of F on G. Obviously, F' is a homothetic copy of $S \cap G$. The homothety ratio is $-1/d$. Since g is the center of the simplex $S \cap G$, every $(d-2)$-dimensional plane parallel to a $(d-2)$-dimensional face of $S \cap G$ and passing through g is d times closer to this face than to the opposite vertex of $S \cap G$. We conclude that F' has nonempty intersections with all the $(d-2)$-dimensional planes through g which are parallel to the $(d-2)$-dimensional faces of F'. Thus $g \in F'$ and $Y = \mathrm{conv}(F \cup F')$. Consequently, the δ-segment of cylinder Y with endpoint g is the δ-segment we are looking for.

Case 2: *all vertices of $-(1/d)S$ are in $G \cup H$, and each of the hyperplanes G and H contains more than one vertex of $-(1/d)S$*. We apply the indirect approach: we assume that $-(1/d)S$ contains no δ-segment of Y and we show that Y is not a subset of S.

Denote by $g_1, \dots, g_m$ those vertices of $-(1/d)S$ which are in G, and by $h_1, \dots, h_n$ the vertices of $-(1/d)S$ which are in H. We have $2 \le m \le d-1$ and $2 \le n \le d-1$. Denote by g_i' the projection of g_i on H, where $i = 1, \dots, m$, and denote by h_i' the (perpendicular) projection of h_i on G, where $i = 1, \dots, n$ (see Fig. 1, where $d = 3$ and $m = n = 2$).

The planes $\mathrm{aff}\{g_1', \dots, g_m'\}$ and $\mathrm{aff}\{h_1, \dots, h_n\}$ are contained in the hyperplane H. Since $m + n = d + 1$ and since the simplex $-(1/d)S$ is nondegenerate, these planes have exactly one common point. Denote this point by o. Let o' denote the projection of o on G. Our assumption that $-(1/d)S$ contains no δ-segment of Y (remember that $Y = \mathrm{cyl}(-(1/d)S, \delta)$) implies that $\mathrm{conv}\{h_1, \dots, h_m\}$ or $\mathrm{conv}\{g_1', \dots, g_n'\}$ does not contain o. Assume the first possibility (for the second possibility further consideration is analogical).

Consider the n-dimensional plane K containing $h_1, \dots, h_n, h_1', \dots, h_n', o, o'$. Denote by S^* the simplex such that $-(1/n)S^*$ is the n-dimensional simplex with vertices $h_1, \dots, h_n$ and o'. Of course, $-(1/n)S^* \subset S^* \subset K$. Let Π denote the projection of the

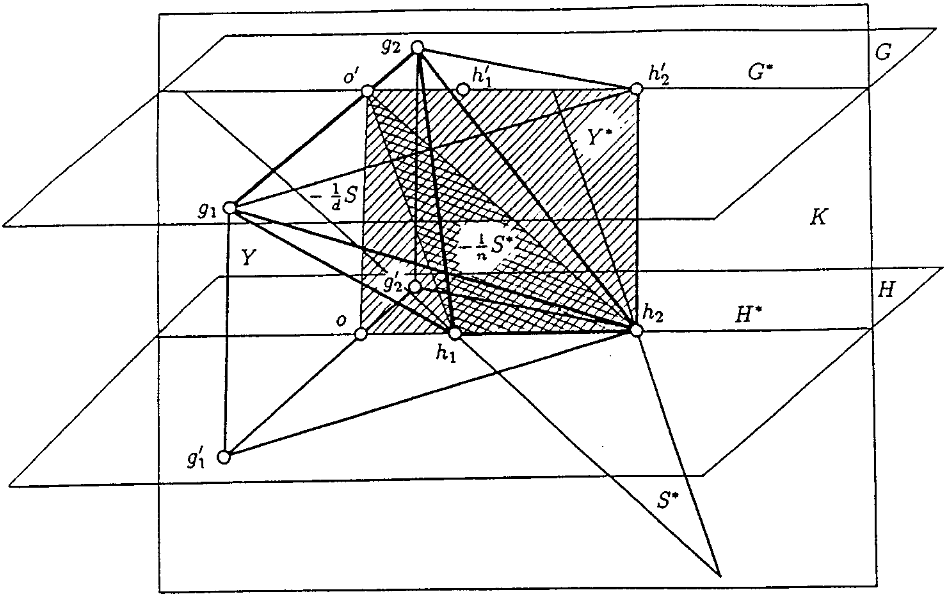

Fig. 1

space E^d on the plane K such that $\Pi(\text{aff}\{g'_1, \ldots, g'_m\}) = \{o\}$. We have $\Pi(S) = S^*$ and $\Pi(-(1/d)S) = -(1/n)S^*$. We apply the n-dimensional version of Case 1 to simplices $-(1/n)S^*$ and S^*. In place of G (as in Case 1) we take the $(n-1)$-dimensional plane $G^* = \text{aff}\{h'_1, \ldots, h'_n\}$. The role of H (as in Case 1) is played by $H^* = \text{aff}\{h_1, \ldots h_n\}$. Since $o \notin \text{conv}\{h_1, \ldots, h_n\}$, we see that $-(1/n)S^*$ contains no δ-segment of the cylinder $Y^* = \text{cyl}(-(1/n)S^*, \delta)$. Applying the n-dimensional version of Case 1 we conclude that S^* does not contain Y^*. Thus an $(n-1)$-dimensional face of S^* does not support Y^*. Since the $(n-1)$-dimensional face of $-(1/n)S^*$ opposite the vertex o' lies in H^*, we conclude that the parallel $(n-1)$-dimensional face of S^* supports Y^* at o'. Thus there is a vertex h_i, where $i \in \{1, \ldots, n\}$, such that the hyperplane carrying the $(n-1)$-dimensional face of S^* containing h_i does not support Y^* (in Fig. 1 both h_1 and h_2 illustrate h_i). This and $\Pi(Y) = Y^*$ imply that the plane carrying the facet of S passing through h_i does not support Y. Thus S does not contain Y.

Case 3: *at least one vertex of* $-(1/d)S$ *is out of* $G \cup H$. Denote by $u_0, \ldots, u_k$ those vertices of $-(1/d)S$ which are in $G \cup H$ and denote the remaining vertices by $u_{k+1}, \ldots, u_d$ (Fig. 2 shows the case when $d = 3$ and $k = 2$). Let $V = \text{aff}\{u_0, \ldots, u_k\}$. Denote by S' the k-dimensional simplex such that $-(1/k)S' = \text{conv}\{u_0, \ldots, u_k\}$. Of course, $-(1/k)S' \subset S' \subset V$ and $-(1/k)S' = V \cap -(1/d)S$.

Since $-(1/d)S \subset Y \subset S$ and since the vertices of $-(1/d)S$ are in the boundary of S, we see that they are boundary points of Y. Thus from the fact that $u_{k+1}, \ldots, u_d$ are not in $G \cup H$ and that $Y = \text{cyl}(-(1/d)S, \delta)$ we conclude that δ is parallel to all those $d-k$ facets of S which contain $u_{k+1}, \ldots, u_d$. Hence δ is parallel to all the $d-k$

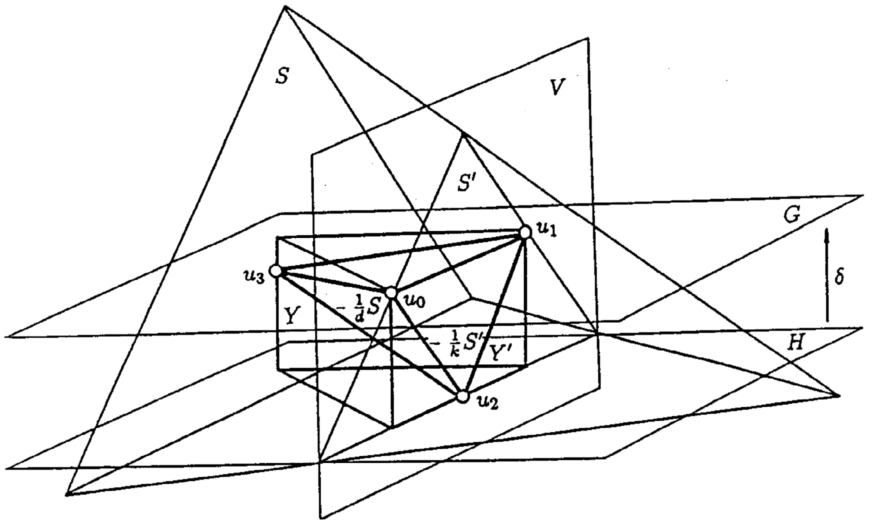

Fig. 2

facets of $-(1/d)S$ that are opposite those vertices. Consequently, δ is parallel to V. Thus $V \cap Y = Y'$, where $Y' = \mathrm{cyl}(-(1/k)S', \delta)$.

Observe that

$$-\frac{1}{k}S' \subset Y' \subset S'. \tag{1}$$

The left inclusion is obvious. Since every $(d-1)$-dimensional face of S supports Y at a vertex of $-(1/d)S$, each $(k-1)$-dimensional face of S' supports Y' at a vertex of $-(1/k)S$. Hence the right inclusion also holds.

We take into account (1) and apply the k-dimensional version of Case 2, where $V \cap G$ and $V \cap H$ play the parts of G and H (as in Case 2), respectively. We conclude that a δ-segment of cylinder Y' is contained in the simplex $-(1/k)S'$. Obviously, this δ-segment is contained in Y as well, and thus is the δ-segment that we are looking for.

This settles Case 3.

We have thus shown the existence of a δ-segment of Y in the simplex $-(1/d)S$.

The promised uniqueness of the δ-segment is a consequence of the following property of a d-dimensional simplex: the sum of dimensions of two faces of a simplex being intersections of this simplex with two opposite supporting hyperplanes is at most $d-1$. It is enough to consider the projection of one of the hyperplanes onto the other. □

From Lemma 1 we immediately obtain the following lemma.

Lemma 2. *Let $S \subset E^d$ be a simplex, let $P \subset E^d$ be a nondegenerate parallelotope, and let $\delta_1, \dots, \delta_d$ be the directions of the edges of P. If $-(1/d)S \subset P \subset S$, then the simplex $-(1/d)S$ contains precisely one δ_i-segment A_i of P for every $i \in \{1, \dots, d\}$.*

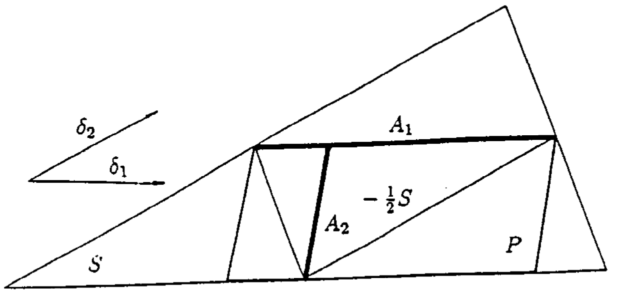

Fig. 3

Figure 3 shows a planar illustration of Lemma 2, and three-dimensional illustrations are given in Figs. 4 and 5.

Let the assumptions of Lemma 2 hold for a parallelotope P and a simplex S. Then we call $A_1, \ldots, A_d$ *the segments connecting the opposite facets of* P, or *the connecting segments* for short. Denote by o a vertex of P. For every $i \in \{1, \ldots, d\}$ we denote by B_i the edge of P which is a translate of A_i and whose endpoint is o. Moreover, for convenience of further considerations, we introduce a coordinate system such that o is the coordinate center, and such that the ith coordinate axis contains B_i in its positive half, where $i = 1, \ldots, d$.

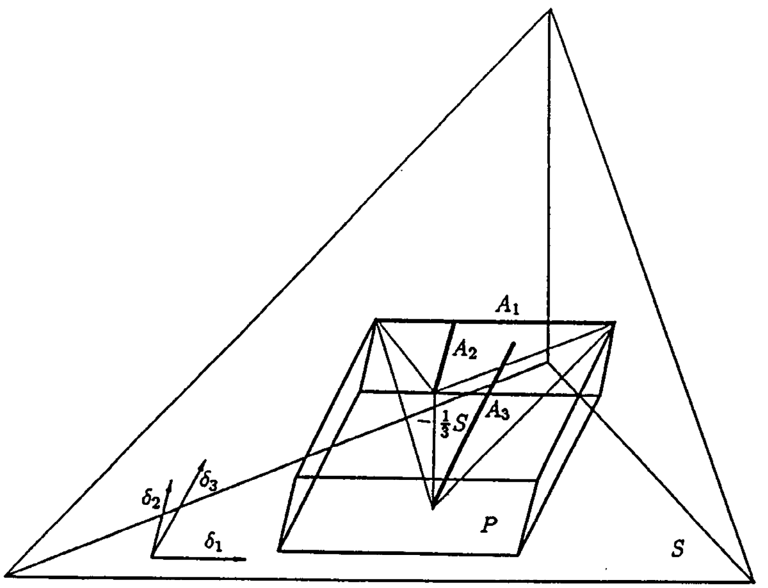

Fig. 4

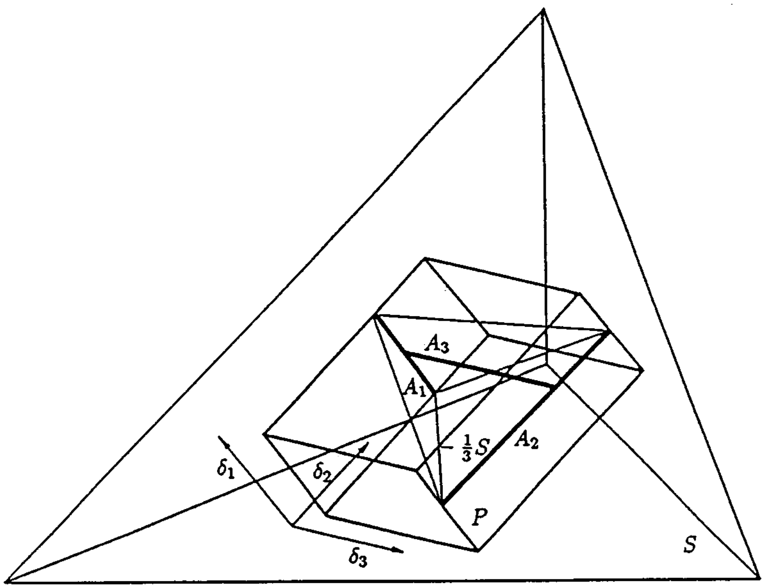

Fig. 5

Obviously, we have

$$\mathrm{vol}(P) \;=\; d! \cdot \mathrm{vol}\left[\mathrm{conv}\left(\bigcup_{i=1}^{d} B_i\right)\right]. \tag{2}$$

Every convex body C can be represented as the union of segments parallel to the ith axis, where $i \in \{1, \ldots, d\}$. We translate these segments parallel to the ith axis in order to get them in the positive half-space $x_i \geq 0$ such that an endpoint of each segment is contained in the hyperplane $x_i = 0$. The union of the translated segments is again a convex body and its volume remains unchanged (see [14]). We denote the obtained body by $\rho_i(C)$. This is *Radziszewski's operation* introduced in [14]. Consider the superposition ρ of operations $\rho_1, \ldots, \rho_d$. Clearly, ρ transforms each convex body into a convex body of the same volume.

Observe that the set $\rho[\mathrm{conv}(\bigcup_{i=1}^{d} A_i)]$ contains $B_1, \ldots, B_d$. Since this set is convex, it contains $\mathrm{conv}(\bigcup_{i=1}^{d} B_i)$ as well. Since ρ does not change the volume of a convex body, we obtain

$$\mathrm{vol}\left[\mathrm{conv}\left(\bigcup_{i=1}^{d} B_i\right)\right] \leq \mathrm{vol}\left[\mathrm{conv}\left(\bigcup_{i=1}^{d} A_i\right)\right]. \tag{3}$$

We conjecture that $\mathrm{vol}[\mathrm{conv}(\bigcup_{i=1}^{d} A_i)] = (1/d!)\mathrm{vol}(P)$.

Lemma 3. *If P is a parallelotope of maximum volume contained in a d-dimensional simplex S, then $-(1/d)S \subset P$.*

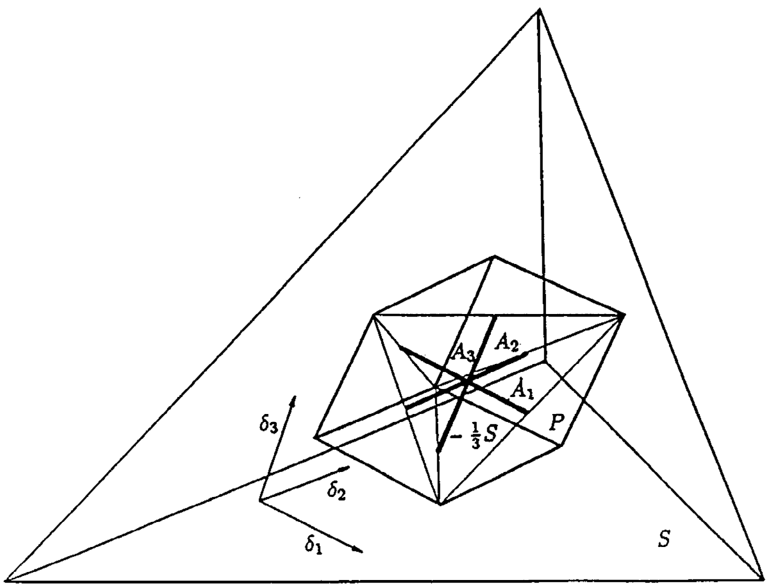

Fig. 6

Proof. From [8] we know that a d times larger homothetic copy R of P contains S. Since R is centrally symmetric, it also contains a translate of $-S$. Consequently, P contains a translate of $-(1/d)S$. Since $P \subset S$, this translate is contained in S. Thus it is nothing other than $-(1/d)S$ itself. □

Lemma 3, and also Lemma 4 and Theorem 1 below, are illustrated in Fig. 3 for $d = 2$, and in Figs. 4 and 5 for $d = 3$ Observe that Fig. 6 does not illustrate Lemmas 3 and 4 and Theorem 1 (but it does illustrate Lemma 2). The reason is that the parallelotope in Fig. 6 is not a parallelotope of maximum volume contained in the simplex. We see that the condition $-(1/d)S \subset P \subset S$ does not characterize parallelotopes of maximum volume contained in S.

Lemma 4. *If P is a parallelotope of maximum volume contained in a d-dimensional simplex S, then* $-(1/d)S = \operatorname{conv}(\bigcup_{i=1}^{d} A_i)$.

Proof. As explained at the beginning of this paper, S contains a parallelotope of volume $(d!/d^d)\operatorname{vol}(S)$. By the assumed maximality of the volume of P we obtain that $\operatorname{vol}(P) \geq (d!/d^d)\operatorname{vol}(S)$. Hence

$$\operatorname{vol}\left(-\frac{1}{d}S\right) \leq \frac{1}{d!}\operatorname{vol}(P). \tag{4}$$

Because of Lemma 3 we can use Lemma 2 which permits us to apply (2) and (3). Moreover, we apply (4). We obtain $\operatorname{vol}(-(1/d)S) \leq \operatorname{vol}[\operatorname{conv}(\bigcup_{i=1}^{d} A_i)]$. This inequality and the inclusion $\mathrm{conv}(\bigcup_{i=1}^{d} A_i) \subset -(1/d)S$ imply that $-(1/d)S = \mathrm{conv}(\bigcup_{i=1}^{d} A_i)$. □

The author conjectures that the inverse implication to this in Lemma 4 holds true as well, i.e., that the equality $-(1/d)S = \mathrm{conv}(\bigcup_{i=1}^{d} A_i)$ characterizes every parallelotope P of maximum volume contained in S; of course, under the assumption that $-(1/d)S \subset P \subset S$.

Theorem 1. If P is a parallelotope of maximum volume in a d-dimensional simplex S, then $\mathrm{vol}(P) = (d!/d^d)\,\mathrm{vol}(S)$.

Proof. We apply Lemma 4. Moreover, owing to Lemmas 2 and 3, we apply (2) and (3). We see that $\mathrm{vol}(-(1/d)S) \geq (1/d!)\,\mathrm{vol}(P)$. This inequality and (4) mean that Theorem 1 holds true. □

Corollary. *If S is a simplex of minimum volume containing a parallelotope P, then* $\mathrm{vol}(S) = (d^d/d!)\,\mathrm{vol}(P)$.

In connection with Theorem 1 it is natural to ask what is the maximum volume of a zonotope contained in a d-dimensional simplex S (or, equivalently, in a regular simplex). It is easy to see that if $d = 2$, then this maximum volume, $\frac{2}{3}\,\mathrm{vol}(S)$, is attained for the zonotope $S \cap (-S)$. The author conjectures that for $d = 3$ this maximum volume is $\frac{4}{9}\,\mathrm{vol}(S)$. The corresponding zonotope is obtained from the octahedron $S \cap (-S)$ by truncating prisms at its vertices by hyperplanes parallel to pairs of opposite edges of S such that one-third of each edge of the octahedron is cut off. The question seems to be more difficult if an analogical construction for higher dimensions also gives a zonotope of maximum volume in S. Paper [5] considers centrally symmetric convex bodies of the maximum volume contained in a simplex.

A natural conjecture concerning Theorem 1 would be that for every convex body $C \subset E^d$ there is a parallelotope $P \subset C$ such that $\mathrm{vol}(P) \geq (d!/d^d)\,\mathrm{vol}(C)$. Surprisingly, such a conjecture is not true in general (see [1]). We only know that it is true for $d = 2$ and $d = 3$, see [14] and [2], and that there exists a parallelotope in C whose volume is at least $d^{-d+1}\,\mathrm{vol}(C)$, see [10] (this is a slight improvement on the estimate $d^{-d}\,\mathrm{vol}(C)$ from [12]).

We pose a problem about the generalization of Theorem 1: evaluate the maximum j-dimensional volume of a j-dimensional parallelotope in a regular d-dimensional simplex and describe all those largest j-dimensional parallelotopes. Of course, the largest one-dimensional parallelotopes are the edges of the regular simplex. The author conjectures that, for $d \geq 3$, the maximum two-dimensional volume of a two-dimensional parallelotope (i.e., the maximum area of a parallelogram) in a regular d-simplex of edge length 1 is $\frac{1}{4}$. If $d = 3$, then this value is attained for the convex hull of two segments, each of which connects the midpoints of the opposite edges of the simplex. For $d \geq 4$, this value is attained for the above described parallelograms contained in the three-dimensional faces of the simplex.

2. Description of the Parallelotopes of Maximum Volume Contained in a Simplex

A sequence Φ of d segments $S_1, \dots, S_d$, called *frame-segments*, contained in a d-dimensional simplex R is said to be a *frame* of R if the following *frame-construction*, consisting of $d + 1$ stages, is possible. At the beginning, i.e., at stage 0, we have 0 frame-segments. We also have $d + 1$ so-called *frame-faces*; they are the vertices of R. When passing from the $(k - 1)$st stage to the kth stage, we add segment S_k. It should connect two disjoint frame-faces. Simultaneously, these two frame-faces are removed and one new frame-face is added. This new frame-face is the convex hull of the union of the removed ones. Clearly, at the jth stage, where $j \in \{0, \dots, d\}$, we have j frame-segments and $d - j + 1$ frame-faces. Thus at the dth stage we have d frame-segments and one frame-face R.

The two frame-faces at stage $d - 1$ are called the *main frame-faces*. By the *main frame-segment* we mean the frame-segment S_d added when passing to the dth stage.

Figures 7–9 show frames of a three-dimensional simplex with vertices a, b, c, d. When passing to successive stages of the frame-construction, we successively add frame-segments ab, ce, df (Fig. 7) and ad, bc, gh (Fig. 8). Sometimes a number of different frame-constructions using the same segments is possible. In Fig. 9 we can make three different frame-constructions by choosing the following orders of using the frame segments: order ab, ac, dk, order ac, ab, dk, and order ac, dk, ab. In Fig. 8 one more order, bc, ad, gh, is also possible. In Fig. 7 only one order is possible.

If frame-construction is possible at least up to the jth stage, where $j \in \{0, \dots, d\}$, then at the jth stage the following two obvious properties hold true.

(i) *Every frame-face of dimension at least* 1 *is the convex hull of the frame-segments contained in it. The number of those frame-segments is equal to the dimension of the frame-face.*

(ii) *Every two different frame-faces at a fixed stage have empty intersection.*

From the definition of the frame we conclude that vectors determined by the frame-segments $S_1, \dots, S_d$ are linearly independent.

By the *parallelotope constructed over a frame* we mean the smallest parallelotope

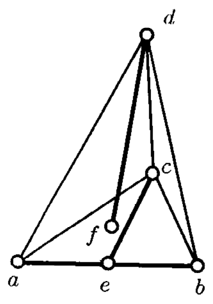

Fig. 7

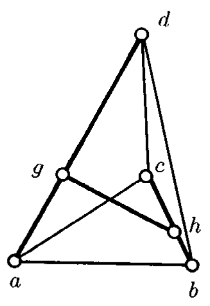

Fig. 8

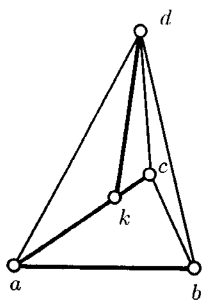

Fig. 9

which contains this frame and whose edges are parallel to the frame-segments. Following are two properties of the parallelotopes constructed over frames.

(iii) *Let* $S_1, \dots, S_d$ *be a frame of a* d*-dimensional simplex* R *and let* P *be the parallelotope constructed over this frame. For every* $k \in \{1, \dots, d\}$, *the pair of faces of* P *parallel to all frame-segments different from* S_k *contains the endpoints of segment* S_k.

Assume the opposite. Then a face F of P parallel to all frame-segments different from S_k contains no endpoint of S_k. Denote by G the greatest (by inclusion) frame-face obtained during the process of frame-construction which is contained in F. Since G contains no endpoint of S_k, all endpoints of the frame-segments not contained in G are out of G. Since G is smaller than R, we see that the frame-construction process cannot be provided to the end. A contradiction.

From (iii) we immediately obtain the following property.

(iv) *The edges of the parallelotope constructed over a frame are translates of the frame-segments*.

In Fig. 3 we see the parallelogram constructed over the frame A_1, A_2 of the simplex $-\frac{1}{2}S$. In Figs. 4 and 5 we have parallelotopes constructed over the frame A_1, A_2, A_3 of $-\frac{1}{3}S$. Figures 3–5 (but not Fig. 6) illustrate the following theorem.

Theorem 2. The parallelotopes of maximum volume in a d-dimensional simplex S coincide with the parallelotopes constructed over the frames of the simplex $-(1/d)S$. The parallelotopes are inscribed in S.

Proof. We divide the proof into three parts.

(I) We show that if P is a parallelotope of maximum volume contained in S and if $A_1, \dots, A_d$ are the connecting segments (see Lemma 3, Lemma 2, and the definition just after), then we can construct a frame of $-(1/d)S$ using $A_1, \dots, A_d$ in place of the frame-segments.

Assume that our frame-construction has been performed up to the jth stage, where $j \in \{0, \dots, d-1\}$. We intend to show that passing to the $(j+1)$st stage is possible.

We start by showing that, in each frame-face K, at the jth stage there is an endpoint of a connecting segment different from the connecting segments contained in this frame-face. Denote the dimension of K by k. If $k = 0$, i.e., if K is a vertex of S, then from Lemma 4 we conclude that a connecting segment has an endpoint at K. Let $k \geq 1$. From (i) we see that K is the convex hull of some k connecting segments. To simplify the notation, assume that these segments are $A_1, \dots, A_k$. Put $K = \operatorname{conv}(A_1 \cup \dots \cup A_k)$ and $L = \operatorname{conv}(A_{k+1} \cup \dots \cup A_d)$. Assume that in K there is no endpoint of a connecting segment different from the connecting segments contained in K. Observe that none of the segments $A_1, \dots, A_k$ has an endpoint in L and that none of the segments $A_{k+1}, \dots, A_d$ has an endpoint in K (the first statement results from the construction of a frame, and the second statement is nothing other than our assumption). This and the fact that the connecting segments are parallel to the edges of P and that they connect pairs of opposite facets of P imply that $\mathrm{conv}(K \cup L)$ has at least $d+2$ vertices. This contradicts the equality (resulting from Lemma 4) that $-(1/d)S = \mathrm{conv}(K \cup L)$.

At the jth stage we have $d - j + 1$ frame-faces. They contain exactly j connecting segments. From (ii) we see that every endpoint of each of the remaining $d - j$ connecting segments is contained in at most one of the above $d - j + 1$ faces. Thus from the fact shown in the preceding paragraph we conclude that there is a pair of frame-faces containing the endpoints of a connecting segment. Thus we can pass to the $(j + 1)$st stage.

We see that a permutation of $\{A_1, \ldots, A_d\}$ is a frame of the simplex $-(1/d)S$.

(II) By induction we show that the parallelotope constructed over an arbitrary frame of $-(1/d)S$ is inscribed in S.

Of course, this property holds true in E^1. Assume that it holds true in $E^1, \ldots, E^{d-1}$ and consider the space E^d.

Let $S_1, \ldots, S_d$ be a frame Φ of $-(1/d)S$. Let S_d be the main frame-segment. Consider the frame Φ^* of S consisting of the successive frame-segments S_i^* that are images of the frame segments S_i, where $i = 1, \ldots d$, under homothety with ratio $-d$ and center in the gravity center of S. Of course, S_d^* is the main frame-segment of frame Φ^*. Let M_1 and M_2 be the main frame-faces of frame Φ^*. Let m_i denote the number of frame-segments in M_i, where $i = 1, 2$. It is clear that $m_1 + m_2 = d - 1$ and that S_d^* has endpoints in M_1 and M_2.

Let T be the parallelotope whose vertices are of the form $(1/d)\sum_{k=1}^{d} s_k$, where s_k is an endpoint of S_k^* for $k = 1, \ldots, d$. Observe that every edge of T is a translate of a segment S_k, where $k \in \{1, \ldots, d\}$. From (iv) we see that the parallelotope T contains a translate of the simplex $-(1/d)S$. This, the obvious inclusion $T \subset S$, and the fact that $-(1/d)S$ is the only homothetic image of S of ratio $-1/d$ that is contained in S, imply that T is nothing other than the parallelotope constructed over frame Φ of $-(1/d)S$.

In order to show that T is inscribed in S, take an arbitrary vertex $w = (1/d)\sum_{k=1}^{d} s_k$ of T. We intend to show that w is on the boundary of S. Clearly, m_i of points $s_1, \ldots, s_d$ are in M_i, where $i = 1, 2$. One among the points $s_1, \ldots, s_d$ is an endpoint of S_d^*. Thus it is in M_1 or in M_2. Without loss of generality we may assume that this endpoint is in M_1. We see that $m_1 + 1$ among the points $s_1, \ldots, s_d$ are in M_1, and that the remaining m_2 of these points are in M_2. Of course, if $m_1 = d-1$, then w is in the boundary of S. Consider the case when $m_1 \le d - 2$. Now $m_2 \ge 1$. Vertex w is in a segment whose one endpoint w_1 is in the convex hull of the aforementioned $m_1 + 1$ points, and the second endpoint w_2 is in the convex hull of the remaining m_2 points. Of course, $w_1 \in M_1$. Moreover, by the inductive assumption, w_2 is in the relative boundary of M_2. Consequently, w is on the boundary of S. We see that T is inscribed in S.

(III) We intend to prove that the parallelotope T constructed over an arbitrary frame of $-(1/d)S$ is a parallelotope of maximum volume contained in S. Because of the inclusion $T \subset S$ shown in (II) and because of Theorem 1, it is sufficient to show that

$$\mathrm{vol}(T) = \frac{d!}{d^d}\mathrm{vol}(S), \tag{5}$$

i.e., that $\mathrm{vol}(-(1/d)S) = (1/d!)\,\mathrm{vol}(T)$. Since every two d-dimensional parallelotopes are affinely equivalent, we do not make our considerations narrower assuming that T is the unit cube. Consequently, we have to show that

$$\operatorname{vol}\left(-\frac{1}{d}S\right) = \frac{1}{d!}. \tag{6}$$

We show this by induction.

Of course, (6) holds true in E^1.

Assume that (6) is true in $E^1, \ldots, E^{d-1}$ and consider the space E^d. By the definition of a frame we see that the main frame-segment connects two frame-faces of dimensions r and $d-r-1$, where $r \in \{0, \ldots, d-1\}$. By the inductive assumption, the r-dimensional volume of the first frame-face is $1/r!$, and the $(d-r-1)$-dimensional volume of the second face is $1/(d-r-1)!$. Presenting $-(1/d)S$ as the union of sections by hyperplanes perpendicular to the main frame-segment, we obtain

$$\operatorname{vol}\left(-\frac{1}{d}S\right) = \int_0^1 \frac{x^r}{r!}\frac{(1-x)^{d-r-1}}{(d-r-1)!}\,dx \;=\; \frac{1}{d!}.$$

This gives (6) and thus confirms (5). Consequently, T is a parallelotope of maximum possible volume in S. □

Let P be a parallelotope of the maximum volume contained in a simplex S. From the definition of a frame and from Theorem 2 we see how the vertices of $-(1/d)S$ are distributed in the boundary of P. There is a pair $P(1)$, $P(2)$ of opposite facets of P which contains all vertices of $-(1/d)S$ (see Figs. 3–5). If $P(j_1)$, where $j_1 \in \{1, 2\}$, contains more than one vertex of $-(1/d)S$, then there is a pair $P(j_1, 1)$, $P(j_1, 2)$ of opposite facets of $P(j_1)$, which contains all vertices of $-(1/d)S$ lying in $P(j_1)$, and so on by induction: if a face $P(j_1, \ldots, j_k)$, where $j_1, \ldots, j_k \in \{1, 2\}$, contains more than one vertex of $-(1/d)S$, then there is a pair $P(j_1, \ldots, j_k, 1)$, $P(j_1, \ldots, j_k, 2)$ of opposite facets of $P(j_1, \ldots, j_k)$, which contains all vertices of $-(1/d)S$ lying in $P(j_1, \ldots, j_k)$.

3. Final Remarks

We give some remarks as requested by the referee, who asked about the motivation of the research, suggested to present a “wider picture,” and asked about connections to the problem of finding large simplices in a cube.

The motivation for this research about large parallelotopes in a simplex was the necessity of an example. In [9] the following analog of the well-known theorem of John [7] about the ellipsoid of maximal volume contained in a convex body was announced. Let C be a convex body and let D be a centrally symmetric convex body in E^d. If D' is an affine image of D of maximal possible volume contained in C, then $C \subset (2d-1)D'$. When writing [9], the author was not sure if the ratio $2d-1$ was the best possible. Candidates, in order to show that the ratio cannot be improved, were a simplex S in place of C, a parallelotope P in place of D, and the parallelotope P_a defined at the beginning of this paper as D'. Since the smallest positive k such that $S \subset kP_a$ is equal to $2d-1$, it was sufficient to show that P_a is a parallelotope of maximum volume in S. This task is performed in this paper. The theorem announced in [9] is proved in [11] which was submitted after the manuscript of the present paper was ready.

Miscellaneous problems of maximizing the volume of an affine image of a convex body that is a subset of another convex body are considered in many papers. The pioneer seems to be Blaschke [3] who showed that every planar convex body C contains a triangle of area $3\sqrt{3}\,\pi^{-1}\,\mathrm{vol}(C)$. This estimate cannot be improved for ellipses. The analogical d-dimensional question remains open. McKinney [13] answered this question for the class of centrally symmetric bodies. He proved that every centrally symmetric convex body contains a simplex of volume at least $(1/d!\,\pi_d)\,\mathrm{vol}(C)$, where π_d denotes the volume of the unit ball in E^d. This value cannot be lessened in the case of an ellipsoid as C. A classic result of this kind is the theorem of John [7], about large ellipsoids in a convex body. Macbeath [12] considered large parallelotopes in C. He proved that every convex body $C \subset E^d$ contains a parallelotope of volume at least $d^{-d}\,\mathrm{vol}(C)$. This estimate has been improved up to $d^{-d+1}\,\mathrm{vol}(C)$ in [10]. Radziszewski [14] obtained the best possible estimate $\frac{2}{9}\,\mathrm{vol}(C)$ for $d = 3$.

Many papers consider large simplices in a parallelotope, or, equivalently, in a cube. See the survey paper of Hudelson et al. [6] and also the survey paper of Brenner and Cummings [4] about the equivalent Hadamard maximum determinant problem. The problem considered in the present paper is in a sense dual to those well-known problems. Observe that *if Δ is a simplex of maximum volume in a parallelotope P, then $P \subset -d\Delta$*. (If it were not true, then we could construct in P a simplex of volume larger than the volume of Δ by moving one vertex of Δ such that it remains in P but that its distance from the hyperplane containing the opposite facet increases.) Thus $\Delta \subset P \subset -d\Delta$. Denote $-d\Delta$ by S. Then $\Delta = -(1/d)S$. We have $-(1/d)S \subset P \subset S$. Consequently, Lemma 2 can be applied here. We see that *every simplex of maximum possible volume contained in the unit d-dimensional cube $K = [0, 1]^d$ contains exactly one unit segment parallel to each given axis*. As a consequence, *if Δ is a simplex of maximum possible volume in K, then for every $i \in \{1, \dots, d\}$ the set $\rho_i(\Delta)$ is a prism*. The base of this prism is the convex hull of the ith projection of the set of all vertices of Δ. The apex of this prism is in the opposite facet of K. Since the operation of Radziszewski does not change the volume, the volume of this prism is equal to the volume of Δ. We see that *the maximal volume of a d-simplex in K is exactly d times smaller than the maximum $(d-1)$-dimensional volume of a convex hull of $d+1$ vertices of a $(d-1)$-dimensional unit cube*. If we take $d+1$ in place of d, we get the equivalent question: how large can the volume of the convex hull of $d+2$ vertices of K be? It is natural to put the more general question: how large a volume can the convex hull of some j vertices of K, where $d+1 \le j \le 2^d$, have? The case $j = d+1$ is just the question about the maximum volume of a d-simplex in K.

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